BEAM THEORY – ELASTIC STABILITY

ADVANCED STRUCTURAL MECHANICS

The ERAMCA Project

Environmental Risk Assessment and Mitigation on Cultural Heritage assets in Central Asia

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ERAMCA
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Environmental Risk Assessment and Mitigation on Cultural Heritage Assets in Central Asia

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LECTURER/STUDENTS OBJECTIVES

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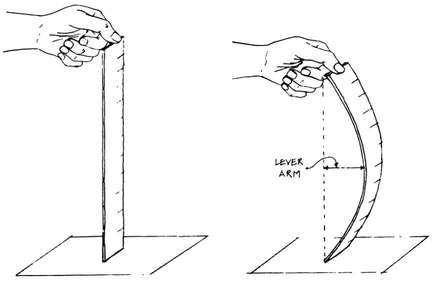
- Present the behaviour of slender compressed structures.
- Understand the mathematical model. Apply the theory to calculate the critical load for beams.

INTRODUCTION



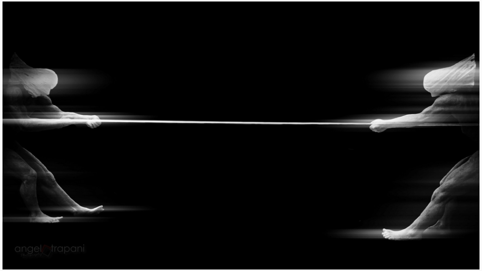
513 BUCKLING OF THIN RULER UNDER AXIAL COMPRESSION

Buckling of Unstiffened shell | Buckling of Stiffened shell with Lozenge Grids | Buckling of Stiffened shell with Triangle grids

Nice video of buckling of a can of Coke.

Tension and compression

The behavior of slender elastic rods subjected to axial force depends on the sign of N (tension or compression)

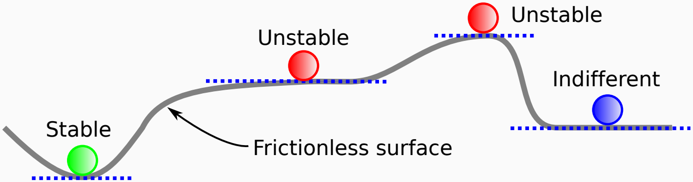


Tension: the rod remains straight up to rupture (left); compression: beyond a certain value of N, it is possible to reach deflected configurations in equilibrium (right)

Equilibrium: stable, unstable and indifferent

A configuration in equilibrium (C) is moved slightly away (C'):

- if the system returns to its original equilibrium position, it is said to be stable
- if the system moves further away from that position, it is said to be unstable



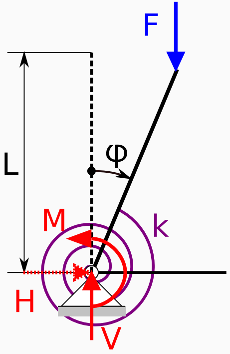
Rigid body (ball) on a frictionless surface, gravitational field. The equilibrium is assured where the tangent to the surface is horizontal, but the green ball only is in a *stable equilibrium state*

EXAMPLES WITH CONCENTRATED ELASTICITY

Equilibrium refers to the deformed configuration!

The equilibrium conditions take into account displacements with respect to the undeformed configuration

Rigid rod, elasticity concentrated in the torsional spring ($M = k\varphi$)



Equilibrium equations:

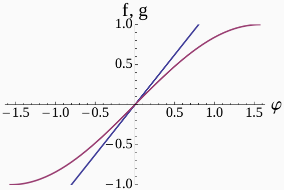
$$\begin{cases} H = 0 \\ V = F \\ M = FL\sin\varphi \end{cases}$$

M is a function of φ, so that:

The latter equation can be rewritten as:

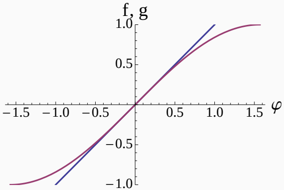
$$\frac{k}{FL}\varphi = \sin\varphi$$

plotting $f(\varphi) = \frac{k}{FL}\varphi$ and $g(\varphi) = \sin\varphi$. The intersections lead to three solutions:



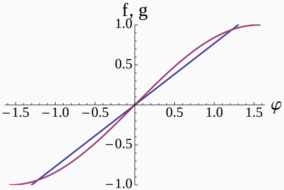
One solution ($\varphi = 0$)

$$\frac{k}{FL} > 1 \iff F < \frac{k}{L}$$



One solution ($\varphi = 0$)

$$\frac{k}{FL} = 1 \iff F = \frac{k}{L}$$



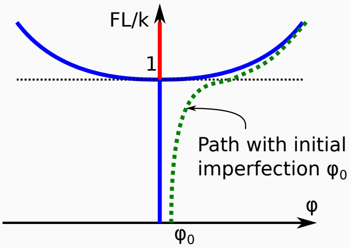
Three solutions

$(\varphi = -\varphi^*, 0, +\varphi^*)$

$$\frac{k}{FL} < 1 \iff F > \frac{k}{L}$$

Every point of the diagram of the non-dimensional force $\frac{FL}{k}$ is in equilibrium. If the load path starts from $F = 0$:

- if $F < k/L$ the rod must be vertical ($\varphi = 0$)
- if $F > k/L$ the rod can be vertical or move to the left or to the right ($\pm\varphi^*$)
- the value $F_{cr} = k/L$ where the load can follows different paths (bifurcation) is the critical load



Blue: stable paths Red: unstable path

Green: stable path with initial imperfection φ_0

Another very simple structure

Rigid rod, elasticity concentrated in the spring ($F_s = kL\sin\varphi$). Equilibrium about A gives:

$$\underbrace{(kL\sin\varphi)}_{\text{Force } F_s}\underbrace{(L\cos\varphi)}_{\text{Arm}} - FL\sin\varphi = 0$$

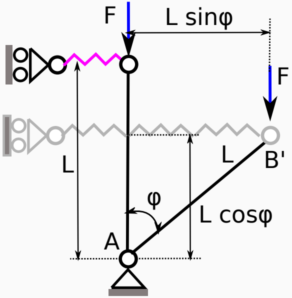
so that $F = kL\cos\varphi$

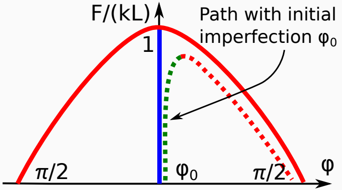
It is again:

Blue: stable paths

Red: unstable path

Green: stable path with initial imperfection φ_0





Elastic rods (elastic constant k); equilibrium of node B':

$$F - 2\, N_{AB} \sin\varphi = 0 \quad \text{where:}$$

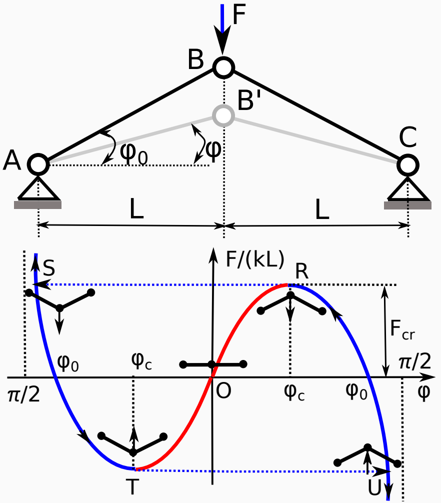
$$N_{AB} = k\, \Delta L_{AB} = k(AB' - AB) =$$

$$k\left(\frac{L}{\cos\varphi} - \frac{L}{\cos\varphi_0}\right)$$

so that:

$$\frac{F}{2kL} = \sin\varphi\left(\frac{1}{\cos\varphi} - \frac{1}{\cos\varphi_0}\right)$$

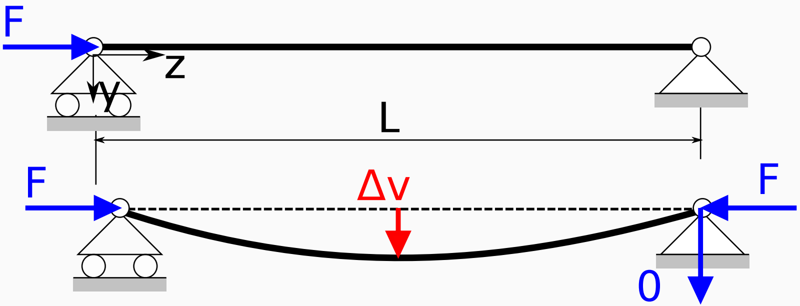
The path starts at φ_0, reaches φ_C (at R) and snaps through S.



