STRUCTURAL ANALYSIS – FORCE (OR FLEXIBILITY) METHOD

ADVANCED STRUCTURAL MECHANICS

The ERAMCA Project

Environmental Risk Assessment and Mitigation on Cultural Heritage assets in Central Asia

v2022317

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Lecturer/students objectives

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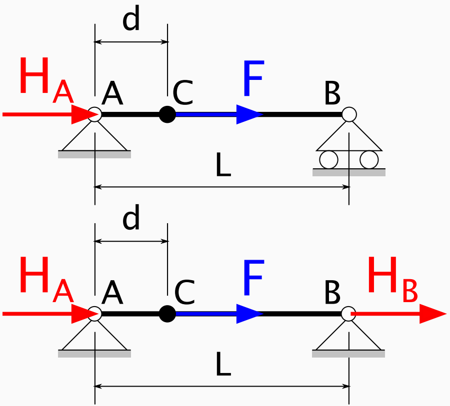
- Present a method of solution of statically indeterminate structures that assumes as unknowns the reactions of redundant constraints.
- Understand the method. Apply the method to structures with one unknown reaction.

Introduction

Statically indeterminate structures (1/2)

Indeterminate structures:

- reactions or internal forces cannot be determined from equations of statics alone (the number of reactions or the number of internal forces exceeds the number of static equilibrium equations)
- it is necessary to satisfy the equilibrium equations (implying that the structure is in equilibrium) and compatibility equations (requirement if for assuring the continuity of the structure without any breaks) and force displacement equations (the way in which displacement are related to forces)



$$g = 3n - \sum v$$

$$g = 3 \times 1 - (2_A + 1_B) = 0$$

$$H_A + F = 0$$

$$\implies H_A = -F$$

$$g = 3 \times 1 - (2_A + 2_B) = -1$$

$$H_A + F + H_B = 0$$

$$\implies H_A = -F - H_B$$

It is possible to satisfy equilibrium with any value to $H_B = X$, with $H_A = -F - X$.

An infinite number of solutions are possible

Force (or flexibility) method

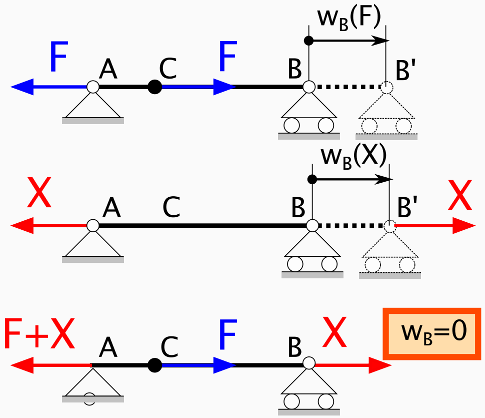
Force (or flexibility) method

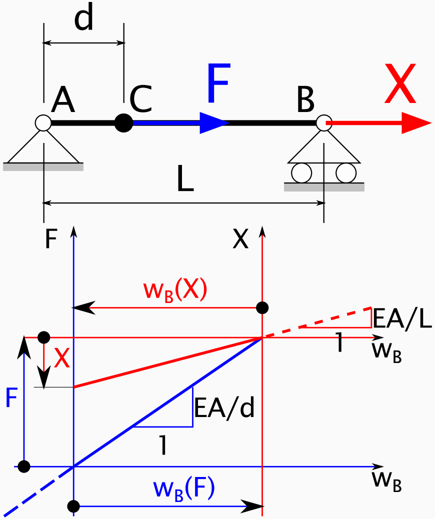
Equation of compatibility: expresses the fact that the deformation of the structure supposed to be elastic and linear must be compatible with the conditions at the supports

On a statically determinate structure, it is possible to calculate:

- displacements due to the external forces
- displacements due to the unknown forces

