STRUCTURAL ANALYSIS – MATRIX STRUCTURAL ANALYSIS

ADVANCED STRUCTURAL MECHANICS

The ERAMCA Project

Environmental Risk Assessment and Mitigation on Cultural Heritage assets in Central Asia

v2022317

Table of Contents

Lecturer/students objectives

Introduction

Description of operations performed by software

Lecturer/students objectives

Lecturer/students objectives

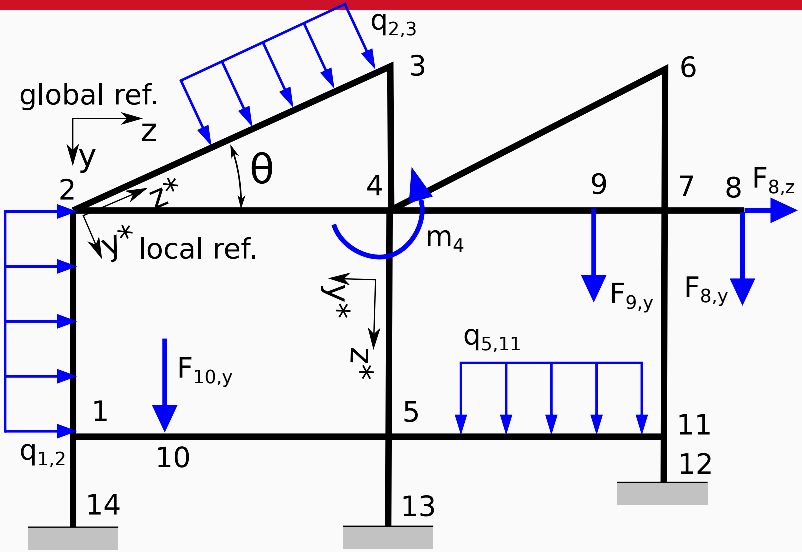
- Present the fundamentals of solution of framed structures through the calculator.
- Understand the procedure that make possible to translate a structural problem into a procedure for digital computers.

Introduction

Presentation

Setup of a procedure (set of instructions carried out by a digital computer) for the solution of:

- 2D structures composed by beams and columns, with internal and external constraints, made of beams with cross sections symmetrical respect to the vertical plane, lying in the same plane
- loaded by:
  - concentrated forces applied in the vertical plane, couples with moment axis perpendicular to vertical plane, applied on nodes
  - concentrated or distributed forces contained on vertical plane, applied along the beams
- and made of linear elastic materials


q2,3
3
6
global ref.
z
y
θ
2
z*
4
9
7
8
F8,z
y*
local ref.
m4
F9,y
F8,y
y*
z*
q5,11
F10,y
1
5
11
q1,2
10
12
14
13

But first...

Now that we are all calm...

... let's add some details!

Now that we are all calm...

... let's add some details!

What is the user U requested to do?
What is the software S doing behind the scenes?
What data are needed for a successful run?

Summary of the procedure (1/2)

1. Description of geometry (coordinates of nodes), by hands or importing a model U

Summary of the procedure (1/2)

1. Description of geometry (coordinates of nodes), by hands or importing a model **U**
2. Application of the external and internal constraints **U**

Summary of the procedure (1/2)

1. Description of geometry (coordinates of nodes), by hands or importing a model **U**
2. Application of the external and internal constraints **U**
3. For each beam e, definition of:

   3.2 geometry of the cross section (centroid G, area A and second order moments I_x and I_y are usually calculated by the software) **U**

1. Description of geometry (coordinates of nodes), by hands or importing a model **U**
2. Application of the external and internal constraints **U**
3. For each beam e, definition of:

   3.2 geometry of the cross section (centroid G, area A and second order moments I_x and I_y are usually calculated by the software) **U**

   3.3 material properties E and G (or E and ν) **U**

Summary of the procedure (1/2)

1. Description of geometry (coordinates of nodes), by hands or importing a model **U**
2. Application of the external and internal constraints **U**
3. For each beam e, definition of:

    3.2 geometry of the cross section (centroid G, area A and second order moments I_x and I_y are usually calculated by the software) **U**

    3.3 material properties E and G (or E and ν) **U**

4. For each loaded node, definition of:

    4.1 forces F_z, F_y and M_x (usually global frame) **U**

Summary of the procedure (1/2)

1. Description of geometry (coordinates of nodes), by hands or importing a model **U**
2. Application of the external and internal constraints **U**
3. For each beam e, definition of:

   3.2 geometry of the cross section (centroid G, area A and second order moments I_x and I_y are usually calculated by the software) **U**

   3.3 material properties E and G (or E and ν) **U**

4. For each loaded node, definition of:

   4.1 forces F_z, F_y and M_x (usually global frame) **U**

5. For each loaded beam, definition of:

1. Description of geometry (coordinates of nodes), by hands or importing a model **U**
2. Application of the external and internal constraints **U**
3. For each beam e, definition of:

   3.2 geometry of the cross section (centroid G, area A and second order moments I_x and I_y are usually calculated by the software) **U**

