BEAM THEORY – CONSTITUTIVE EQUATIONS

STRUCTURAL MECHANICS

The ERAMCA Project

Environmental Risk Assessment and Mitigation on Cultural Heritage assets in Central Asia

v2022317

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Lecturer/students objectives

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- Link static and kinematic behavior of beams.
- Understand the relationships between the internal forces and deformations.

Introduction

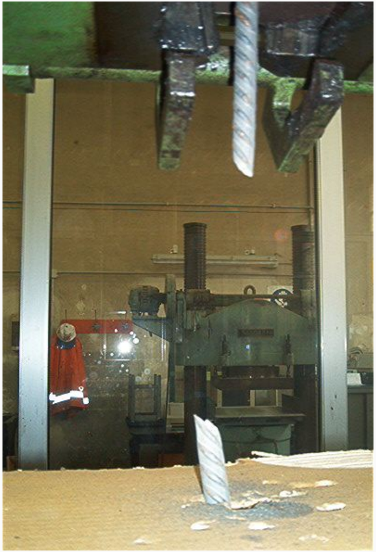
Aim of the lesson

The aim of the lecture is to provide the information necessary to understand:

- relations between internal forces and deformations, i.e., global constitutive equations

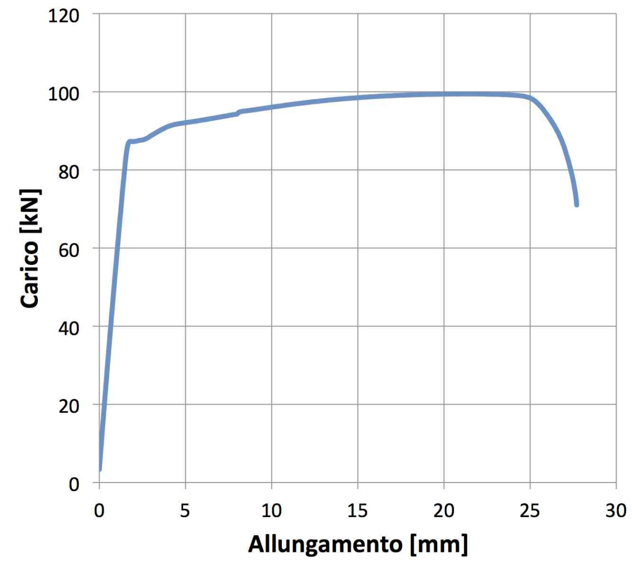
Constitutive equations for beams

Experiments (normal force, tension) – Steel (1/3)

