A SHORT MATH PRIMER

Structural Mechanics

The ERAMCA Project

Environmental Risk Assessment and Mitigation on Cultural Heritage assets in Central Asia

v2022317

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Lecturer/students objectives

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- Present some mathematical tools used in the course.
- Convince the classroom to revise them as soon as possible!

Introduction

The objective of the lecture is to revise some topics useful for the course.

- Sum/difference $\boldsymbol{C} = \boldsymbol{A} \pm \boldsymbol{B}$.
  Terms of $\boldsymbol{C}$ are given by:
  $$c_{ij} = a_{ij} \pm b_{ij}$$
  The sizes of $\boldsymbol{A}$ and $\boldsymbol{B}$ must be identical.
- Product $\boldsymbol{C} = \boldsymbol{AB}$.
  Terms of $\boldsymbol{C}$ are given by:
  $$c_{ij} = \sum_{k=1}^{m} a_{ik}\, b_{kj}$$
  It has a meaning only if the number of columns in $\boldsymbol{A}$ is equal to the number of rows in $\boldsymbol{B}$ (i.e., $\underset{(q\times p)}{\boldsymbol{C}} = \underset{(q\times m)}{\boldsymbol{A}}\underset{(m\times p)}{\boldsymbol{B}}$).
  Multiplication of matrices is not commutative as in ordinary algebra (i.e., $\boldsymbol{AB} \neq \boldsymbol{BA}$)!

- Transpose of $\boldsymbol{A}$.
  The transpose $\boldsymbol{A}^T$ is obtained by changing rows and columns of $\boldsymbol{A}$. If $\boldsymbol{A} = \boldsymbol{A}^T$ square matrix $\boldsymbol{A}$ is symmetrical. In case of a product, it is $(\boldsymbol{AB})^T = \boldsymbol{B}^T\boldsymbol{A}^T$.
- Determinant $a = \det\boldsymbol{A}$ of $\boldsymbol{A}$ (square matrix only):

$$a = \det \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix} = \sum_{ijk} \epsilon_{ijk} a_{1i} a_{2j} a_{3k}$$

  where:

$$\epsilon_{ijk} = \begin{cases} +1 & \text{if } ijk \text{ appears in the sequence 12312} \\ -1 & \text{in the sequence 32132} \\ 0 & \text{if in any other sequence} \end{cases}$$

  If $\det\boldsymbol{A} = 0$, matrix $\boldsymbol{A}$ is singular.

- Inverse $\boldsymbol{A}^{-1}$ of $\boldsymbol{A}$ (square matrix only).
  Matrix $\boldsymbol{A}^{-1}$ is such that:
  $$\boldsymbol{A}\boldsymbol{A}^{-1} = \boldsymbol{I}$$
  where $\boldsymbol{I}$ is an identity matrix having zero on all off-diagonal positions and unity on each of the diagonal positions.
  It is useful for the solutions of linear systems of equations $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$ where $\boldsymbol{x} = \boldsymbol{A}^{-1}\boldsymbol{b}$.
  If $\boldsymbol{A}^{-1} = \boldsymbol{A}^{T}$ matrix $\boldsymbol{A}$ is called orthogonal.
  The inversion procedure requires appropriate numerical procedures (for instance, the Cholesky method).

- Eigenvalues λ_i of a symmetric matrix of $\boldsymbol{A}$ $(n \times n)$:

$$(\boldsymbol{A} - \lambda_i \boldsymbol{I})\boldsymbol{\Phi}_i = \boldsymbol{0} \quad \text{where} \quad \det|\boldsymbol{A} - \lambda_i \boldsymbol{I}| = 0$$

  $\boldsymbol{\Phi}_i$ represent the eigenvector related to eigenvalue λ_i. There are n such eigenvalues λ_i to each of which corresponds an eigenvector $\boldsymbol{\Phi}_i$. Such vectors can be shown to be orthonormal ($\boldsymbol{\Phi}_i^T \boldsymbol{\Phi}_i = \delta_{ij}$, where $\delta_{ij} = 1$ if $i = j$, 0 if $i \neq j$). The physical meaning is to find a vector $\boldsymbol{A\Phi}$ parallel to the vector $\boldsymbol{\Phi}$.