   3.3 material properties E and G (or E and ν) **U**

4. For each loaded node, definition of:

   4.1 forces F_z, F_y and M_x (usually global frame) **U**

5. For each loaded beam, definition of:

   5.1 distributed loads q_{z^*} and q_{y^*} (usually local frame) **U**

1. Description of geometry (coordinates of nodes), by hands or importing a model **U**
2. Application of the external and internal constraints **U**
3. For each beam e, definition of:
   3.1 topology, i.e., node i and j (i.e., length L and angle θ_e), slide 24 **S**
   3.2 geometry of the cross section (centroid G, area A and second order moments I_x and I_y are usually calculated by the software) **U**
   3.3 material properties E and G (or E and ν) **U**
4. For each loaded node, definition of:
   4.1 forces F_z, F_y and M_x (usually global frame) **U**
5. For each loaded beam, definition of:
   5.1 distributed loads q_{z^*} and q_{y^*} (usually local frame) **U**

1. Description of geometry (coordinates of nodes), by hands or importing a model **U**
2. Application of the external and internal constraints **U**
3. For each beam e, definition of:
   - 3.1 topology, i.e., node i and j (i.e., length L and angle θ_e), slide 24 **S**
   - 3.2 geometry of the cross section (centroid G, area A and second order moments I_x and I_y are usually calculated by the software) **U**
   - 3.3 material properties E and G (or E and ν) **U**
   - 3.4 stiffness $\boldsymbol{K}_e$, matrix $\boldsymbol{N}_e$ and $\boldsymbol{K}^x$, slide 14 **S**
4. For each loaded node, definition of:
   - 4.1 forces F_z, F_y and M_x (usually global frame) **U**
5. For each loaded beam, definition of:
   - 5.1 distributed loads q_{z^*} and q_{y^*} (usually local frame) **U**

Summary of the procedure (2/2)

6. Run the software (and cross the fingers!)

U

See the checklist, slide 39!

Summary of the procedure (2/2)

6. Run the software (and cross the fingers!) U

10. Look at the output for results (lists, diagrams of normal force, moment, shear; deformed shape…); to be critically checked U

See the checklist, slide 39!

Summary of the procedure (2/2)

6. Run the software (and cross the fingers!) U
7. Assembling procedure and determination of $\boldsymbol{K}$ and $\boldsymbol{R}$ S

10. Look at the output for results (lists, diagrams of normal force, moment, shear; deformed shape...); to be critically checked U

See the checklist, slide 39!

Summary of the procedure (2/2)

6. Run the software (and cross the fingers!) U
7. Assembling procedure and determination of $\boldsymbol{K}$ and $\boldsymbol{R}$ S
8. Solution and determination of displacements δ_L and forces $\boldsymbol{V}_V$, slide 36 S

10. Look at the output for results (lists, diagrams of normal force, moment, shear; deformed shape...); to be critically checked U

See the checklist, slide 39!

6. Run the software (and cross the fingers!) U
7. Assembling procedure and determination of $\boldsymbol{K}$ and $\boldsymbol{R}$ S
8. Solution and determination of displacements $\boldsymbol{\delta}_L$ and forces $\boldsymbol{V}_V$, slide 36 S
9. For each beam e, determination of (slide 36):
   - 9.1 end forces $\boldsymbol{Q}_i^*$ and $\boldsymbol{Q}_j^*$ S
   - 9.2 external reactions $\boldsymbol{V}$ S
   - 9.3 normal force, moment and shear S
10. Look at the output for results (lists, diagrams of normal force, moment, shear; deformed shape…); to be critically checked U

See the checklist, slide 39!

Description of operations performed by software

Displacement (of stiffness) method

The displacements or every node k are assumed as kinematical parameters in the local reference frame

$$\boldsymbol{\delta}_i^* = \begin{bmatrix} \varphi_i \\ v_i \\ w_i \end{bmatrix}, \quad \boldsymbol{\delta}_j^* = \begin{bmatrix} \varphi_j \\ v_j \\ w_j \end{bmatrix}$$

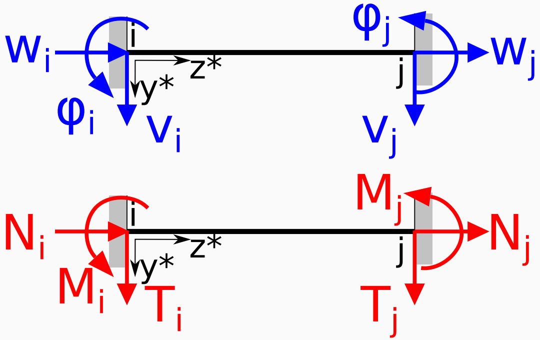
The reactions at the ends i e j of every beams (local reference frame) induce by the applied displacements are evaluated:

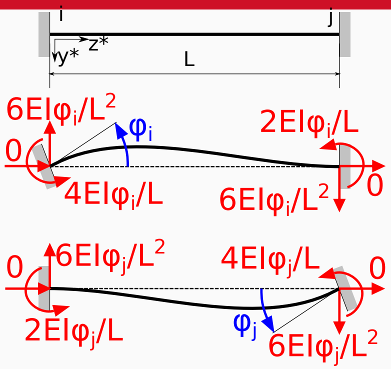
$$\boldsymbol{Q}_i^* = \begin{bmatrix} M_i \\ T_i \\ N_i \end{bmatrix}, \quad \boldsymbol{Q}_j^* = \begin{bmatrix} M_j \\ T_j \\ N_j \end{bmatrix},$$

Adding the forces applied in every nodes, it is possible to impose equilibrium and to find the unknown displacements

Sign conventions

Displacements and forces at both ends of a beam are positive if directed in the same direction of the local reference frame. Rotations and couples are positive if counterclockwise.



