BEAM THEORY – DEFLECTION OF BEAMS

STRUCTURAL MECHANICS

The ERAMCA Project

Environmental Risk Assessment and Mitigation on Cultural Heritage assets in Central Asia

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Lecturer/students objectives

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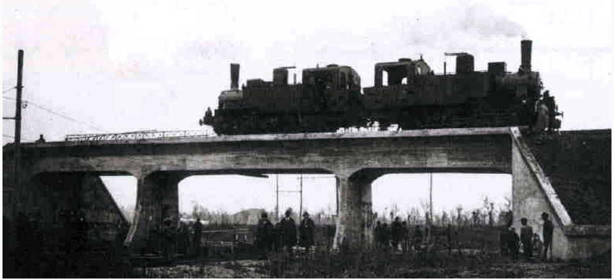
- Present the deformation of plane elastic beams in terms of displacements of the longitudinal axis.
- Understand the mathematical model for the deflection and the solution method taking into account the proper boundary conditions. Apply the theory to calculate displacements and rotations to beams under different loading and boundary conditions.

Introduction

Aims

In engineering work, it is frequently necessary to estimate the displacements on a structure subjected to loads:

- at the design stage, in order to assess if the displacements under ordinary loads are acceptable or not
- during the final static testing, in order to assess if the structure, under design loads, behaves as expected without exiting elastic bounds



Bridge on Lorgana Canal on Bologna-Portomaggiore railway. Bridge testing.

Goals

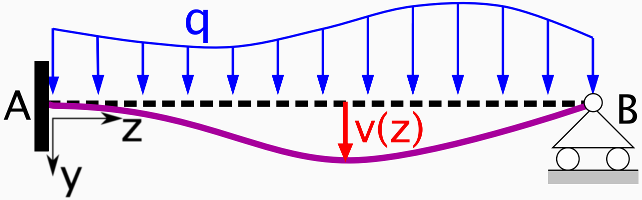
To develop a theory (i.e., a mathematical model) that allows to compute the displacements v, perpendicular to the beam axis, as a function of:

- beam length
- shape and size of beam cross section
- material properties
- position and magnitude of the loads
- supports

To find a mathematical expression that assigns to z, i.e., the abscissa along the beam axis, a value of the displacement function $v(z)$. This expression is called deflection

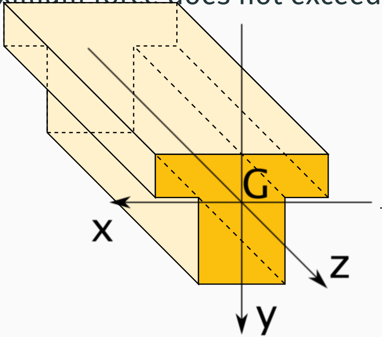
In order to treat the problem in a simple way, let us suppose that:

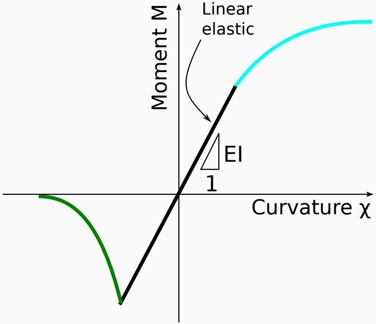
- the longitudinal axis of the beam, z, is straight
- the distributed load is a continuous function and acts perpendicular to beam axis, i.e., vertically
- the lateral displacements $v(z)$ are small with respect to the size of the cross section and the rotations $\varphi(z)$ of the axis are omitted
- shear strain, γ, is neglected



In addition, let us suppose that:

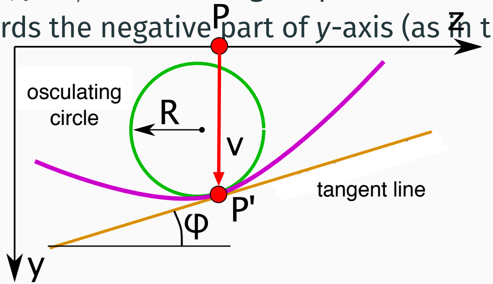
- the cross section is constant along the length of the beam and symmetric with respect to the vertical plane
- the maximum force does not exceed elastic limits





For any curve in the plane, it is possible to define:

- the grade, i.e., the angle φ that the tangent line forms with z axis, positive if anti-clockwise (as in the sketch)
- the curvature χ, which modulus is equal to the inverse of the radius R of the osculating circle ($\chi = 1/R$) and the sign is positive if the concavity of the curve looks towards the negative part of y-axis (as in the sketch)



