STRUCTURAL ANALYSIS – DISPLACEMENT (OR STIFFNESS) METHOD

ADVANCED STRUCTURAL MECHANICS

The ERAMCA Project

Environmental Risk Assessment and Mitigation on Cultural Heritage assets in Central Asia

v2022317

Table of Contents

Lecturer/students objectives

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- Present a method of solution of statically indeterminate structures that assumes as unknowns displacements or rotations.
- Understand the method. Apply the method to structures with one unknown kinematic parameter.

Introduction

- Statically indeterminate structures are structures whose reactions cannot be determined using only equations of equilibrium; analysis of such structures requires, in addition to equilibrium, consideration of compatibility of displacements, and therefore of the relative stiffness of structural elements
- Such structures are also described as redundant, in that they contain elements, or constraints, beyond what is required for equilibrium

Displacement (or stiffness) mehod

Procedure

- Certain displacements called kinematic parameters are chosen as unknown. When these parameters are calculated, the solution in terms of internal forces and deformation follows
- It is possible to find infinite compatible solutions but only one assuring the equilibrium of the structure

Nodes and kinematic unknowns

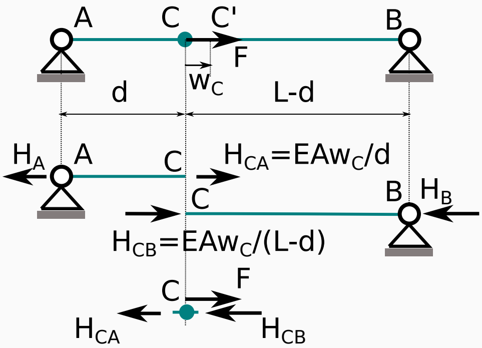
A number of nodes are to be chosen; their displacements are the unknowns of the problem (kinematic unknowns)

Loadings on nodes

The nodes are loaded by external forces and couples and by forces and couples from beams connected to the same nodes; their equilibrium allows to obtain the unknown displacements

The beam is statically indeterminate with respect to axial forces; all displacements will be parallel to the longitudinal axis

The external force is applied at point C, for this reason, such point is chosen as node and its displacement as main parameter



Equilibrium

Is written with respect to the undeformed configuration; for small displacements and deformation C is closed to C’

Recalling the relationship between axial force and elongation (or shortening) for a beam in tension (or in compression) applied to AC and CB:

$$w_{C,CA} = \varepsilon_{z,CA}\, d = \frac{H_{CA}}{EA} d \implies H_{CA} = \frac{EA}{d} w_{C,CA}$$

$$w_{C,CB} = \varepsilon_{z,CB}\,(L-d) = \frac{H_{CB}}{EA}(L-d) \implies H_{CB} = \frac{EA}{L-d} w_{C,CB}$$

where $w_{C,CA} = w_{C,CB} = w_C$, the equilibrium for node C gives:

$$-H_{CA} + F - H_{CB} = 0. \quad \text{i.e.} \quad -\frac{EA}{d} w_C + F - \frac{EA}{L-d} w_C = 0$$

that make possible to obtain w_C:

