BEAM THEORY – KINEMATICS

STRUCTURAL MECHANICS

The ERAMCA Project

Environmental Risk Assessment and Mitigation on Cultural Heritage assets in Central Asia

V2O22317

This work is licensed under a Creative Commons "Attribution-ShareAlike 4.0 International" license.







Lecturer/students objectives

Introduction

Kinematics





LECTURER/STUDENTS OBJECTIVES





Present the main aspects of the kinematic behaviour of beams.
 Understand the relationships between displacements of the beam axis and deformations.





INTRODUCTION





The aim of this lecture is to provide the information necessary to understand the relations between displacements of the longitudinal axis and deformations, i.e., kinematical equations





It is a plane problem where the external loadings are applied in the vertical (y, z) plane, and the cross section (perpendicular to z) is symmetrical respect to y. This means that the deformed shape is lying in the (y, z) plane.

Small displacements and rotations are assumed.

- the effect of curvature is neglected
- the elongation for an elemental length is calculated neglecting the rotation



KINEMATICS





KINEMATICS





Simple deformation modes of a small element of length *h* are examined:

- longitudinal deformation or elongation
- shear deformation
- rigid rotation
- curvature

A variable deformation along the longitudinal axis is taken into account through a limit process: the beam is considered made of elements of length *h* approaching to zero



Deformation ε_0 is the ratio between its elongation and the original length

$$\varepsilon_{o} = rac{\Delta w}{h}$$

Considering the variability of *w* along *z*:

$$\varepsilon_{o}(z) = \lim_{h \to o} \frac{\Delta w(z)}{h} =$$
$$\lim_{h \to o} \frac{w(z+h) - w(z)}{h} =$$
$$w'(z) = \frac{dw(z)}{dz}$$







The displacement along y is given by the sum of

- shear deformation γ
- rigid rotation φ
- curvature χ





SHEAR DEFORMATION AND ROTATION

Shear deformation:

$$\Delta \mathbf{v}_{s} = \mathbf{h} \sin \mathbf{\gamma}$$

Rigid rotation:

 $\Delta \mathbf{v}_r = -h \sin \varphi$

(displacement Δv_r should be negative when φ is positive, i.e., counterclockwise)





CURVATURE

AINA

of the European

Curvature (cross sections remain plane and undistorted, so that ε_0 is linear):

$$\Delta \mathbf{v}_c = -\left[\mathbf{R} - \mathbf{R}\cos(\Delta \varphi)\right] =$$
$$= -\frac{1 - \cos(\chi h)}{\chi}$$

$$\chi = rac{1}{R} = rac{\Delta arphi}{h}, \quad R \Delta arphi = h$$

 $(\chi = 1/R$ represents the curvature)





CURVATURE AND DEFORMATION OF THE CROSS SECTION

$$\frac{\overline{JJ'}}{\overline{JK}} = \frac{\overline{HK}}{\overline{CH}} \quad \text{i.e.} \quad \frac{\varepsilon \, \mathrm{d}z}{y} = \frac{\mathrm{d}z}{R}$$
$$\varepsilon(y) = \frac{y}{R} = \chi y$$

Notice that ε_0 was used to indicate $\varepsilon(y = 0, z)$, i.e., the deformation of the longitudinal axis z. Due to curvature, ε changes in the cross section! The whole deformation is:



R

н

Cross

section

G

 $\mathbf{x} =$

ź

Adding the previous three contributions:

$$\Delta \mathbf{v} = \Delta \mathbf{v}_{s} + \Delta \mathbf{v}_{r} + \Delta \mathbf{v}_{c} = h \sin \gamma - h \sin \varphi - \frac{1 - \cos(\chi h)}{\chi}$$

and, using the limit:

$$\lim_{h \to 0} \frac{\Delta \mathbf{v}(\mathbf{z})}{h} = \lim_{h \to 0} \frac{h \sin \gamma(\mathbf{z})}{h} - \lim_{h \to 0} \frac{h \sin \varphi(\mathbf{z})}{h} - \lim_{h \to 0} \frac{1 - \cos(\chi(\mathbf{z})h)}{\chi(\mathbf{z})h}$$
$$= \sin \gamma(\mathbf{z}) - \sin \varphi(\mathbf{z}) - \mathbf{0} = \sin \gamma(\mathbf{z}) - \sin \varphi(\mathbf{z}) = \mathbf{v}'(\mathbf{z}) = \frac{\mathrm{d}\mathbf{v}(\mathbf{z})}{\mathrm{d}\mathbf{z}}$$

Hôpital's rule $\lim_{X \to 0} \frac{1 - \cos X}{X} = \lim_{X \to 0} \frac{0 + \sin X}{1} = 0$ Environmental Risk Assessment and Mitigation on Cultural Heritage Assets in Central Asia

For infinitesimal displacements and rotations, such that $\sin\gamma \approx \gamma$ and $\sin\varphi \approx \varphi$, is:

$$\mathbf{v}'(\mathbf{z}) = -\sin \mathbf{\varphi}(\mathbf{z}) + \sin \mathbf{\gamma}(\mathbf{z})$$

it is obtained:

$$\mathbf{v}'(z)=oldsymbol{\gamma}(z)-oldsymbol{arphi}(z)$$
 i.e., $oldsymbol{\gamma}(z)=\mathbf{v}'(z)+oldsymbol{arphi}(z)=rac{\mathsf{d}\mathbf{v}(z)}{\mathsf{d}z}+oldsymbol{arphi}(z)$

The curvature $\chi(z)$ is now:

$$\chi(z) = \lim_{h \to 0} \frac{\Delta \varphi(z)}{h} = \lim_{h \to 0} \frac{\varphi(z+h) - \varphi(z)}{h} = \varphi'(z) = \frac{d\varphi(z)}{dz}$$



Relationship between w, v, φ and ε_0 , γ , χ :

$$\begin{aligned} \varepsilon_{o}(z) &= \frac{\mathrm{d}w(z)}{\mathrm{d}z} \\ \gamma(z) &= \frac{\mathrm{d}v(z)}{\mathrm{d}z} + \varphi(z) \\ \chi(z) &= \frac{\mathrm{d}\varphi(z)}{\mathrm{d}z} \end{aligned}$$

Timošenko and Eulero-Bernoulli models (simplified)

 $\gamma(z)
eq$ 0: Timošenko model

$$m{\gamma}(z)=$$
 0: Eulero-Bernoulli model, i.e., $m{arphi}(z)=-rac{{
m d}{
m v}(z)}{{
m d}z}$





Homogeneity of dimensions

Base quantities: force (F), lengths (L) and mass (M)

Any physically meaningful equation will have the same dimensions on the left and right sides (dimensional homogeneity)





Quantity	Physical dimension	SI unit
$arepsilon, arepsilon_0, \gamma$	-	-
χ	L^{-1}	m^{-1}
V, W	L	m
arphi	-	rad



Signs

- v, w: positive if directed along the positive direction of (y, z), respectively
- ε , ε_0 : positive if directed along the positive direction of z
- φ : positive if counterclockwise
- χ : positive if concavity of the curve faces the left of z



- 1. With the hand flat, point the fingers in the direction of *X*.
- Curl all fingers but your index finger to point towards Y.
- 3. Raise the thumb. That's Z.