If the mathematical expression of the curve is $v = v(z)$, with the sign conventions previously denoted, the curvature in any point is

$$\chi(z) = -\frac{v''(z)}{\sqrt{\left(1 + v'(z)^2\right)^3}}$$

If the rotation of φ is small (negligible with respect to the unity), the following approximations are valid

$$\varphi(z) = -v'(z) = -\frac{\mathrm{d}v(z)}{\mathrm{d}z}$$

$$\chi(z) = -v''(z) = -\frac{\mathrm{d}^2 v(z)}{\mathrm{d}z^2}$$

Second-order differential equation

The relationship between bending moment and curvature in a given point Q, at an abscissa z, is:

$$\chi(z) = \frac{M(z)}{K_{B,e}} = \frac{M(z)}{E\, I_x}$$

where the product $E\, I_x$ is the flexural stiffness of the beam

The approximation for small rotations leads to the

2nd-order differential equation for the elastic curve

$$\frac{\mathrm{d}^2 v(z)}{\mathrm{d}z^2} = -\frac{M(z)}{E\, I_x}$$

Second order differential equation:

$$\frac{\mathrm{d}^2 v(z)}{\mathrm{d}z^2} = -\frac{M(z)}{E\, I_x}$$

- determine the deflection $v(z)$ when the bending moment $M(z)$ is known ($M(z)$ has to be determined in a previous step)
- need a double integration
- needs 2 boundary conditions (one for each end or both on a single end)

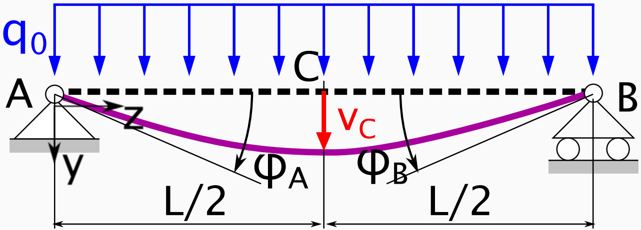
If P is one of the two ends, A or B, of the beam:
Kinematic boundary conditions: on displacements or rotations (two)

- if the displacements is prevented, $v|_P = 0$
- if the rotation is prevented, $v'|_P = 0$ (since $\varphi = -v'$)

In general, the known quantities ($v|_P$, $\varphi|_P$ can be different from zero (settlements); the external reactions and bending moment $M(z)$ are determined previously.

Example

Problem: Determine the expression of the displacement function for a simply supported beam, of length L and flexural stiffness $E\,I_x$, under a uniform distributed load q_0; compute the deflection in the middle point v_C, and the rotations at the ends φ_A, φ_B



Elaboration: the moment for this beam is $M(z) = {}^{qLz}/_{2} - {}^{qz^2}/_{2}$. The stiffness $E\,I_x$, which is constant, exits the integration. The integration constants C_1 and C_2 enter in the final expression:

$$E\,I_x\,v''(z) = -M(z) = \frac{qz^2}{2} - \frac{qLz}{2} \tag{1}$$

$$E\,I_x\,v'(z) = E\,I_x\,(-\varphi(z)) = \frac{qz^3}{6} - \frac{qLz^2}{4} + C_1 \tag{2}$$

$$E\,I_x\,v(z) = \frac{qz^4}{24} - \frac{qLz^3}{12} + C_1 z + C_2 \tag{3}$$

At the ends A and B, the vertical displacements (both null) are the known quantities.

For getting the values of the integration constants, the boundary conditions are imposed on the displacements

- at A ($z = 0$) it is $v|_A = v(0) = 0$
- at B ($z = L$) it is $v|_B = v(L) = 0$

that is:

$$v(0) = 0 \implies 0 = C_2$$

$$v(L) = 0 \implies 0 = \frac{qL^4}{24} - \frac{qL^4}{12} + C_1 L, \quad C_1 = +\frac{qL^3}{24}$$

Substituting in Eq. 3, the displacement function $v(z)$ is obtained:

$$v(z) = \frac{1}{E\,I_x}\left(\frac{qz^4}{24} - \frac{qLz^3}{12} + \frac{qL^3}{24}z\right)$$

The deflection at midspan is:

$$c = v(L/2) = +\frac{5}{384}\frac{q_0L^4}{E\,I_x} \quad \text{(positive, i.e., same as } y\text{)}$$

The rotations $\varphi_A = -\varphi_B$ are computed from the first derivative:

$$\varphi_A = -v'(0) = -\frac{1}{24}\frac{q_0L^3}{E\,I_x} \quad \text{(negative, i.e., clockwise)}$$