BEAM THEORY – DISPLACEMENTS CALCULATION AND STATICALLY INDETERMINATE STRUCTURES

ADVANCED STRUCTURAL MECHANICS

The ERAMCA Project

Environmental Risk Assessment and Mitigation on Cultural Heritage assets in Central Asia

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LECTURER/STUDENTS OBJECTIVES

Lecturer/students objectives

- Present a method to calculate displacements and solve statically indeterminate structures using the theorem of virtual work.
- Calculate displacements for simple schemes and solve simple statically indeterminate structures.

INTRODUCTION

The theorem of virtual work, for a beam, states that $\mathcal{L}_{ve} = \mathcal{L}_{vi}$:

- external work (by all forces acting on the beam):

$$\mathcal{L}_{ve} = \int_0^L (p_a\, w_b + q_a\, v_b)\, dz +$$

$$+ (N_a\, w_b + T_a\, v_b + M_a\, \varphi_b)\,|_L + (-N_a\, w_b - T_a\, v_b - M_a\, \varphi_b)\,|_0 =$$

$$= \int_0^L (p_a\, w_b + q_a\, v_b)\, dz + (N_a\, w_b + T_a\, v_b + M_a\, \varphi_b)\,|_0^L$$

- internal work (by internal forces):

$$\mathcal{L}_{vi} = \int_0^L (N_a\, \varepsilon_b + T_a\, \gamma_b + M_a\, \chi_b)\, dz$$

- The theorem of virtual work is valid if the system a is in equilibrium and the system b is kinematically admissible
- The two systems are independent

In the following...

... the contribution of the work of normal force and shear is neglected so that $\mathcal{L}_{vi} = \int_0^L M_a \chi_b \, dz$, i.e., the deformability due to bending only is considered

CALCULATION OF DISPLACEMENTS (UNIT-LOAD METHOD)

Calculation of Displacements (Unit-Load Method) (1/3)

Structure loaded by the:

- external loads (subscript " r")
- Unit force (or a force *U*) at a point and in the direction of the displacement to be found

In the virtual work of the real displacements (in blue) are associated to the forces of the structure loaded by the unit force (in red), in such a way that the displacement to be found is the only unknown of the equation.

Vertical displacement of point B of a cantilever with uniform stiffness ($E\,I_x = \text{const}$):

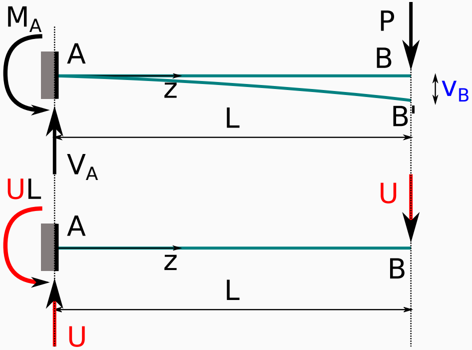
$$(U)\,\cancel{v_A}^{0} + (UL)\,\cancel{\varphi_A}^{0} + (0)\,\cancel{w_A}^{0} + (U)\,v_B =$$

$$\int_0^L M_U \chi_r \mathrm{d}z =$$

$$\int_0^L M_U \frac{M_r}{E\,I_x}\mathrm{d}z = \int_0^L \frac{M_r\,M_U}{E\,I_x}\mathrm{d}z =$$

$$\frac{1}{E\,I_x}\int_0^L [-P\,(L-z)]\,[-U\,(L-z)]\,\mathrm{d}z =$$

$$\frac{PU}{E\,I_x}\int_0^L (L-z)^2\mathrm{d}z \quad \text{i.e.,} \quad v_B = +\frac{P\,L^3}{3E\,I_x}$$



Rotation of point B of a cantilever with uniform stiffness ($E\,I_x = \text{const}$):

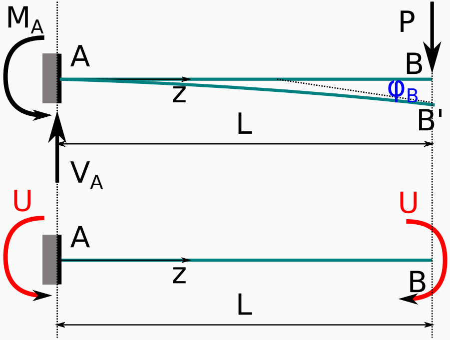
$$(0)\cancel{v_A}^{0} + (U)\cancel{\varphi_A}^{0} + (0)\cancel{w_A}^{0} + (U)\,\varphi_B =$$