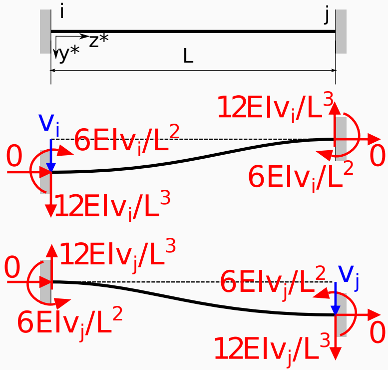


The rotation of φ of one end is applied and the corresponding reactions are calculated (local frame):

- rotation φ_i of node i
- rotation φ_j of node j

$$M_i = \frac{4EI_x}{L}\varphi_i, T_i = -\frac{6EI_x}{L^2}\varphi_i, N_i = 0, M_j = \frac{2EI_x}{L}\varphi_i, T_j = \frac{6EI_x}{L^2}\varphi_i, N_j = 0$$

$$M_i = \frac{2EI_x}{L}\varphi_j, T_i = -\frac{6EI_x}{L^2}\varphi_j, N_i = 0, M_j = \frac{4EI_x}{L}\varphi_j, T_j = \frac{6EI_x}{L^2}\varphi_j, N_j = 0$$



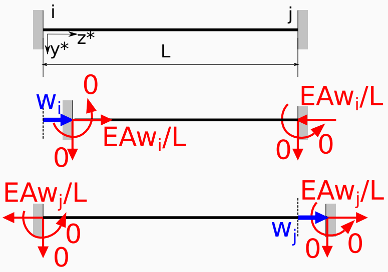
The displacement v of one end is applied and the corresponding reactions are calculated (local frame):

- displacement v_i of node i
- displacement v_j of node j

$$M_i = -\frac{6EI_x}{L^2} v_i,\ T_i = \frac{12EI_x}{L^3} v_i,\ N_i = 0,\ M_j = -\frac{6EI_x}{L^2} v_i,\ T_j = -\frac{12EI_x}{L^3} v_i,\ N_j = 0$$

$$M_i = \frac{6EI_x}{L^2} v_j,\ T_i = -\frac{12EI_x}{L^3} v_j,\ N_i = 0,\ M_j = \frac{6EI_x}{L^2} v_j,\ T_j = \frac{12EI_x}{L^3} v_j,\ N_j = 0$$

Imposed displacements at the ends (3/3)



The displacement w of one end is applied and the corresponding reactions are calculated (local frame):

- displacement w_i of node i
- displacement w_j of node j

$$M_i = 0, T_i = 0, N_i = \frac{EA}{L}w_i, M_j = 0, T_j = 0, N_j = -\frac{EA}{L}w_i$$

$$M_i = 0, T_i = 0, N_i = -\frac{EA}{L}w_j, M_j = 0, T_j = 0, N_j = \frac{EA}{L}w_j$$

Combining in one matrix the previous relationships, rotations φ_i and φ_j:

$$\begin{bmatrix} M_i \\ T_i \\ N_i \\ M_j \\ T_j \\ N_j \end{bmatrix} = \begin{bmatrix} \frac{4EI_x}{L} & 0 & 0 & \frac{2EI_x}{L} & 0 & 0 \\ -\frac{6EI_x}{L^2} & 0 & 0 & -\frac{6EI_x}{L^2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{2EI_x}{L} & 0 & 0 & \frac{4EI_x}{L} & 0 & 0 \\ \frac{6EI_x}{L^2} & 0 & 0 & \frac{6EI_x}{L^2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \varphi_i \\ v_i \\ w_i \\ \varphi_j \\ v_j \\ w_j \end{bmatrix}$$

The effects of displacements are superposed, displacements v_i and v_j:

$$\begin{bmatrix} M_i \\ T_i \\ N_i \\ M_j \\ T_j \\ N_j \end{bmatrix} = \begin{bmatrix} \frac{4EI_x}{L} & -\frac{6EI_x}{L^2} & 0 & \frac{2EI_x}{L} & \frac{6EI_x}{L^2} & 0 \\ -\frac{6EI_x}{L^2} & \frac{12EI_x}{L^3} & 0 & -\frac{6EI_x}{L^2} & -\frac{12EI_x}{L^3} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{2EI_x}{L} & -\frac{6EI_x}{L^2} & 0 & \frac{4EI_x}{L} & \frac{6EI_x}{L^2} & 0 \\ \frac{6EI_x}{L^2} & -\frac{12EI_x}{L^3} & 0 & \frac{6EI_x}{L^2} & \frac{12EI_x}{L^3} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \varphi_i \\ v_i \\ w_i \\ \varphi_j \\ v_j \\ w_j \end{bmatrix}$$