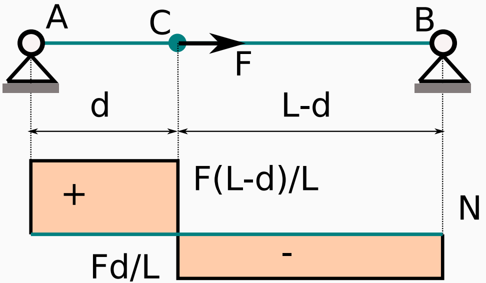
$$w_C = \frac{F}{EA} \frac{d(L-d)}{L}$$

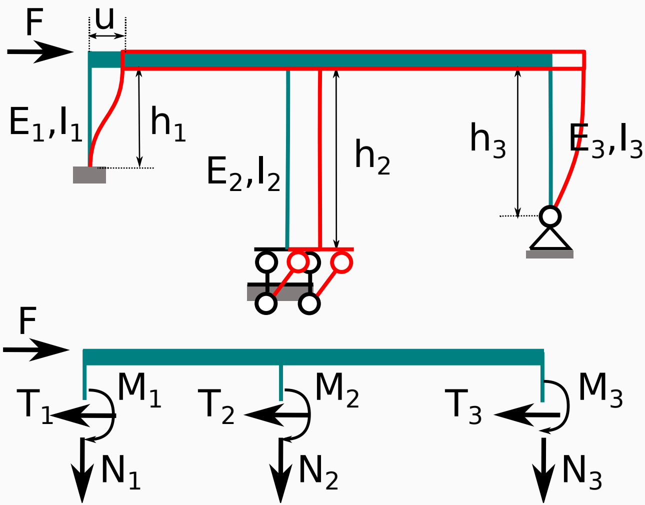
The normal force is given by:

$$N_{CA} = H_{CA} = \frac{EA}{d} w_C = \left(\frac{EA}{d}\right)\left(\frac{F}{EA}\frac{d(L-d)}{L}\right) = F\frac{L-d}{L}$$

$$N_{CB} = -H_{CB} = -\frac{EA}{L-d} w_C = -\left(\frac{EA}{L-d}\right)\left(\frac{F}{EA}\frac{d(L-d)}{L}\right) = -F\frac{d}{L}$$

The AC portion is subjected to tension while BC to compression.





F
u
E_1, I_1
h_1
E_2, I_2
h_2
h_3
E_3, I_3
F
T_1
M_1
T_2
M_2
T_3
M_3
N_1
N_2
N_3

The horizontal beam is assumed to be undeformable (infinitely rigid $E_t I_{xt} \to \infty$ and $E_t A_t \to \infty$). The columns are considered infinitely rigid with respect to axial deformation ($E_t A_t \to \infty$). The only kinematic parameter is the horizontal displacement u (node of the structure).

The equilibrium for the horizontal beam is:

$$-\sum_{i=1}^{3} T_i + F = 0 \implies -K_1 u_1 - K_2 u_2 - K_3 u_3 + F = 0$$

where K_1, K_2 and K_3 represent the bending stiffnesses of three columns with respect to the horizontal displacement (see tables at the end):

$$K_1 = 12\frac{E_1 I_{x1}}{h_1^3}, \quad K_2 = 0, \quad K_3 = 3\frac{E_3 I_{x3}}{h_3^3}$$

Observing that $u = u_1 = u_2 = u_3$, it is obtained:

$$u = \frac{F}{K_1 + K_2 + K_3}$$

and:

$$T_1 = K_1 u = \frac{K_1}{K_1 + K_2 + K_3} F$$

$$T_2 = K_2 u = \frac{K_2}{K_1 + K_2 + K_3} F = 0$$

$$T_3 = K_3 u = \frac{K_3}{K_1 + K_2 + K_3} F$$

It can be seen that F is distributed in proportion to the stiffnesses K_1, K_2 and K_3 of the columns.

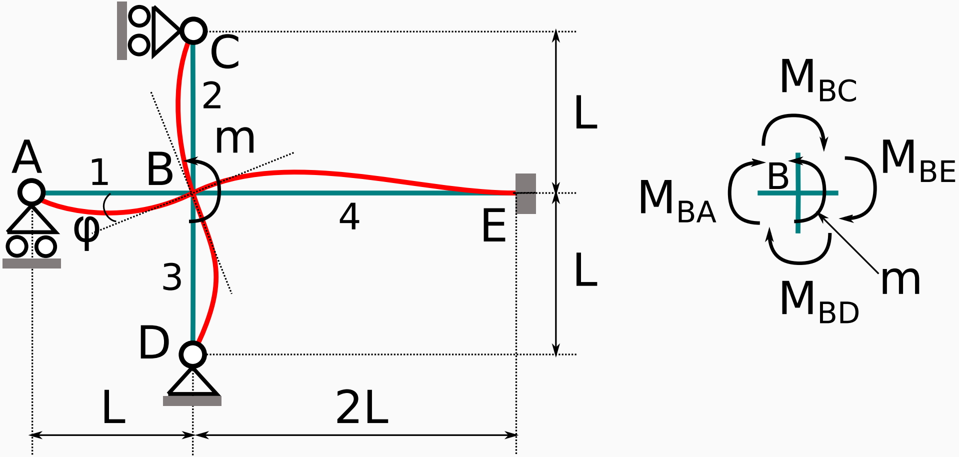
If follows that:

$$u = \frac{F}{12\frac{E_1 I_{x1}}{h_1^3} + 3\frac{E_3 I_{x3}}{h_3^3}}$$

$$T_1 = 12\frac{E_1 I_{x1}}{h_1^3} u = \frac{12\frac{E_1 I_{x1}}{h_1^3}}{12\frac{E_1 I_{x1}}{h_1^3} + 3\frac{E_3 I_{x3}}{h_3^3}} F$$

$$T_2 = 0$$

$$T_3 = 3\frac{E_3 I_{x3}}{h_3^3} u = \frac{3\frac{E_3 I_{x3}}{h_3^3}}{12\frac{E_1 I_{x1}}{h_1^3} + 3\frac{E_3 I_{x3}}{h_3^3}} F$$


C
2
m
A
1
B
φ
4
E
3
D
L
L
L
2L
M_BC
M_BE
M_BA
B
M_BD
m

The beams are assumed to be undeformable, i.e., infinitely rigid with respect to axial deformation ($E_t A_t \rightarrow \infty$). The axial deformation is thus neglected. The only kinematic parameter is the rotation of node B: $\varphi = \varphi_B^{(BA)} = \varphi_B^{(BC)} = \varphi_B^{(BE)} = \varphi_B^{(BD)}$.

Moment equilibrium of the node gives:

$$\sum_{i=1}^{4} M_i - m = 0 \implies K_1\varphi + K_2\varphi + K_3\varphi + K_4\varphi - m = 0$$

where K_1, K_2 and K_3 are the bending stiffnesses of four beams with respect to the rotation of one end (see tables at the end):

$$K_1 = 3\frac{EI_x}{L}, \quad K_2 = 3\frac{EI_x}{L}, \quad K_3 = 3\frac{EI_x}{L}, \quad K_4 = 4\frac{EI_x}{2L} = 2\frac{EI_x}{L}$$

It can be obtained:

$$\varphi = \frac{m}{K_1 + K_2 + K_3 + K_4}$$

and:

$$M_1 = M_B^{(BA)} = K_1\,\varphi = \frac{K_1}{K_1 + K_2 + K_3 + K_4} m$$
$$M_2 = M_B^{(BC)} = K_2\,\varphi = \frac{K_2}{K_1 + K_2 + K_3 + K_4} m$$
$$M_3 = M_B^{(BD)} = K_3\,\varphi = \frac{K_3}{K_1 + K_2 + K_3 + K_4} m$$
$$M_4 = M_B^{(BE)} = K_4\,\varphi = \frac{K_4}{K_1 + K_2 + K_3 + K_4} m$$

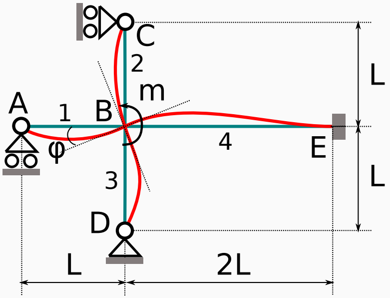
Couple m is distributed in proportion to the stiffnesses K_1, K_2, K_3 and K_4 of the beams.