$$\int_0^L M_U \chi_r \,\mathrm{d}z =$$

$$\int_0^L M_U \frac{M_r}{E\,I_x}\mathrm{d}z = \int_0^L \frac{M_r\,M_U}{E\,I_x}\mathrm{d}z =$$

$$\frac{1}{E\,I_x}\int_0^L [-P\,(L-z)]\,[-U]\,\mathrm{d}z =$$

$$\frac{PU}{E\,I_x}\int_0^L (L-z)\mathrm{d}z \quad \text{i.e.,} \quad \varphi_B = +\frac{P\,L^2}{2E\,I_x}$$



STATICALLY INDETERMINATE STRUCTURES

Choice of the statically determined structure and:

- statically determined structure loaded by the external loads (subscript "0")
- statically determined structure loaded by the redundant force equal to 1 loads (subscript "1")

In the virtual work of the real displacements (in blue) are associated to the forces of the statically determined structure loaded by the redundant force equal to 1 (in red).

Superposition principle

The external reactions, normal force, shear, moment… for the original structure are found as $F = F_0 + X_1 F_1$

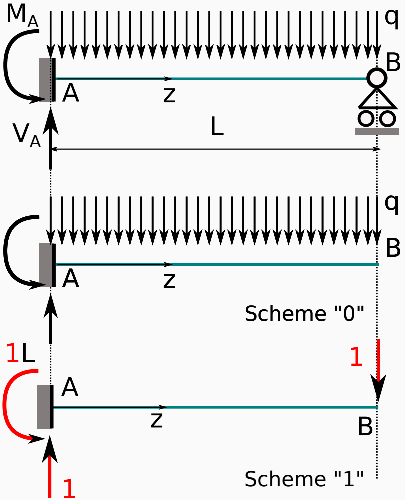
The moment and the curvature for the original structure are:

$$M_r = M_0 + X_1 M_1$$

$$\chi_r = \frac{M_r}{E\,I_x} = \frac{M_0 + X_1 M_1}{E\,I_x}$$

so that the TWV is:

$$(1)\cancelto{0}{v_A} + (1L)\cancelto{0}{\varphi_A} + (0)\cancelto{0}{w_A} + (1)\cancelto{0}{v_B} = \int_0^L M_1 \chi_r \mathrm{d}z = \int_0^L M_1 \frac{M_r}{E\,I_x} \mathrm{d}z$$



That is:

$$0 = \int_0^L M_1 \frac{M_0 + X_1 M_1}{E I_x} dz = \int_0^L \frac{M_0 M_1}{E I_x} dz + X_1 \int_0^L \frac{M_1{}^2}{E I_x} dz$$

where:

$$\int_0^L \frac{M_0 M_1}{E I_x} dz = \cdots = + \frac{qL^4}{8E I_x} \qquad \int_0^L \frac{M_1{}^2}{E I_x} dz = \ldots = + \frac{1L^3}{3E I_x}$$

i.e.,

$$0 = + \frac{qL^4}{8E I_x} + X_1 \frac{1L^3}{3E I_x} \quad \text{so that} \quad X_1 = -\frac{3}{8} qL$$

Finally, for example: $M_A = M_0^A + X_1 M_1{}^A = \left(-\frac{qL^2}{2}\right) + \left(-\frac{3}{8}qL\right)(-1L) = -\frac{qL^2}{8}$

Some remarks:

- term $+\frac{qL^4}{8E\,I_x}$ represents the displacement of the point B due to the external load q, i.e., $v_B^{(q)}$
- term $+\frac{1L^3}{3E\,I_x}$ represents the displacement of the point B due to the $X_1 = V_B = 1$, i.e., $v_B^{(1)}$
- the final equation is $v_B^{(q)} + v_B^{(X_1)} = v_B^{(q)} + X_1\, v_B^{(1)} = 0$ (compatibility of displacement at B)
- the negative value obtained for $X_1 = -\frac{3}{8}qL$ means that V_B is in the opposite direction of the force 1 applied at B

If the degree of indeterminacy n is larger than one, a similar procedure can be applied (a $n \times n$ system of linear equations is obtained)

ADDITIONAL READING

Simpson's rule for numerical integration:

$$p_i = \int_a^b f(z)\, g(z) \mathrm{d}z =$$

$$= \frac{b-a}{6}\left[f(a)\, g(a) + 4\, f\left(\frac{a+b}{2}\right)\, g\left(\frac{a+b}{2}\right) + f(b)\, g(b)\right]$$

This formula provides exact results...