Displacements w_i and w_j:

$$\begin{bmatrix} M_i \\ T_i \\ N_i \\ M_j \\ T_j \\ N_j \end{bmatrix} = \begin{bmatrix} \frac{4EI_x}{L} & -\frac{6EI_x}{L^2} & 0 & \frac{2EI_x}{L} & \frac{6EI_x}{L^2} & 0 \\ -\frac{6EI_x}{L^2} & \frac{12EI_x}{L^3} & 0 & -\frac{6EI_x}{L^2} & -\frac{12EI_x}{L^3} & 0 \\ 0 & 0 & \frac{EA}{L} & 0 & 0 & -\frac{EA}{L} \\ \frac{2EI_x}{L} & -\frac{6EI_x}{L^2} & 0 & \frac{4EI_x}{L} & \frac{6EI_x}{L^2} & 0 \\ \frac{6EI_x}{L^2} & -\frac{12EI_x}{L^3} & 0 & \frac{6EI_x}{L^2} & \frac{12EI_x}{L^3} & 0 \\ 0 & 0 & -\frac{EA}{L} & 0 & 0 & \frac{EA}{L} \end{bmatrix} \begin{bmatrix} \varphi_i \\ v_i \\ w_i \\ \varphi_j \\ v_j \\ w_j \end{bmatrix}$$

The matrix that links displacements and internal forces at the ends, is the stiffness matrix (local reference frame):

$$\begin{bmatrix} M_i \\ T_i \\ N_i \\ M_j \\ T_j \\ N_j \end{bmatrix} = \begin{bmatrix} \frac{4EI_x}{L} & -\frac{6EI_x}{L^2} & 0 & \frac{2EI_x}{L} & \frac{6EI_x}{L^2} & 0 \\ -\frac{6EI_x}{L^2} & \frac{12EI_x}{L^3} & 0 & -\frac{6EI_x}{L^2} & -\frac{12EI_x}{L^3} & 0 \\ 0 & 0 & \frac{EA}{L} & 0 & 0 & -\frac{EA}{L} \\ \frac{2EI_x}{L} & -\frac{6EI_x}{L^2} & 0 & \frac{4EI_x}{L} & \frac{6EI_x}{L^2} & 0 \\ \frac{6EI_x}{L^2} & -\frac{12EI_x}{L^3} & 0 & \frac{6EI_x}{L^2} & \frac{12EI_x}{L^3} & 0 \\ 0 & 0 & -\frac{EA}{L} & 0 & 0 & \frac{EA}{L} \end{bmatrix} \begin{bmatrix} \varphi_i \\ v_i \\ w_i \\ \varphi_j \\ v_j \\ w_j \end{bmatrix}$$

it is symmetrical, terms on the diagonal are positive, and is singular (represents a beam free in space)

It is assumed:

$$\boldsymbol{\delta}_i^* = \begin{bmatrix} \varphi_i \\ v_i \\ w_i \end{bmatrix}, \quad \boldsymbol{\delta}_j^* = \begin{bmatrix} \varphi_j \\ v_j \\ w_j \end{bmatrix}, \quad \boldsymbol{Q}_i^* = \begin{bmatrix} M_i \\ T_i \\ N_i \end{bmatrix}, \quad \boldsymbol{Q}_j^* = \begin{bmatrix} M_j \\ T_j \\ N_j \end{bmatrix}$$

$$\boldsymbol{K}_{ii} = \begin{bmatrix} \frac{4EI_x}{L} & -\frac{6EI_x}{L^2} & 0 \\ -\frac{6EI_x}{L^2} & \frac{12EI_x}{L^3} & 0 \\ 0 & 0 & \frac{EA}{L} \end{bmatrix}, \quad \boldsymbol{K}_{jj} = \begin{bmatrix} \frac{4EI_x}{L} & \frac{6EI_x}{L^2} & 0 \\ \frac{6EI_x}{L^2} & \frac{12EI_x}{L^3} & 0 \\ 0 & 0 & \frac{EA}{L} \end{bmatrix}$$

$$\boldsymbol{K}_{ij} = \boldsymbol{K}_{ji} = \begin{bmatrix} \frac{2EI_x}{L} & \frac{6EI_x}{L^2} & 0 \\ -\frac{6EI_x}{L^2} & -\frac{12EI_x}{L^3} & 0 \\ 0 & 0 & -\frac{EA}{L} \end{bmatrix}$$

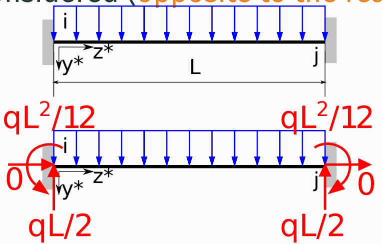
The relationship between displacements and reactions is:

$$\begin{bmatrix} M_i \\ T_i \\ N_i \\ M_j \\ T_j \\ N_j \end{bmatrix} = \begin{bmatrix} \frac{4EI_x}{L} & -\frac{6EI_x}{L^2} & 0 & \frac{2EI_x}{L} & \frac{6EI_x}{L^2} & 0 \\ -\frac{6EI_x}{L^2} & \frac{12EI_x}{L^3} & 0 & -\frac{6EI_x}{L^2} & -\frac{12EI_x}{L^3} & 0 \\ 0 & 0 & \frac{EA}{L} & 0 & 0 & -\frac{EA}{L} \\ \frac{2EI_x}{L} & -\frac{6EI_x}{L^2} & 0 & \frac{4EI_x}{L} & \frac{6EI_x}{L^2} & 0 \\ \frac{6EI_x}{L^2} & -\frac{12EI_x}{L^3} & 0 & \frac{6EI_x}{L^2} & \frac{12EI_x}{L^3} & 0 \\ 0 & 0 & -\frac{EA}{L} & 0 & 0 & \frac{EA}{L} \end{bmatrix} \begin{bmatrix} \varphi_i \\ v_i \\ w_i \\ \varphi_j \\ v_j \\ w_j \end{bmatrix}$$

or, in a more compact form:

$$\begin{bmatrix} \mathbf{Q}_i^* \\ \mathbf{Q}_j^* \end{bmatrix} = \begin{bmatrix} \mathbf{K}_{ii} & \mathbf{K}_{ij} \\ \mathbf{K}_{ji} & \mathbf{K}_{jj} \end{bmatrix} \begin{bmatrix} \boldsymbol{\delta}_i^* \\ \boldsymbol{\delta}_j^* \end{bmatrix}$$

If concentrated or distributed loads are acting on a beam, the equivalent forces on the nodes are to be considered (opposite to the reactions on fixed-end scheme):



For instance, for an uniform load q:

$$M_i^0 = -\frac{qL^2}{12}, T_i^0 = +\frac{qL}{2}, N_i^0 = 0, M_j^0 = +\frac{qL^2}{12}, T_j^0 = +\frac{qL}{2}, N_j^0 = 0$$

Equivalent nodal loadings

The relationship between displacements and reactions, accounting equivalent nodal loads is:

$$\begin{bmatrix} M_i \\ T_i \\ N_i \\ M_j \\ T_j \\ N_j \end{bmatrix} = \begin{bmatrix} \frac{4EI_x}{L} & -\frac{6EI_x}{L^2} & 0 & \frac{2EI_x}{L} & \frac{6EI_x}{L^2} & 0 \\ -\frac{6EI_x}{L^2} & \frac{12EI_x}{L^3} & 0 & -\frac{6EI_x}{L^2} & -\frac{12EI_x}{L^3} & 0 \\ 0 & 0 & \frac{EA}{L} & 0 & 0 & -\frac{EA}{L} \\ \frac{2EI_x}{L} & -\frac{6EI_x}{L^2} & 0 & \frac{4EI_x}{L} & \frac{6EI_x}{L^2} & 0 \\ \frac{6EI_x}{L^2} & -\frac{12EI_x}{L^3} & 0 & \frac{6EI_x}{L^2} & \frac{12EI_x}{L^3} & 0 \\ 0 & 0 & -\frac{EA}{L} & 0 & 0 & \frac{EA}{L} \end{bmatrix} \begin{bmatrix} \varphi_i \\ v_i \\ w_i \\ \varphi_j \\ v_j \\ w_j \end{bmatrix} - \begin{bmatrix} M_i^0 \\ T_i^0 \\ N_i^0 \\ M_j^0 \\ T_j^0 \\ N_j^0 \end{bmatrix}$$

or:

$$\begin{bmatrix} \boldsymbol{Q}_i^* \\ \boldsymbol{Q}_j^* \end{bmatrix} = \begin{bmatrix} \boldsymbol{K}_{ii} & \boldsymbol{K}_{ij} \\ \boldsymbol{K}_{ji} & \boldsymbol{K}_{jj} \end{bmatrix} \begin{bmatrix} \delta_i^* \\ \delta_j^* \end{bmatrix} - \begin{bmatrix} \boldsymbol{F}_i^* \\ \boldsymbol{F}_j^* \end{bmatrix}$$

where: $\boldsymbol{F}_i^* = [M_i^0 \; T_i^0 \; N_i^0]^T, \quad \boldsymbol{F}_j^* = [M_j^0 \; T_j^0 \; N_j^0]^T$

Beam equilibrium

For a single beam $i - j$ in the local reference frame:

$$\begin{bmatrix} \boldsymbol{Q}_i^* \\ \boldsymbol{Q}_j^* \end{bmatrix} = \begin{bmatrix} \boldsymbol{K}_{ii} & \boldsymbol{K}_{ij} \\ \boldsymbol{K}_{ji} & \boldsymbol{K}_{jj} \end{bmatrix} \begin{bmatrix} \boldsymbol{\delta}_i^* \\ \boldsymbol{\delta}_j^* \end{bmatrix} - \begin{bmatrix} \boldsymbol{F}_i^* \\ \boldsymbol{F}_j^* \end{bmatrix}$$

In a compact fashion, where subscript $_e$ indicates the beam $i - j$, is:

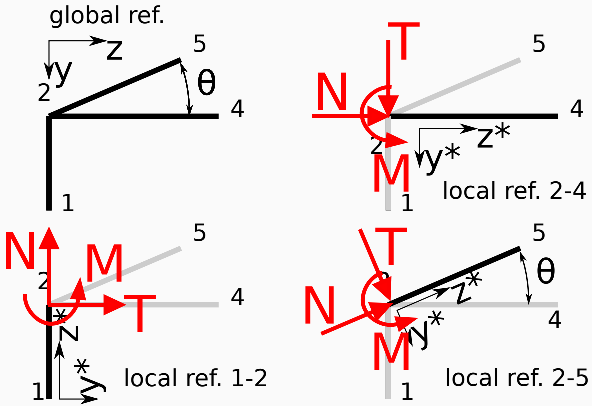
$$\boldsymbol{Q}_e^* = \boldsymbol{K}_e \boldsymbol{\delta}_e^* - \boldsymbol{F}_e^*$$

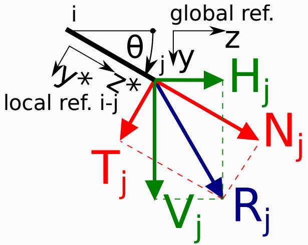
or:

$$\underset{(6\times6)}{\boldsymbol{K}_e} \underset{(6\times1)}{\boldsymbol{\delta}_e^*} = \underset{(6\times1)}{\boldsymbol{Q}_e^*} + \underset{(6\times1)}{\boldsymbol{F}_e^*}$$

where, under every matrix, the corresponding dimensions are shown (number of rows $\times$ number of columns).

To add the components of forces transferred by beams to nodes, the angle between the local and global reference frame has to be taken into account:





The components of resultant R_j are T_j, N_j (local reference) and V_j, H_j (global reference). Moment M_j is the same in both systems. Similar transformation for v_j and w_j (rotations do not change)

To move from local to global reference frame, the rotation matrix $\boldsymbol{N}_e$ is introduced:

$$\boldsymbol{N}_e = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \cos\theta_e & \sin\theta_e & 0 & 0 & 0 \\ 0 & -\sin\theta_e & \cos\theta_e & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cos\theta_e & \sin\theta_e \\ 0 & 0 & 0 & 0 & -\sin\theta_e & \cos\theta_e \end{bmatrix}$$

It is possible to move from the local reference frame (superscript *) to the global frame:

$$\boldsymbol{\delta}_e^* = \boldsymbol{N}_e \boldsymbol{\delta}_e$$
$$\boldsymbol{Q}_e^* = \boldsymbol{N}_e \boldsymbol{Q}_e$$
$$\boldsymbol{F}_e^* = \boldsymbol{N}_e \boldsymbol{F}_e$$

The equilibrium equation for the beam:

$$\boldsymbol{K}_e \boldsymbol{\delta}_e^* = \boldsymbol{Q}_e^* + \boldsymbol{F}_e^*$$

is now:

$$\boldsymbol{K}_e \boldsymbol{N}_e \boldsymbol{\delta}_e = \boldsymbol{N}_e \left(\boldsymbol{Q}_e + \boldsymbol{F}_e \right)$$

By a left multiplication (multiplication of matrices is not commutative as in ordinary algebra!) by $\boldsymbol{N}_e^T$ for both sides:

$$\boldsymbol{N}_e^T \boldsymbol{K}_e \boldsymbol{N}_e \boldsymbol{\delta}_e = \boldsymbol{N}_e^T \boldsymbol{N}_e \left(\boldsymbol{Q}_e + \boldsymbol{F}_e\right)$$

Matrix $\boldsymbol{N}_e$ is such that $\boldsymbol{N}_e^T \boldsymbol{N}_e = \boldsymbol{I}$, where $\boldsymbol{I}$ is a matrix with zero excepts for the diagonal, where the values are equal to one (unit matrix):

$$\left(\boldsymbol{N}_e^T \boldsymbol{K}_e \boldsymbol{N}_e\right) \boldsymbol{\delta}_e = \boldsymbol{I} \left(\boldsymbol{Q}_e + \boldsymbol{F}_e\right)$$

A matrix does not change if multiplied by $\boldsymbol{I}$:

$$\left(\boldsymbol{N}_e^T \boldsymbol{K}_e \boldsymbol{N}_e\right) \boldsymbol{\delta}_e = \boldsymbol{Q}_e + \boldsymbol{F}_e$$

Now, the equilibrium equation of the beam (global reference frame) is:

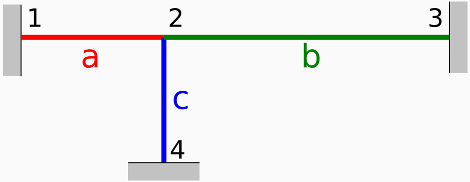
$$\left(\underset{(6\times6)}{\boldsymbol{N}_e^T}\ \underset{(6\times6)}{\boldsymbol{K}_e}\ \underset{(6\times6)}{\boldsymbol{N}_e}\right)\underset{(6\times1)}{\boldsymbol{\delta}_e} = \underset{(6\times1)}{\boldsymbol{Q}_e} + \underset{(6\times1)}{\boldsymbol{F}_e}$$

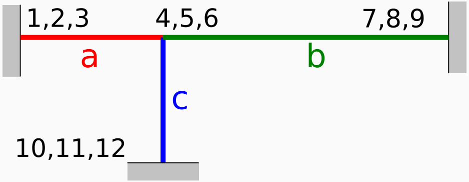
- $\boldsymbol{N}_e^T\boldsymbol{K}_e\boldsymbol{N}_e$: stiffness matrix for the beam
- $\boldsymbol{\delta}_e$: nodal displacement vector
- $\boldsymbol{Q}_e$: external reactions vector
- $\boldsymbol{F}_e$: equivalent nodal force vector

The stiffness matrices and reaction vectors for each beam must be added, taking into account the correspondence between the unknowns of the beam and the global ones (assembling). The DOF are obtained numbering from 1 to n.