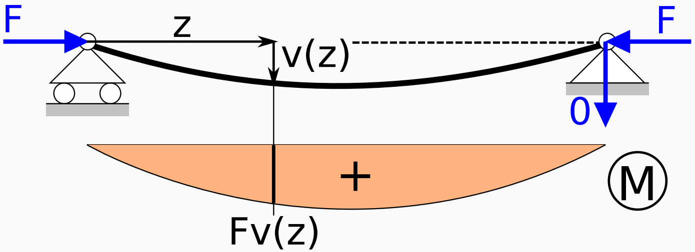
Distributed elasticity

Beam loaded in compression

A straight beam is subjected to axial compression. If the original configuration is moved slightly away by an infinitesimal displacement Δv, will the system return to its original equilibrium position, or will it move further away from that position?



A moment equal to $M(z) = F\,v(z)$ is necessary for equilibrium:



The moment inside a beam is given by the elastic curve equation:

$$\frac{d^2 v(z)}{dz^2} = -\frac{M(z)}{E\,I_x}$$

Equilibrium in the deformed configuration (2/2)

Summing up, the two moments are:

- $M_{\mathrm{stab}} = -E\,I_x\,v''(z)$: stabilizing (or restoring) moment
- $M_{\mathrm{dest}} = F\,v(z)$: destabilizing moment

For equilibrium $M_{\mathrm{stab}} = M_{\mathrm{dest}}$:

$$-E\,I_x\,\frac{\mathrm{d}^2 v(z)}{\mathrm{d}z^2} = F\,v(z)$$

or:

$$\frac{\mathrm{d}^2 v(z)}{\mathrm{d}z^2} + \frac{F}{E\,I_x} v(z) = 0$$

This equation is a linear, homogeneous differential equation of the second order with constant coefficients:

$$\frac{d^2 v(z)}{dz^2} + \frac{F}{E\,I_x} v(z) = 0$$

the general solution is:

$$v(z) = C_1 \sin\left(\sqrt{\frac{F}{E\,I_x}}\,z\right) + C_2 \cos\left(\sqrt{\frac{F}{E\,I_x}}\,z\right)$$

Recalling the boundary conditions that must be satisfied at both ends, it is possible to find C_1 and C_2:

$$v(0) = 0$$

$$v(L) = 0$$

The first condition is:

$$v(0) = 0 \implies C_2 = 0$$

hence

$$v(z) = C_1 \sin\left(\sqrt{\frac{F}{E\,I_x}}\,z\right)$$

$$v(L) = 0 \implies C_1 \sin\left(\sqrt{\frac{F}{E\,I_x}}\,L\right) = 0 \implies C_1 = 0$$

Recalling that $C_1 = C_2 = 0$, the solution is $v(z) = 0$. It means that the deformed configuration overlap the undeformed configuration!

But...

Solution of the differential equation (3/3)

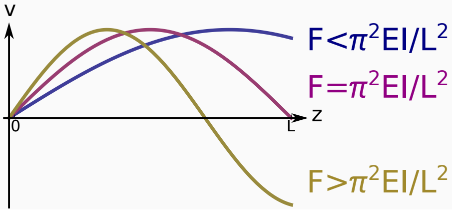
It seems no other solutions different from the trivial one are possible.

Reconsidering the equation obtained imposing the end condition for $z = 0$:

$$v(z) = C_1 \sin\left(\sqrt{\frac{F}{E\,I_x}}\,z\right)$$

it satisfies the end condition for $z = L$ if the force F is such that $\sin\left(\sqrt{\frac{F}{E\,I_x}}\,L\right) = 0$,

e.g., for $\sqrt{\frac{F}{E\,I_x}}\,L = \pi$, or for $F = \frac{\pi^2 E\,I_x}{L^2}$



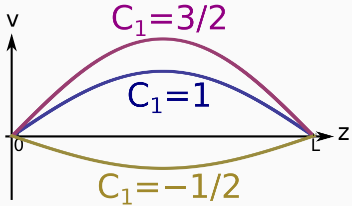
There is no uniqueness!