... if $f(z)$ and $g(z)$ are polynomials such that the sum of their degrees is ≤ 3

Numerical integral calculation – example

If both diagrams are triangular:

$$p_i = \int_a^b f(z)\,g(z)\mathrm{d}z =$$

$$= \frac{b-a}{6}\left[f(a)\,g(a) + 4\,f\left(\frac{a+b}{2}\right)\,g\left(\frac{a+b}{2}\right) + f(b)\,g(b)\right] =$$

$$\frac{L}{6}\left[(0)(0) + 4\left(\frac{M}{2}\right)\left(\frac{M'}{2}\right) + (M)(M')\right] = \frac{M\,M'}{3}L$$

that is the same result of row 3, column 3 of the table of slide 14.

If $M = -P\,L$ and $M' = -U\,L$ (exercise 1, slide 6) $\frac{PU\,L^3}{3}$ is obtained.

TABLES FOR INTEGRAL CALCULATION

ALCUNI VALORI DI $\frac{1}{l}\int M' M \, dx$

l	M — M	M	M_1 — M_2	M — $-M$	M — $-\frac{M}{2}$	αl, βl, M
M' — M'	$M' M$	$\frac{1}{2} M' M$	$\frac{1}{2} M' (M_1 + M_2)$	0	$\frac{1}{4} M' M$	$\frac{1}{2} M' M$
M'	$\frac{1}{2} M' M$	$\frac{1}{3} M' M$	$\frac{1}{6} M' (M_1 + 2 M_2)$	$-\frac{1}{6} M' M$	0	$\frac{1}{6} M' M (1 + \alpha)$
M'	$\frac{1}{2} M' M$	$\frac{1}{6} M' M$	$\frac{1}{6} M' (2 M_1 + M_2)$	$\frac{1}{6} M' M$	$\frac{1}{4} M' M$	$\frac{1}{6} M' M (1 + \beta)$
M'_1 — M'_2	$\frac{1}{2} M (M'_1 + M'_2)$	$\frac{1}{6} M (M'_1 + 2 M'_2)$	$\frac{1}{6} \left[M'_1 (2 M_1 + M_2) + M'_2 (M_1 + 2 M_2) \right]$	$\frac{1}{6} M (M'_1 - M'_2)$	$\frac{1}{4} M'_1 M$	$\frac{1}{6} M \left[M'_1 (1 + \beta) + M'_2 (1 + \alpha) \right]$
M' — $-M'$	0	$-\frac{1}{6} M' M$	$\frac{1}{6} M' (M_1 - M_2)$	$\frac{1}{3} M' M$	$\frac{1}{4} M' M$	$\frac{1}{6} M' M (1 - 2\alpha)$
M' — $-\frac{M'}{2}$	$\frac{1}{4} M' M$	0	$\frac{1}{4} M' M_1$	$\frac{1}{4} M' M$	$\frac{1}{4} M' M$	$\frac{1}{4} M' M \beta$
$-\frac{M'}{2}$ — M'	$\frac{1}{4} M' M$	$\frac{1}{4} M' M$	$\frac{1}{4} M' M_2$	$-\frac{1}{4} M' M$	$-\frac{1}{8} M' M$	$\frac{1}{4} M' M \alpha$
$l/2$, M'	$\frac{1}{2} M' M$	$\frac{1}{4} M' M$	$\frac{1}{4} M' (M_1 + M_2)$	0	$\frac{1}{8} M' M$	$\frac{M' M}{12 \beta} (3 - 4\alpha^2)$
γl, δl, M'	$\frac{1}{2} M' M$	$\frac{1}{6} M' M (1 + \gamma)$	$\frac{1}{6} M' \left[M_1 (1 + \delta) + M_2 (1 + \gamma) \right]$	$\frac{1}{6} M' M (1 - 2\gamma)$	$\frac{1}{4} M' M \delta$	$\frac{M' M}{6 \beta \gamma} (2\gamma - \gamma^2 - \alpha^2)$ $\gamma \geq \alpha$
M'	$\frac{2}{3} M' M$	$\frac{1}{3} M' M$	$\frac{1}{3} M' (M_1 + M_2)$	0	$\frac{1}{6} M' M$	$\frac{1}{3} M' M (1 + \alpha \beta)$
M' — M'	$\frac{1}{3} M' M$	$\frac{1}{6} M' M$	$\frac{1}{6} M' (M_1 + M_2)$	0	$\frac{1}{12} M' M$	$\frac{1}{6} M' M (1 - 2\alpha\beta)$