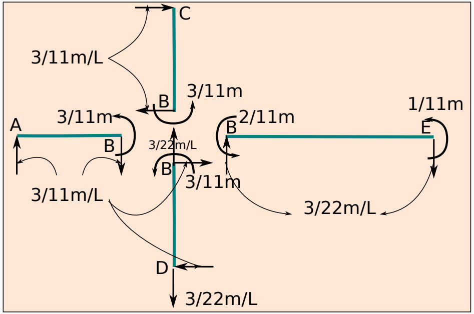
By substituting:

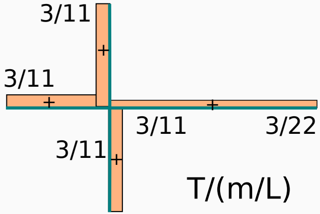
$$\varphi = \frac{mL}{11EI_x}$$

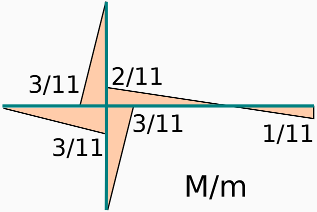
$$M_1 = M_B^{(BA)} = \frac{3}{11}m$$
$$M_2 = M_B^{(BC)} = \frac{3}{11}m$$
$$M_3 = M_B^{(BD)} = \frac{3}{11}m$$
$$M_4 = M_B^{(BE)} = \frac{2}{11}m$$

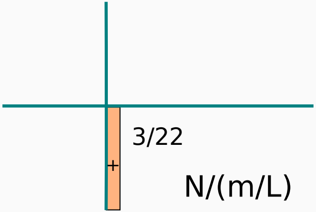
Example – fixed-ends frame











Comparison between force and displacement method

Force (flexibility) method

- Choice of a *statically determinate structure* by elimination of redundant reactions
- The statically determinate structure is *equilibrated* but *not compatible* with supports (the displacements of supports are generally equal to zero)

Displacement (stiffness) method

- Choice of a *geometrical determinate structure* adding constraints to obtain displacements equla to zero for all nodes
- The geometrical determinate structure is *compatible* but *not equilibrated*; to obtain nodes with zero displacements it is necessary to apply forces and couples different from real ones

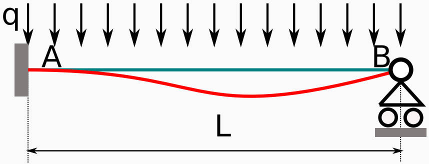
Force (flexibility) method

- It is possible to obtain a compatible system by applying forces and couples X_i to the nodes
- Forces and couples X_i are obtained by imposing that displacements and rotations are compatible with supports

Displacement (stiffness) method

- The system can be equilibrated by imposing displacements and rotations Y_i to the nodes
- Displacements and rotations Y_i are determined by imposing that forces an couples are equilibrated

The beam represented below is examined.



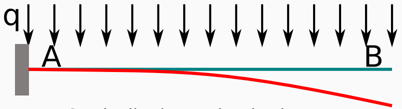
Force (flexibility) method

The statically determinate system is obtained by eliminating the support at B.

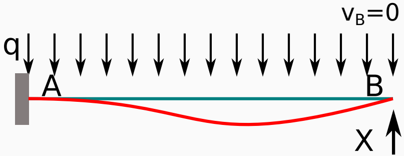
Displacement (stiffness) method

The system with displacements equal to zero is obtained by replacing the roller at B with fixed-end support.

