BEAM THEORY – KINEMATICS

STRUCTURAL MECHANICS

The ERAMCA Project

Environmental Risk Assessment and Mitigation on Cultural Heritage assets in Central Asia

v2022317

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Lecturer/students objectives

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- Present the main aspects of the kinematic behaviour of beams.
- Understand the relationships between displacements of the beam axis and deformations.

Introduction

The aim of this lecture is to provide the information necessary to understand the relations between displacements of the longitudinal axis and deformations, i.e., kinematical equations

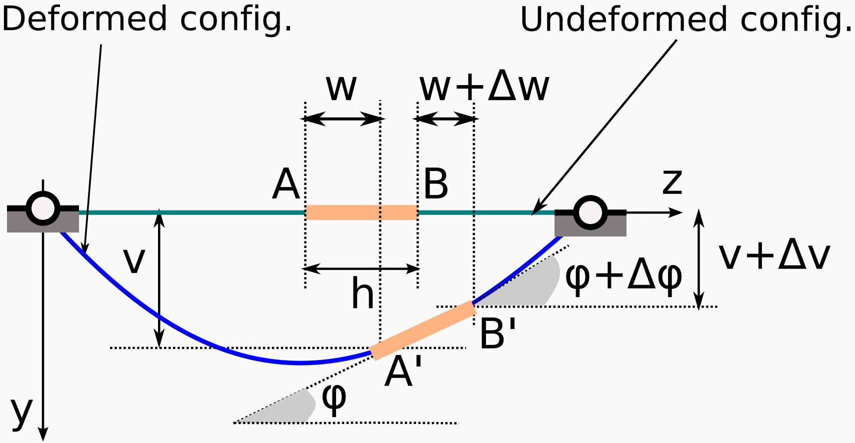
Some hypotheses

It is a plane problem where the external loadings are applied in the vertical (y, z) plane, and the cross section (perpendicular to z) is symmetrical respect to y. This means that the deformed shape is lying in the (y, z) plane.

Small displacements and rotations are assumed.

- the effect of curvature is neglected
- the elongation for an elemental length is calculated neglecting the rotation

Kinematics


Deformed config.
Undeformed config.
w
w+Δw
A
B
z
v
h
v+Δv
φ+Δφ
B'
A'
φ
y

Simple deformations

Simple deformation modes of a small element of length h are examined:

- longitudinal deformation or elongation
- shear deformation
- rigid rotation
- curvature

A variable deformation along the longitudinal axis is taken into account through a limit process: the beam is considered made of elements of length h approaching to zero

Deformation ε_0 is the ratio between its elongation and the original length

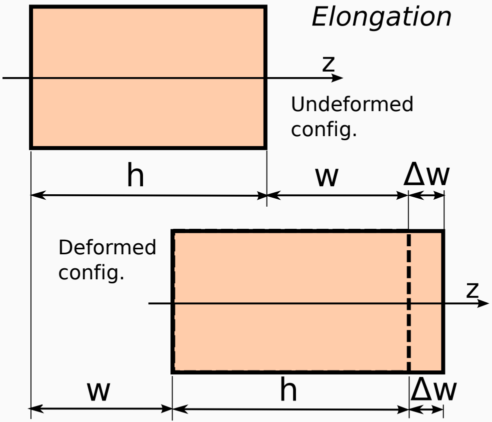
$$\varepsilon_0 = \frac{\Delta w}{h}$$

Considering the variability of w along z:

$$\varepsilon_0(z) = \lim_{h \to 0} \frac{\Delta w(z)}{h} =$$

$$\lim_{h \to 0} \frac{w(z+h) - w(z)}{h} =$$

$$w'(z) = \frac{\mathrm{d}w(z)}{\mathrm{d}z}$$



Transverse displacement

The displacement along y is given by the sum of

- shear deformation γ
- rigid rotation φ
- curvature χ

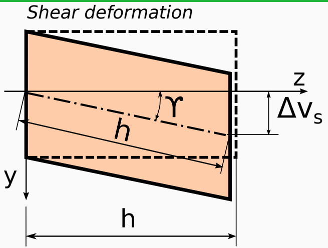
Shear deformation:

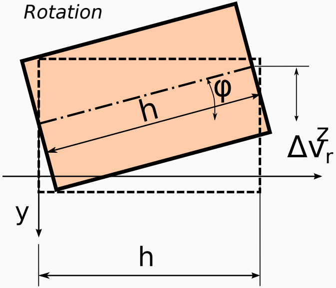
$$\Delta v_s = h \sin \gamma$$

Rigid rotation:

$$\Delta v_r = -h \sin \varphi$$

(displacement Δv_r should be negative when φ is positive, i.e., counterclockwise)



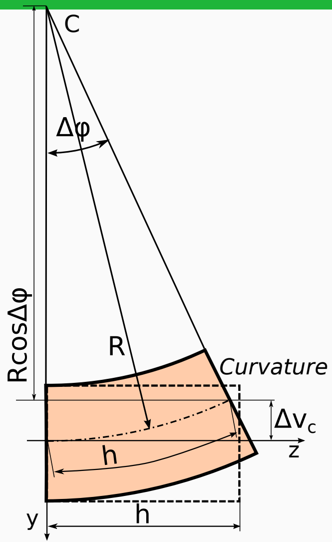


Curvature (cross sections remain plane and undistorted, so that ε_0 is linear):

$$\Delta v_c = -\left[R - R\cos(\Delta\varphi)\right] = $$
$$= -\frac{1-\cos(\chi\, h)}{\chi}$$

$$\chi = \frac{1}{R} = \frac{\Delta\varphi}{h}, \quad R\,\Delta\varphi = h$$

($\chi = {}^{1}/_{R}$ represents the curvature)