ALCUNI VALORI DI $\frac{1}{l}\int M' M \, dx$

l	M — M	M	M_1 — M_2	M — $-M$	M — $-\frac{M}{2}$	αl, βl, M
M'	$\frac{2}{3} M' M$	$\frac{1}{4} M' M$	$\frac{1}{12} M' (5 M_1 + 3 M_2)$	$\frac{1}{6} M' M$	$\frac{7}{24} M' M$	$\frac{1}{12} M' M (5 - \alpha - \alpha^2)$
M'	$\frac{2}{3} M' M$	$\frac{5}{12} M' M$	$\frac{1}{12} M' (3 M_1 + 5 M_2)$	$-\frac{1}{6} M' M$	$\frac{1}{24} M' M$	$\frac{1}{12} M' M (5 - \beta - \beta^2)$
M'	$\frac{1}{3} M' M$	$\frac{1}{4} M' M$	$\frac{1}{12} M' (M_1 + 3 M_2)$	$-\frac{1}{6} M' M$	$-\frac{1}{24} M' M$	$\frac{1}{12} M' M (1 + \alpha + \alpha^2)$
M'	$\frac{1}{3} M' M$	$\frac{1}{12} M' M$	$\frac{1}{12} M' (3 M_1 + M_2)$	$\frac{1}{6} M' M$	$\frac{5}{24} M' M$	$\frac{1}{12} M' M (1 + \beta + \beta^2)$
$l/2$, $-M'$, M'	$\frac{1}{6} M' M$	$\frac{1}{6} M' M$	$\frac{1}{6} M' M_2$	$-\frac{1}{6} M' M$	$-\frac{1}{12} M' M$	$\frac{1}{6} M' M \alpha (1 + 2\beta)$
$l/2$, M', $-M'$	$\frac{1}{6} M' M$	0	$\frac{1}{6} M' M_1$	$\frac{1}{6} M' M$	$\frac{1}{6} M' M$	$\frac{1}{6} M' M \beta (1 + 2\alpha)$
$l/2$, M'_1, M'_2, M'_3	$\frac{1}{6} M (M'_1 + 4 M'_2 + M'_3)$	$\frac{1}{6} M (2 M'_2 + M'_3)$	$\frac{1}{6} \left[M'_1 M_1 + 2 M'_2 (M_1 + M_2) + M'_3 M_2 \right]$	$\frac{1}{6} M (M'_1 - M'_3)$	$\frac{1}{12} M (2 M'_1 + 2 M'_2 - M'_3)$	$\frac{1}{6} M \left[M'_1 \beta + 2 M'_2 + M_3 \alpha - \alpha\beta (M'_1 - 2 M'_2 + M'_3) \right]$
M' parabola cubica	$\frac{1}{4} M' M$	$\frac{1}{5} M' M$	$\frac{1}{20} M' (M_1 + 4 M_2)$	$-\frac{3}{20} M' M$	$-\frac{1}{20} M' M$	$\frac{1}{20} M' M (1 + \alpha)(1 + \alpha^2)$
M' parabola cubica	$\frac{1}{4} M' M$	$\frac{1}{20} M' M$	$\frac{1}{20} M' (4 M_1 + M_2)$	$\frac{3}{20} M' M$	$\frac{7}{40} M' M$	$\frac{1}{20} M' M (1 + \beta)(1 + \beta^2)$
M' parabola cubica	$\frac{1}{4} M' M$	$\frac{2}{15} M' M$	$\frac{1}{60} M' (7 M_1 + 8 M_2)$	$-\frac{1}{60} M' M$	$\frac{1}{20} M' M$	$\frac{1}{20} M' M (1 + \alpha) \left(\frac{7}{3} - \alpha^2\right)$
M' parabola cubica	$\frac{1}{4} M' M$	$\frac{7}{60} M' M$	$\frac{1}{60} M' (8 M_1 + 7 M_2)$	$\frac{1}{60} M' M$	$\frac{3}{40} M' M$	$\frac{1}{20} M' M (1 + \beta) \left(\frac{7}{3} - \beta^2\right)$

Externally on internally statically indeterminate structures

Possible indeterminate structures:

- externally statically indeterminate (*top*)
- externally statically determinate and internally statically indeterminate (*middle*)
- externally and internally statically indeterminate (*bottom*)