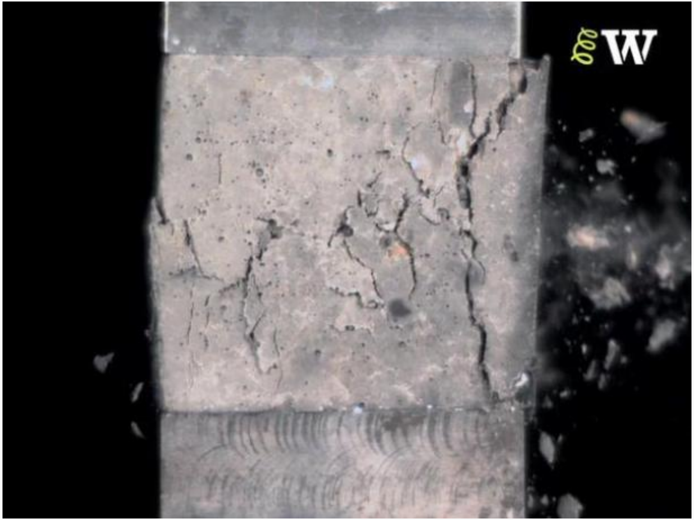


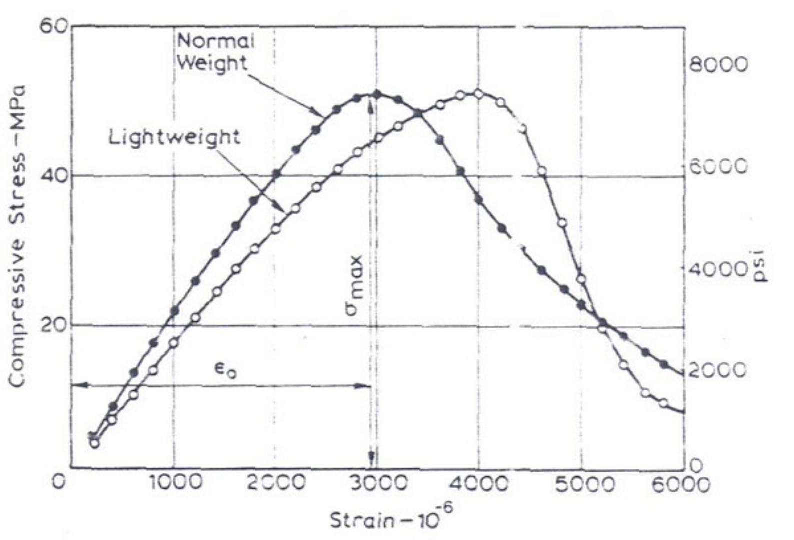
Necking (2/3)

Experiments (normal force, tension) – Steel (3/3)

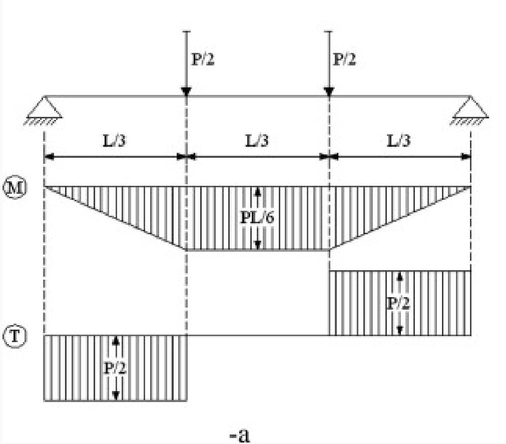


Experiments (normal force, compression) – Concrete (1/2)



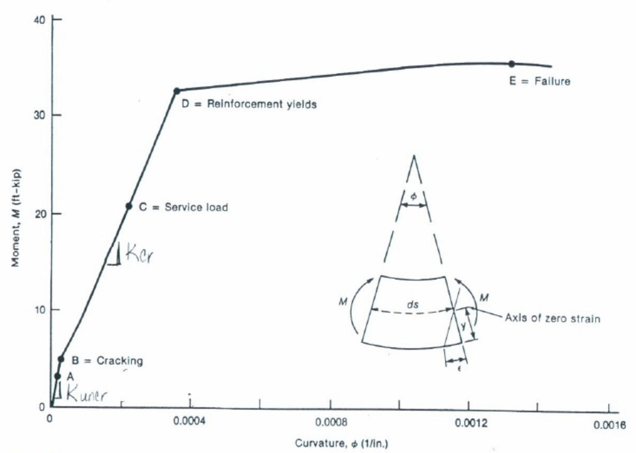

60
40
20
0
Compressive Stress – MPa
Normal Weight
Lightweight
σ_{max}
ϵ_o
8000
6000
4000
2000
0
psi
1000
2000
3000
4000
5000
6000
Strain – 10^{-6}

Experiments (bending) – Reinforced concrete (1/2)

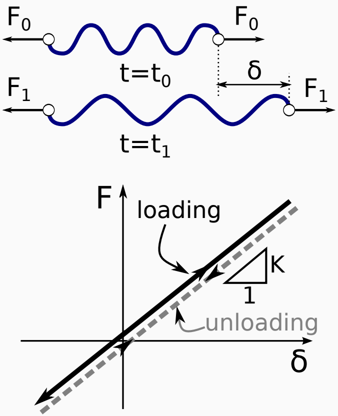


http://www.dtu.dk/subsites/Wind%20Engineering/Education/Beam_Testing_Concrete.aspx

Experiments (bending) – Reinforced concrete (2/2)