Force (flexibility) method



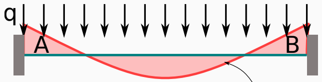
Statically determined scheme



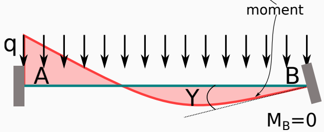
Force V_B is calculated so that displacement at B is the actual one (i.e., $v_B = 0$):

$$v_B = v_B^{(q)} + v_B^{(X)} = 0$$

Displacement (stiffness) method



Fixed end scheme



Rotation at B $\varphi_B = Y$ is calculated so that the moment at B is the actual one (i.e., $M_B = 0$):

$$M_B = M_B^{(q)} + M_B^{(Y)} = 0$$

Force (flexibility) method

The displacements of the right-end of the cantilever are well known (from deflection curve). It is $v_B^{(q)} = \frac{1}{8}\frac{qL^4}{EI_x}$ and $v_B^{(X)} = -\frac{1}{3}\frac{XL^3}{EI_x}$ so that:

$$v_B = v_B^{(q)} + v_B^{(X)} =$$

$$\frac{1}{8}\frac{qL^4}{EI_x} - \frac{1}{3}\frac{XL^3}{EI_x} = 0 \implies$$

$$X = V_B = \frac{3}{8}qL$$

Displacement (stiffness) method

From the summary at the end it is possible to find $M_B^{(q)} = -\frac{qL^2}{12}$ and $M_B^{(Y)} = \frac{4EI_x}{L}Y$ so that:

$$M_B = M_B^{(q)} + M_B^{(Y)} =$$

$$-\frac{qL^2}{12} + \frac{4EI_x}{L}Y = 0 \implies$$

$$Y = \varphi_B = \frac{1}{48}\frac{qL^3}{EI_x}$$

The internal forces are obtained by superposition

Force (flexibility) method

$$V_A = V_A^{(q)} + V_A^{(X)} = qL - X =$$

$$qL - \frac{3}{8}qL = \frac{5}{8}qL$$

$$V_B = V_B^{(q)} + V_B^{(X)} = 0 + X =$$

$$0 + \frac{3}{8}qL = \frac{3}{8}qL$$

$$M_A = M_A^{(q)} + M_A^{(X)} = \frac{1}{2}qL^2 - XL =$$

$$\frac{1}{2}qL^2 - \frac{3}{8}qL^2 = \frac{1}{8}qL^2$$

$$\dots$$

Displacement (stiffness) method

$$V_A = V_A^{(q)} + V_A^{(Y)} =$$

$$\frac{1}{2}qL + \left(+\frac{6EI_x}{L^2}\right)\left(\frac{1}{48}\frac{qL^3}{EI_x}\right) = \frac{5}{8}qL$$

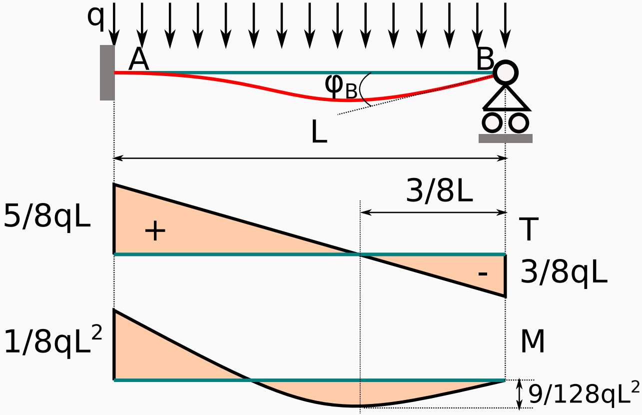
$$V_B = V_B^{(q)} + V_B^{(Y)} =$$

$$\frac{1}{2}qL + \left(-\frac{6EI_x}{L^2}\right)\left(\frac{1}{48}\frac{qL^3}{EI_x}\right) = \frac{3}{8}qL$$

$$M_A = M_A^{(q)} + M_A^{(Y)} =$$

$$\frac{1}{12}qL^2 + \left(+\frac{2EI_x}{L}\right)\left(\frac{1}{48}\frac{qL^3}{EI_x}\right) = \frac{1}{8}qL^2$$

$$\dots$$


q
A
B
φB
L
3/8L
5/8qL
+
T
3/8qL
-
1/8qL2
M
9/128qL2