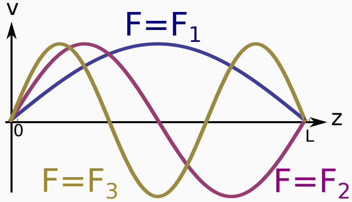
- Constant C_1 can assume any value
- There exist infinite values of F such that $\sin\left(\sqrt{\frac{F}{E\,I_x}}\,L\right) = 0$, i.e.:

$$\sqrt{\frac{F}{E\,I_x}}\,L = n\pi, \quad n = 1, 2, \ldots$$

or

$$F_1 = 1\,\frac{\pi^2 E\,I_x}{L^2}, \; F_2 = 4\,\frac{\pi^2 E\,I_x}{L^2}, \; F_3 = 9\,\frac{\pi^2 E\,I_x}{L^2}, \; \ldots$$





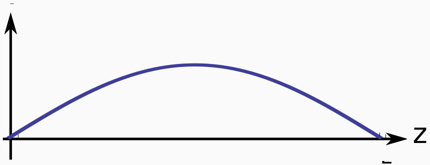
Critical load (buckling load)

- Infinite solutions of the problem corresponding to sinusoidal curves with different periods, are possible (it is not possible to find the magnitude of the deflection)
- The load F should belong to the set $\{F_1, F_2, F_3 \ldots\}$
- The load corresponding to $n = 1$ is the Euler buckling load

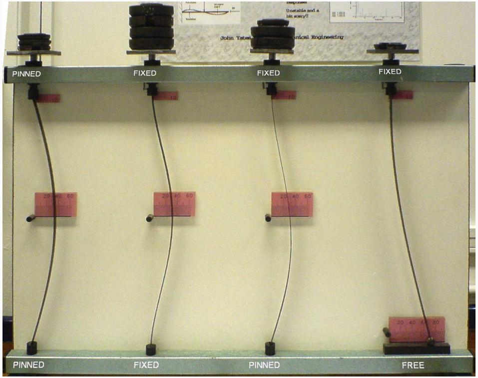
$$F_{cr} = \frac{\pi^2 E I_x}{L^2}$$

- The corresponding elastic curve is called critical deflection

$$F = F_{cr}$$



The deformation and the critical load can change with different support conditions

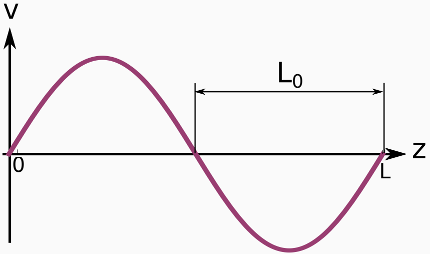


Effective length

When the supports at the end are different from the one examined before, the critical load is similar:

$$F_{cr} = \frac{\pi^2 E\, I_x}{L_o^2}$$

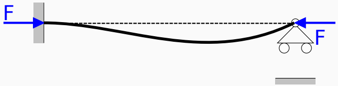
where L_o is the effective length, i.e., the distance between two subsequent points with zero curvature of the sinusoidal curve



Different end conditions

$$L_0 = L \qquad F_{cr} = \frac{\pi^2 E\, I_x}{L^2}$$

$$L_0 = L/2 \qquad F_{cr} = 4\frac{\pi^2 E\, I_x}{L^2}$$



$$L_0 \approx 0.7L \qquad F_{cr} \approx 2\frac{\pi^2 E\, I_x}{L^2}$$

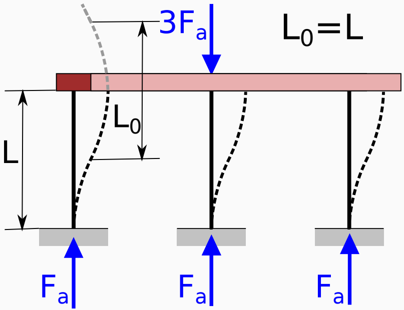
$$L_0 = 2L \qquad F_{cr} = \frac{\pi^2 E\, I_x}{4L^2}$$