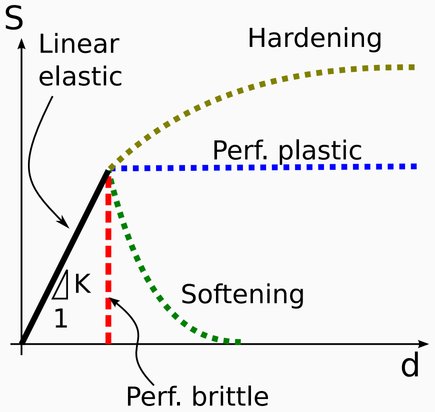
- The linear elastic relationship is such that:
  1. the force is an unique function of deformation
  2. if the force is removed, the original (usually undeformed) state is obtained without residual or permanent deformations. It is a reversible behavior.
- The linear elastic behavior is a particular elastic relationship expressed by a linear law (Hooke's law)



Linear elastic constitutive equations (2/4)

Given an internal force $S(z)$, the corresponding deformation $d(z)$ can be given as $d(z) = \frac{S(z)}{K}$

Internal force S	Deformation d	Stiffness $K = \frac{\Delta S}{\Delta d}$
N	ε_0	$K_{A,e} = E\,A$
T	γ	$K_{S,e} = G\,A_T$
M	χ	$K_{B,e} = E\,I_x$



For an elementary beam (length dz):

- $K_{A,e} = E\,A$: axial stiffness
- $K_{S,e} = G\,A_T$: shear stiffness
- $K_{B,e} = E\,I_x$: bending stiffness

Internal forces and deformations

Proportion between internal forces (N, M, T) and deformations (ε_0, χ, γ) through the stiffness

The stiffness (slope of $S - d$ diagram) depends on material (E, G) and on the cross section shape $\left(A, I_x, A_T = \frac{A}{t}\right)$

$$\begin{aligned}
\varepsilon_0(z) &= \frac{N(z)}{K_{A,e}} = \frac{N(z)}{E\,A} \\
\gamma(z) &= \frac{T(z)}{K_{S,e}} = \frac{T(z)}{G\,A_T} = t\,\frac{T(z)}{G\,A} \\
\chi(z) &= \frac{M(z)}{K_{B,e}} = \frac{M(z)}{E\,I_x}
\end{aligned}$$

Geometrical properties of the cross section

- A: area
- $t > 1$: shear factor $\left(A_T = \frac{A}{t}\right)$
- I_x: second moment of area (or moment of inertia)

The lecture about the geometrical properties of areas deepens this topic.

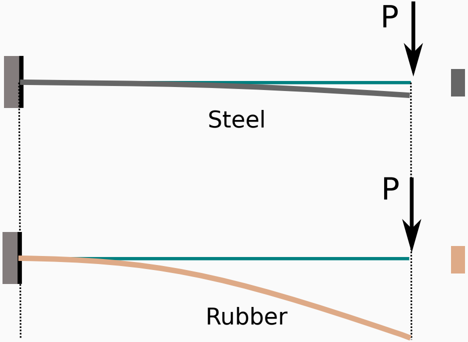
Material properties

- E: modulus of elasticity (Young's modulus)
- G: shear modulus

The lecture about the material properties deepens this topic.

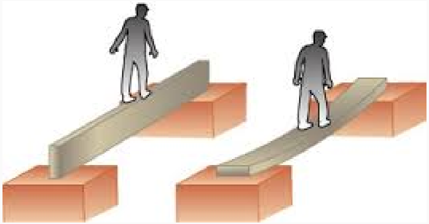
Stiffness: influence of material and section

Different material, same cross section (with respect to the same vertical loading direction)



Top: $E = E_s = 210$ GPa, bottom: $E = E_r \approx 0.1$ GPa

Same material, different cross section (different inertia with respect to the same vertical loading direction)



Left: $I_x = \frac{b\,h^3}{12}$, right: $I_x = \frac{h\,b^3}{12}$ (where x is the horizontal axis, orthogonal to the direction of gravity)

Physical dimensions

Quantity	Physical dimension	SI unit
E, G	FL^{-2}	Pa
A	L^2	m^2
I_x	L^4	m^4
t	–	–
$K_{A,e}$, $K_{S,e}$	F	N
$K_{B,e}$	FL^2	Nm^2