The compatibility with the original supports are used





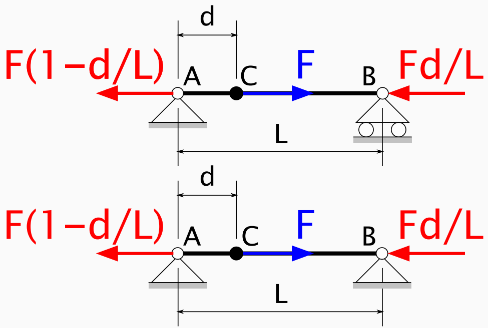
- One of the reactions is designated as redundant and it is eliminated to obtain a statically determinate structure
- superposition
- compatibility

$$w_B(F) + w_B(X) = 0$$

$$\frac{F}{EA/d} + \frac{X}{EA/L} = 0 \implies X = -\frac{d}{L}F$$

This value of X respects both compatibility and equilibrium: it is the solution

The forces calculated as presented above are interpreted as support reactions acting on the statically determinate structure

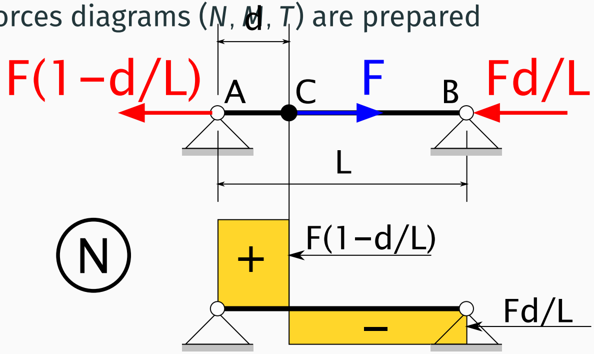


To apply the method, it is necessary to calculate displacements of points where compatibility conditions are imposed

Support reactions and internal forces

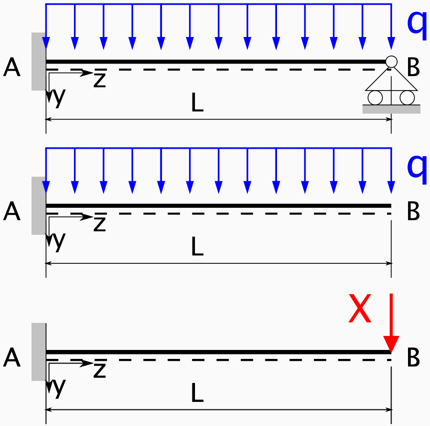
When the X unknown is determined:

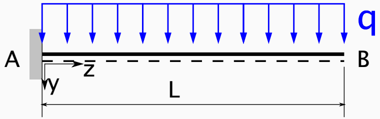
- the support reactions (on the statically determinate structure) from external loads and X are calculated
- the internal forces diagrams (N, M, T) are prepared



Examples

Determine support reactions and internal forces for the statically indeterminate structure to the first degree, assuming as unknown the vertical reaction at point B





Deflection for q uniform:

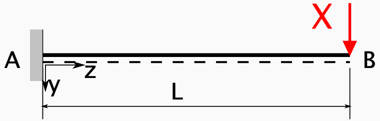
$$v = q\frac{z^4}{24EI_x} + C_1 z^3 + C_2 z^2 + C_3 z + C_4$$

with boundary conditions:

- for A ($z = 0$): $v|_A = v(0) = 0$ and $v'|_A = v'(0) = 0$
- for B ($z = L$): $v''|_B = v''(L) = 0$ and $v'''|_B = v'''(L) = 0$

it is found: $C_1 = -\frac{qL}{6EI_x}$, $C_2 = -\frac{qL^2}{4EI_x}$, $C_3 = 0$, $C_4 = 0$, hence

$$v(z) = \frac{qz^2\left(6L^2 - 4Lz + z^2\right)}{24EI_x}, \qquad v_B(q) = \frac{qL^4}{8EI_x}$$



Deflection for $q = 0$:

$$v = C_1 z^3 + C_2 z^2 + C_3 z + C_4$$

with boundary conditions:

- for A $(z = 0)$: $v|_A = v(0) = 0$ and $v'|_A = v'(0) = 0$
- for B $(z = L)$: $v''|_B = v''(L) = 0$ and $v'''|_B = -\frac{T_B}{EI_x} \Rightarrow v'''(L) = -\frac{X}{EI_x}$

so that: $C_1 = -\frac{X}{6EI_x}$, $C_2 = \frac{XL}{2EI_x}$, $C_3 = 0$, $C_4 = 0$, hence

$$v(z) = -\frac{Xz^2(z-3L)}{6EI_x}, \qquad v_B(X) = \frac{XL^3}{3EI_x}$$

Compatibility:

$$v_B(q) + v_B(X) = 0$$

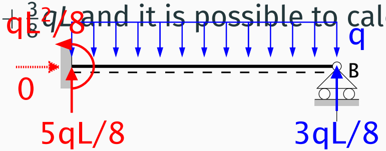
i.e.:

$$\frac{qL^4}{8EI_x} + \frac{XL^3}{3EI_x} = 0$$

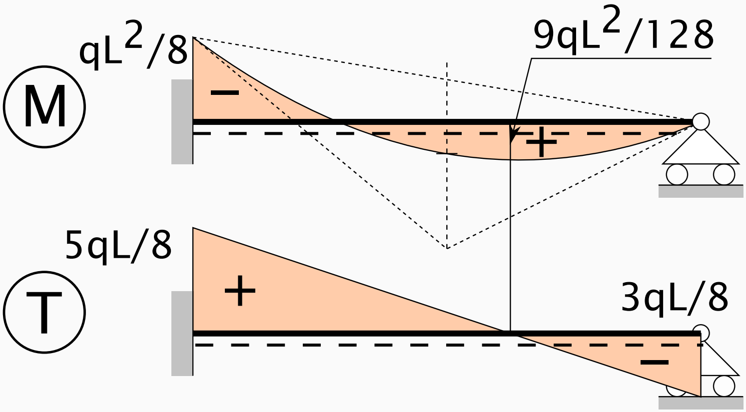
it is obtained the shear force acting at point B ($T_B = X$):

$$X = -\frac{3}{8}qL$$

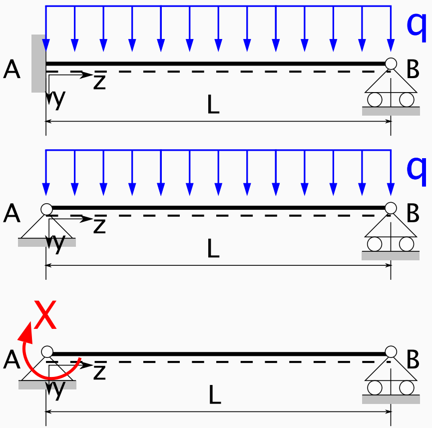
Assuming V_B upward, it is $V_B = -X = +\frac{3}{8}qL$ and it is possible to calculate the support reaction at point A

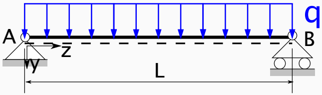


Finally, it is possible to determine the internal force diagrams



Determine support reactions and internal forces for the statically indeterminate structure to the first degree, assuming as unknown the moment at point A





Deflection for q uniform:

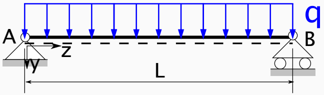
$$v = q\frac{z^4}{24EI_x} + C_1 z^3 + C_2 z^2 + C_3 z + C_4$$

boundary conditions:

- for A ($z = 0$): $v|_A = v(0) = 0$ and $v''|_A = v''(0) = 0$
- for B ($z = L$): $v|_B = v(L) = 0$ and $v''|_B = v''(L) = 0$

and: $C_1 = -\frac{qL}{12EI_x}$, $C_2 = 0$, $C_3 = \frac{qL^3}{24EI_x}$, $C_4 = 0$, hence

$$v(z) = \frac{qz\left(L^3 - 2Lz^2 + z^3\right)}{24EI_x}$$



The deflection is:

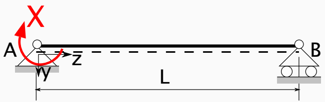
$$v(z) = \frac{qz\left(L^3 - 2Lz^2 + z^3\right)}{24EI_x}$$

The rotation is approximate with the firs derivative:

$$v'(z) = \frac{q\left(L^3 - 6Lz^2 + 4z^3\right)}{24EI_x}$$

in particular, rotation at point A is obtained for $z = 0$:

$$\varphi_A(q) = -v'(0) = -\frac{qL^3}{24EI_x}$$



Deflection for $q = 0$:

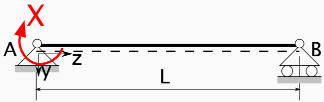
$$v = C_1 z^3 + C_2 z^2 + C_3 z + C_4$$

boundary conditions:

- for A $(z = 0)$: $v|_A = v(0) = 0$ and $v''|_A = -\frac{M_A}{EI_x} \Rightarrow v''(0) = -\frac{X}{EI_x}$
- for B $(z = L)$: $v|_B = v(L) = 0$ and $v''|_B = -\frac{M_B}{EI_x} \Rightarrow v''(L) = 0$

it is found: $C_1 = \frac{X}{6EI_xL}$, $C_2 = -\frac{X}{2EI_x}$, $C_3 = \frac{XL}{3EI_x}$, $C_4 = 0$, hence

$$v(z) = \frac{Xz(z^2 - 3Lz + 2L^2)}{6EI_xL}$$



Deflection is now:

$$v(z) = \frac{Xz(z^2 - 3Lz + 2L^2)}{6EI_xL}$$

rotation is:

$$v'(z) = \frac{X\left(3z^2 - 6Lz + 2L^2\right)}{6EI_xL}$$

in particular, rotation at point A is obtained for $z = 0$:

$$\varphi_A(X) = -v'(0) = -\frac{XL}{3EI_x}$$

Compatibility:

$$\varphi_A(q) + \varphi_A(X) = 0$$

or:

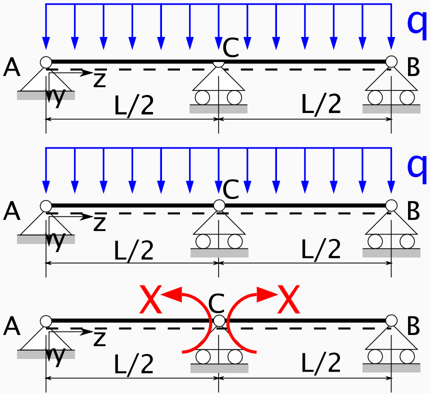
$$-\frac{qL^3}{24EI_x} - \frac{XL}{3EI_x} = 0$$

where it is possible to obtain the bending moment at A ($M_A = X$)

$$X = -\frac{1}{8}qL^2$$

this results (and the corresponding internal forces) is the same of the previous example.

Determine support reactions and internal forces for the statically indeterminate structure to the first degree, assuming as unknown the moment at point C



The rotation at one end of the beam of span L due to an uniform load q has been already calculated in the previous example:

$$\varphi_A(q) = -\frac{qL^3}{24EI_x}$$

For the beam CB (span $L/2$) it is:

$$\varphi_{CB}(q) = -\frac{q(L/2)^3}{24EI_x} = -\frac{qL^3}{192EI_x}$$

The rotation due to a couple X, applied to the same end is (previous example):

$$\varphi_A(X) = -\frac{XL}{3EI_x}$$

For a span equal to $L/2$:

$$\varphi_{CB}(X) = -\frac{X(L/2)}{3EI_x} = -\frac{XL}{6EI_x}$$

The same for the beam CA (paying attention to the signs for the right end). Load q effect:

$$\varphi_{CA}(q) = \frac{q(L/2)^3}{24EI_x} = +\frac{qL^3}{192EI_x}$$

Couple X effect:

$$\varphi_{CA}(X) = \frac{X(L/2)}{3EI_x} = +\frac{XL}{6EI_x}$$

Compatibility: the right rotation on the left of C must be equal to the right, so that

$$\varphi_{CA}(q) + \varphi_{CA}(X) = \varphi_{CB}(q) + \varphi_{CB}(X)$$

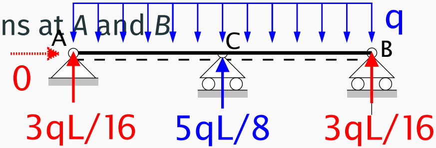
or:

$$\frac{qL^3}{192EI_x} + \frac{XL}{6EI_x} = -\frac{qL^3}{192EI_x} - \frac{XL}{6EI_x}$$

where it is possible to obtain the bending moment at the middle support

$$X = M_C = -\frac{1}{32}qL^2$$

and the support reactions at A and B



Finally, it is possible to draw the internal forces diagrams.