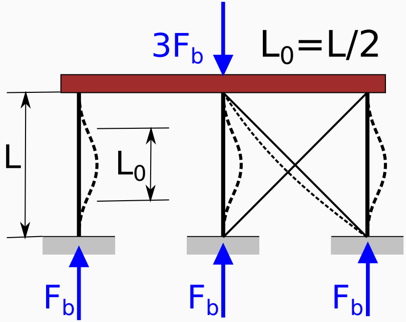
$$L_0 = L \qquad F_{cr} = \frac{\pi^2 E\, I_x}{L^2}$$

$$L_0 = 2L \qquad F_{cr} = \frac{\pi^2 E\, I_x}{4L^2}$$

EXAMPLE: SHEAR-TYPE FRAME

Horizontal displacements are prevented by diagonal members (bracing)



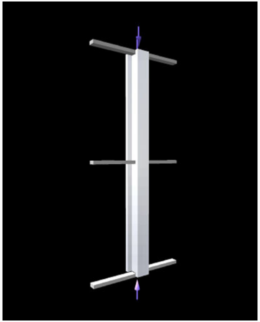


$$F_{a,cr} = \frac{\pi^2 E I_x}{L^2}$$

$$F_{b,cr} = 4\frac{\pi^2 E I_x}{L^2}$$

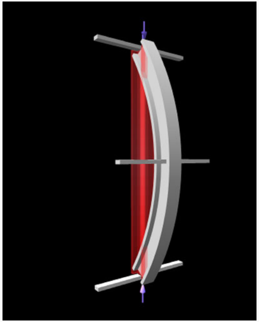
Remember the out-of-plane buckling where the constraints can behave in a different way!

The critical load is $F_{cr} = \frac{\pi^2 E I_x}{L_0^2}$, and depends on the deflection plane, defined by the smaller from $I_{xx}/L_{0,x}^2$ and $I_{yy}/L_{0,y}^2$

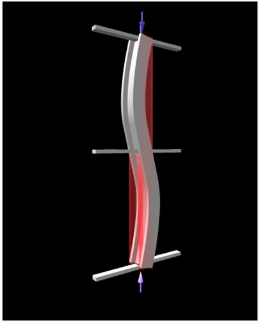


Undeformed configuration

(©Kirk Martini)



Deflection in the strong axis



Deflection in the weak axis

Introducing the radius of gyration $\rho_x = \sqrt{\frac{I_x}{A}}$, and the non-dimensional slenderness:

$$\lambda = \frac{L_0}{\rho_x}$$

the critical load $F_{cr} = \frac{\pi^2 E\, I_x}{L_0^2}$ can be rewritten as:

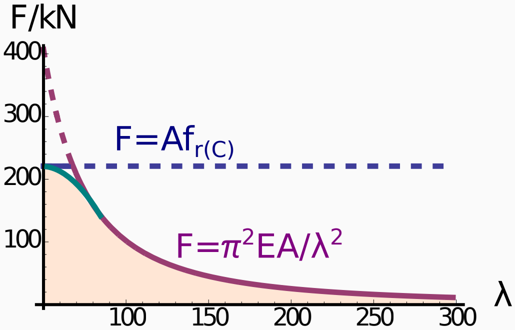
$$F_{cr} = \frac{\pi^2 E\, A}{\lambda^2}$$

It can be seen that the critical load (that must be larger than the applied load for safety reasons), depends on:

- A: cross section area
- E: modulus of elasticity of the beam
- λ: maximum slenderness (in the actual deflection plane, see slide 30)

Transition from the buckling failure to compression failure:

- slender columns: elastic instability (buckling failure)
- stubby columns: compression failure

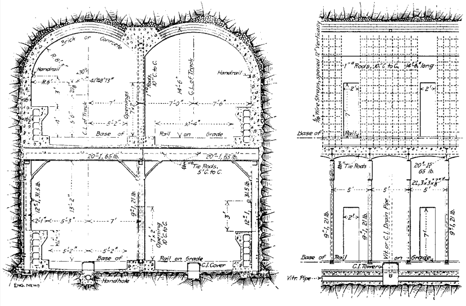


Steel rod (circular cross section, 25 mm diameter), $E = 210$ GPa, $f_{r(C)} = 450$ MPa