Beam	Node i	Node j
a	1	2
b	2	3
c	2	4

Node	DOF
1	1, 2, 3
2	4, 5, 6
3	7, 8, 9
4	10, 11, 12





Assembling the global stiffness (2/5)

Beam a:

	1	2	3	4	5	6	7	8	9	10	11	12
1	●	●	●	●	●	●	0	0	0	0	0	0
2	●	●	●	●	●	●	0	0	0	0	0	0
3	●	●	●	●	●	●	0	0	0	0	0	0
4	●	●	●	●	●	●	0	0	0	0	0	0
5	●	●	●	●	●	●	0	0	0	0	0	0
6	●	●	●	●	●	●	0	0	0	0	0	0
7	0	0	0	0	0	0	0	0	0	0	0	0
8	0	0	0	0	0	0	0	0	0	0	0	0
9	0	0	0	0	0	0	0	0	0	0	0	0
10	0	0	0	0	0	0	0	0	0	0	0	0
11	0	0	0	0	0	0	0	0	0	0	0	0
12	0	0	0	0	0	0	0	0	0	0	0	0

Beams a and b:

	1	2	3	4	5	6	7	8	9	10	11	12
1	●	●	●	●	●	●	0	0	0	0	0	0
2	●	●	●	●	●	●	0	0	0	0	0	0
3	●	●	●	●	●	●	0	0	0	0	0	0
4	●	●	●	●●	●●	●●	●	●	●	0	0	0
5	●	●	●	●●	●●	●●	●	●	●	0	0	0
6	●	●	●	●●	●●	●●	●	●	●	0	0	0
7	0	0	0	●	●	●	●	●	●	0	0	0
8	0	0	0	●	●	●	●	●	●	0	0	0
9	0	0	0	●	●	●	●	●	●	0	0	0
10	0	0	0	0	0	0	0	0	0	0	0	0
11	0	0	0	0	0	0	0	0	0	0	0	0
12	0	0	0	0	0	0	0	0	0	0	0	0

Beams a, b and c:

	1	2	3	4	5	6	7	8	9	10	11	12
1	🔴	🔴	🔴	🔴	🔴	🔴	0	0	0	0	0	0
2	🔴	🔴	🔴	🔴	🔴	🔴	0	0	0	0	0	0
3	🔴	🔴	🔴	🔴	🔴	🔴	0	0	0	0	0	0
4	🔴	🔴	🔴	🔴🟢🔵	🔴🟢🔵	🔴🟢🔵	🟢	🟢	🟢	🔵	🔵	🔵
5	🔴	🔴	🔴	🔴🟢🔵	🔴🟢🔵	🔴🟢🔵	🟢	🟢	🟢	🔵	🔵	🔵
6	🔴	🔴	🔴	🔴🟢🔵	🔴🟢🔵	🔴🟢🔵	🟢	🟢	🟢	🔵	🔵	🔵
7	0	0	0	🟢	🟢	🟢	🟢	🟢	🟢	0	0	0
8	0	0	0	🟢	🟢	🟢	🟢	🟢	🟢	0	0	0
9	0	0	0	🟢	🟢	🟢	🟢	🟢	🟢	0	0	0
10	0	0	0	🔵	🔵	🔵	0	0	0	🔵	🔵	🔵
11	0	0	0	🔵	🔵	🔵	0	0	0	🔵	🔵	🔵
12	0	0	0	🔵	🔵	🔵	0	0	0	🔵	🔵	🔵

The assembling operation is an expansion of 6×6 matrices or 6×1 vectors to the global dimensions, $n \times n$ and $n \times 1$ and their sum. Superscript ^{x} indicates the eXpanded contribution of beam e.

$$\left(\sum_e \underset{(n\times n)}{\boldsymbol{K}^x}\right) \underset{(n\times 1)}{\boldsymbol{\delta}} = \sum_e \underset{(n\times 1)}{\boldsymbol{Q}^x} + \sum_e \underset{(n\times 1)}{\boldsymbol{F}^x}$$

Posing:

- $\boldsymbol{K} = \sum_e \boldsymbol{K}^x$: global stiffness matrix of the structure
- $\boldsymbol{F} = \sum_e \boldsymbol{F}^x$: equivalent nodal force vector

it is obtained:

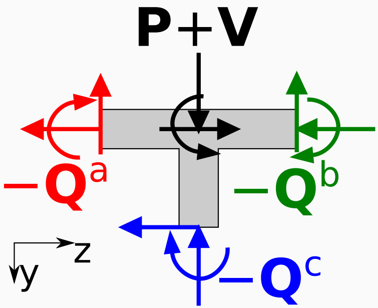
$$\boldsymbol{K}\boldsymbol{\delta} = \sum_e \boldsymbol{Q}^x + \boldsymbol{F}$$