Curvature and deformation of the cross section

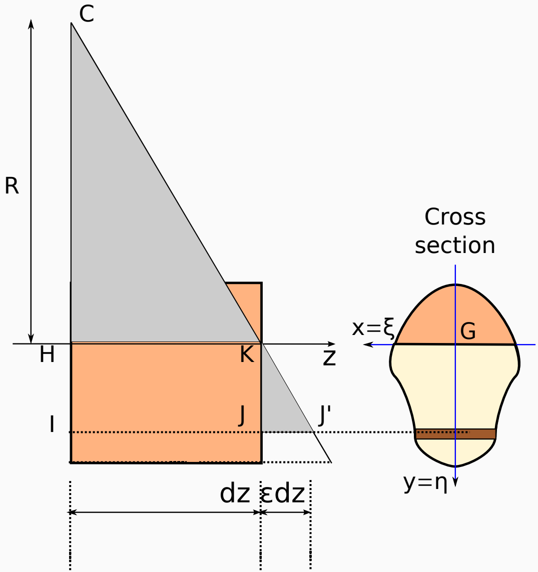
$$\frac{\overline{JJ'}}{\overline{JK}} = \frac{\overline{HK}}{\overline{CH}} \quad \text{i.e.} \quad \frac{\varepsilon\,\mathrm{d}z}{y} = \frac{\mathrm{d}z}{R}$$

$$\varepsilon(y) = \frac{y}{R} = \chi\, y$$

Notice that ε_0 was used to indicate $\varepsilon(y = 0, z)$, i.e., the deformation of the longitudinal axis z. Due to curvature, ε changes in the cross section!

The whole deformation is:

$$\varepsilon(y, z) = \underbrace{\varepsilon_0(z)}_{\text{Longit.}} + \underbrace{\chi y}_{\text{Curvat.}}$$



Adding the previous three contributions:

$$\Delta v = \Delta v_s + \Delta v_r + \Delta v_c = h\,\sin\gamma - h\,\sin\varphi - \frac{1-\cos(\chi\,h)}{\chi}$$

and, using the limit:

$$\lim_{h\to 0}\frac{\Delta v(z)}{h} = \lim_{h\to 0}\frac{h\,\sin\gamma(z)}{h} - \lim_{h\to 0}\frac{h\,\sin\varphi(z)}{h} - \lim_{h\to 0}\frac{1-\cos(\chi(z)\,h)}{\chi(z)\,h}$$

$$= \sin\gamma(z) - \sin\varphi(z) - 0 = \sin\gamma(z) - \sin\varphi(z) = v'(z) = \frac{\mathrm{d}v(z)}{\mathrm{d}z}$$

Hôpital's rule

$$\lim_{x\to 0}\frac{1-\cos x}{x} = \lim_{x\to 0}\frac{0+\sin x}{1} = 0$$

Infinitesimal displacements and rotations

For infinitesimal displacements and rotations, such that $\sin\gamma \approx \gamma$ and $\sin\varphi \approx \varphi$, is:

$$v'(z) = -\sin\varphi(z) + \sin\gamma(z)$$

it is obtained:

$$v'(z) = \gamma(z) - \varphi(z) \quad \text{i.e.,}$$

$$\gamma(z) = v'(z) + \varphi(z) = \frac{\mathrm{d}v(z)}{\mathrm{d}z} + \varphi(z)$$

The curvature $\chi(z)$ is now:

$$\chi(z) = \lim_{h\to 0} \frac{\Delta\varphi(z)}{h} = \lim_{h\to 0} \frac{\varphi(z+h) - \varphi(z)}{h} = \varphi'(z) = \frac{\mathrm{d}\varphi(z)}{\mathrm{d}z}$$

Relationship between w, v, φ and ε_0, γ, χ:

$$\begin{aligned}\varepsilon_0(z) &= \frac{\mathrm{d}w(z)}{\mathrm{d}z}\\ \gamma(z) &= \frac{\mathrm{d}v(z)}{\mathrm{d}z} + \varphi(z)\\ \chi(z) &= \frac{\mathrm{d}\varphi(z)}{\mathrm{d}z}\end{aligned}$$

Timošenko and Eulero-Bernoulli models (simplified)

$\gamma(z) \neq 0$: Timošenko model

$\gamma(z) = 0$: Eulero-Bernoulli model, i.e., $\varphi(z) = -\frac{\mathrm{d}v(z)}{\mathrm{d}z}$

Homogeneity of dimensions

Base quantities: force (F), lengths (L) and mass (M)

Any physically meaningful equation will have the same dimensions on the left and right sides (dimensional homogeneity)

Quantity	Physical dimension	SI unit
ε, ε_0, γ	–	–
χ	L^{-1}	m^{-1}
v, w	L	m
φ	–	rad

Sign conventions

Signs

- v, w: positive if directed along the positive direction of (y, z), respectively
- ε, ε_0: positive if directed along the positive direction of z
- φ: positive if counterclockwise
- χ: positive if concavity of the curve faces the left of z

Reference system (right handed system)

1. With the hand flat, point the fingers in the direction of *X*.
2. Curl all fingers but your index finger to point towards *Y*.
3. Raise the thumb. That's *Z*.