Real world examples

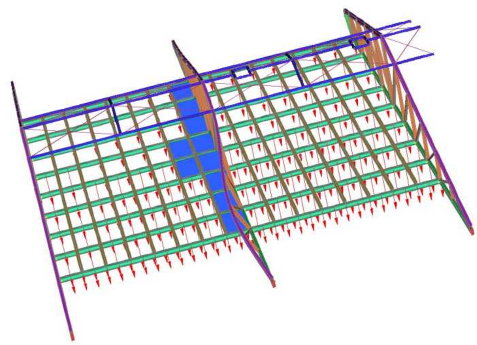
STEEL COLUMNS

Instability of steel columns (composite section) subjected to train loads (New York City subway)



http://www.nycsubway.org

- lateral-torsional buckling of a steel truss frame



http://www.hsh.info

- buckling of the web of a steel beam

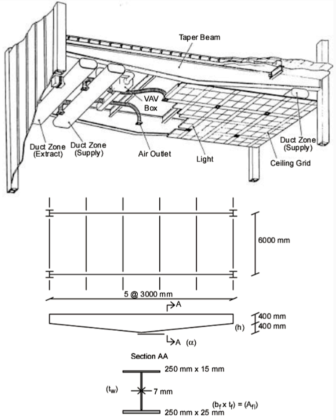
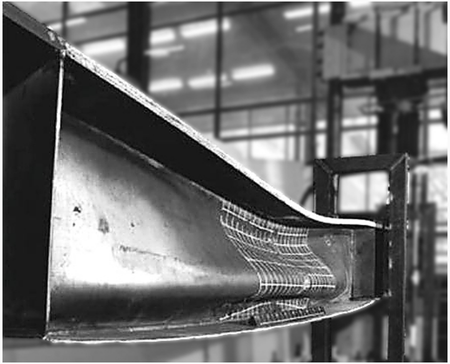


Figure 1. Typical composite tapered floor system.



EXERCISE

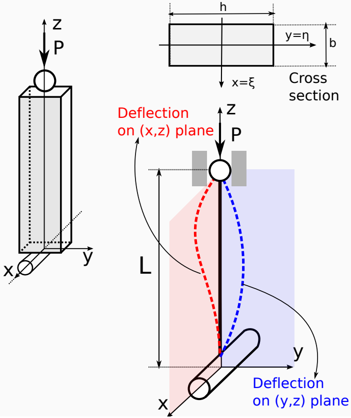
Column with a rectangular cross section: the upper support restraints displacements along *x* and *y*; the lower support restraints displacements along *x* and *y* and rotations about *y* and *z*. Find the critical load.

- (x, z) plane of buckling: hinge–fixed-end, $L_0 \approx 0.7L$:

$$N_{cr}^{(x,z)} \approx \frac{\pi^2 E I_{yy}}{(0.7L)^2} \approx 2\frac{\pi^2 E I_{yy}}{L^2}$$

- (y, z) plane of buckling: simple support–hinge, $L_0 = L$:

$$N_{cr}^{(y,z)} = \frac{\pi^2 E I_{xx}}{L^2}$$



The critical load is the minimum between $N_{cr}^{(x,z)}$ and $N_{cr}^{(y,z)}$:

$$N_{cr} = \min(N_{cr}^{(x,z)},\ N_{cr}^{(y,z)}) = \min\left(2\,\frac{\pi^2\,E\,I_{yy}}{L^2},\ \frac{\pi^2\,E\,I_{xx}}{L^2}\right)$$

and depends on I_{xx} and I_{yy}.

If, for example, it is assumed $h = 2b$, it is $I_{xx} = 4\,I_{yy}$:

$$N_{cr}^{(x,z)} = 2\,\frac{\pi^2\,E\,\frac{I_{xx}}{4}}{L^2} = \frac{1}{2}\,\frac{\pi^2\,E\,I_{xx}}{L^2} \quad \text{and} \quad N_{cr}^{(y,z)} = \frac{\pi^2\,E\,I_{xx}}{L^2}$$

It means that the minimum critical load is equal to $N_{cr}^{(x,z)}$:

$$N_{cr} = N_{cr}^{(x,z)} = \frac{1}{2}\,\frac{\pi^2\,E\,I_{xx}}{L^2}$$