P represents the vector of forces applied directly to the nodes and ***V*** the vector of external reactions (constrained nodes), so that equilibrium gives:

$$-\sum_{e} \boldsymbol{Q}^{x} + \boldsymbol{P} + \boldsymbol{V} = \boldsymbol{0}$$

or:

$$\sum_{e} \boldsymbol{Q}^{x} = \boldsymbol{P} + \boldsymbol{V}$$



For the whole structure, assuming $\boldsymbol{R} = \boldsymbol{P} + \boldsymbol{V} + \boldsymbol{F}$, is:

$$\boldsymbol{K}\,\boldsymbol{\delta} = \boldsymbol{R}$$

$\boldsymbol{K}$ is symmetrical, positive definite (i.e., $\boldsymbol{x}\,\boldsymbol{K}\,\boldsymbol{x}^T > 0$ for $\forall\ \boldsymbol{x}$) and terms on the diagonal are positive

Calculation of displacements

Changing properly rows and columns, $\boldsymbol{K}\boldsymbol{\delta} = \boldsymbol{R}$ can be partitioned as:

$$\begin{bmatrix} \boldsymbol{K}_{LL} & \boldsymbol{K}_{LV} \\ \boldsymbol{K}_{VL} & \boldsymbol{K}_{VV} \end{bmatrix} \begin{bmatrix} \boldsymbol{\delta}_L \\ \boldsymbol{\delta}_V \end{bmatrix} = \begin{bmatrix} \boldsymbol{R}_L \\ \boldsymbol{R}_V \end{bmatrix}$$

to separate the L displacements of free nodes (unknowns) from V displacements of constrained nodes (known, equal to zero or assigned).

From the first row, a linear equation system is obtained:

$$\boldsymbol{K}_{LL}\boldsymbol{\delta}_L = \boldsymbol{R}_L - \boldsymbol{K}_{LV}\boldsymbol{\delta}_V \quad \text{i.e.} \quad \boldsymbol{\delta}_L = \boldsymbol{K}_{LL}^{-1}(\boldsymbol{R}_L - \boldsymbol{K}_{LV}\boldsymbol{\delta}_V)$$

Its solution can be effectively carried out through numerical procedures, and gives the unknown vector $\boldsymbol{\delta}_L$

External reactions calculation

Using equation:

$$\begin{bmatrix} \boldsymbol{K}_{LL} & \boldsymbol{K}_{LV} \\ \boldsymbol{K}_{VL} & \boldsymbol{K}_{VV} \end{bmatrix} \begin{bmatrix} \boldsymbol{\delta}_L \\ \boldsymbol{\delta}_V \end{bmatrix} = \begin{bmatrix} \boldsymbol{R}_L \\ \boldsymbol{R}_V \end{bmatrix}$$

being known $\boldsymbol{\delta}_L$, from the second row it is possible to obtain $\boldsymbol{R}_V$:

$$\boldsymbol{R}_V = \boldsymbol{K}_{VL}\boldsymbol{\delta}_L + \boldsymbol{K}_{VV}\boldsymbol{\delta}_V$$

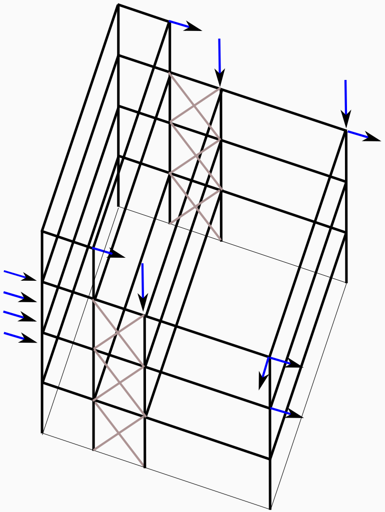
and, the external reactions:

$$\boldsymbol{V}_V = \boldsymbol{R}_V - \boldsymbol{P}_V - \boldsymbol{F}_V$$

where $\boldsymbol{F}_V$ is the vector of the equivalent nodal forces (constrained nodes) and $\boldsymbol{P}_V$ the forces directly applied to the nodes.

The procedure presented above can extended to three dimensional frames

A space frame is a structure system assembled of linear elements so arranged that forces are transferred in a three-dimensional manner. In some cases, the constituent element may be two-dimensional



Here the main differences:

- Each node *i* or *j* of a 3D beam possess 6 degree of freedom (3 displacements and 3 rotations)
- Six forces (normal forces, two shear forces, two bending moments and one torsional moment) are acting at each node
- The matrix $\boldsymbol{K}_e$ will be a 12×12 matrix
- $\boldsymbol{Q}_e$ and $\boldsymbol{F}_e$ will be vectors of dimension 12×1

The dimensions of matrices and vectors are increased, but the automatic procedure (to be carried out by a computer!) remains basically the same

Some advice to catch errors:

- check for a correct geometry: for instance, the stiffness matrix cannot be created for a beam with zero length!
- check if all data needed by the software are given (geometry of the structure, geometrical and material properties)
- pay attention to unstable structures (the global stiffness matrix cannot be managed)
- beams free in space are unstable, so that the global stiffness matrix cannot be obtained
- in a three-dimensional space, the degree of freedom of a beam are six!