In Structural Mechanics...

... matrices are used to describe the state of strain and stress; the eigenvalues to find the principal stresses an deformations

Vectors

A vector is a segment with an orientation, i.e., a segment with an arrow in one of its ends, described through:

- modulus: the length of the vector (representing, in a determined scale, the quantity)
- bearing: identified bundle of lines parallel to the one onto which the vector lies
- direction: identified by the arrow

A unit vector is a vector with modulus equal to 1.

Vectors are used to represent...

... positions of points, forces, velocity and the displacements in a three-dimensional space

Three mutually orthogonal unit vectors (for example, $\boldsymbol{i}$, $\boldsymbol{j}$ and $\boldsymbol{k}$), applied in point O, i.e., the origin of the 3D space, represent a base.

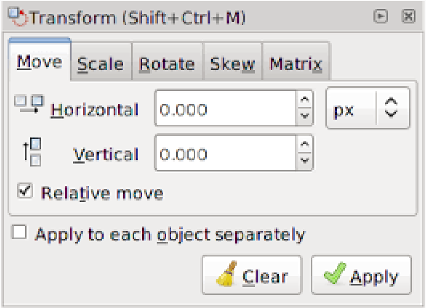
Any generic vector $\boldsymbol{u}$ can be expressed as $\boldsymbol{u} = u_x\boldsymbol{i} + u_y\boldsymbol{j} + u_z\boldsymbol{k}$, where u_x, u_y, u_z are the components of the vector.

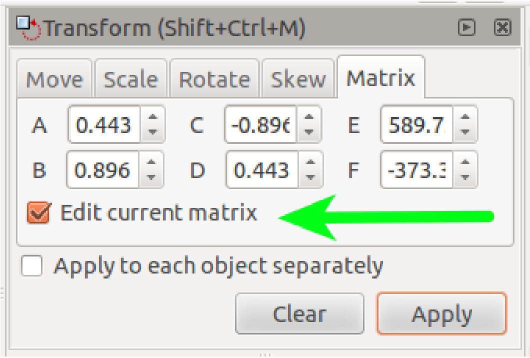
A vector applied in O, with the tip of the arrow in P...

... can have different mathematical representations:

$$\overrightarrow{OP} = (P - O) = \vec{u} = \{u\} = \underline{u} = \boldsymbol{u} = \{u_x, u_y, u_z\} == \begin{Bmatrix} u_x \\ u_y \\ u_z \end{Bmatrix}$$

Vectors: a modern way to use them?





- Sum/difference $\boldsymbol{c} = \boldsymbol{a} \pm \boldsymbol{b}$.

$$\boldsymbol{c} = (a_x \pm b_x)\boldsymbol{i} + (a_y \pm b_y)\boldsymbol{j} + (a_z \pm b_z)\boldsymbol{k}$$

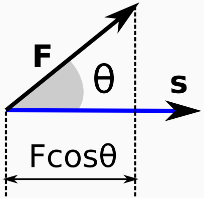
  The sizes of $\boldsymbol{a}$ and $\boldsymbol{b}$ must be identical.
  The sum can be done geometrically (parallelogram law).
- Length (or modulus) of $\boldsymbol{u}$.

$$u = ||\boldsymbol{u}|| = \sqrt{\sum_i u_i^2} = \sqrt{u_x^2 + u_y^2 + u_z^2}$$

- Dot product (symbol "·").
  Links two vectors, for example $\boldsymbol{F}$ and $\boldsymbol{s}$, to a scalar quantity a:

$$a = \boldsymbol{F} \cdot \boldsymbol{s} = ||\boldsymbol{F}||\ ||\boldsymbol{s}||\ \cos\vartheta = F\,s\,\cos\vartheta$$

  where $F = ||\boldsymbol{F}||$ represents the modulus of $\boldsymbol{F}$, $s = ||\boldsymbol{s}||$ the modulus of $\boldsymbol{s}$ and ϑ the angle between $\boldsymbol{F}$ and $\boldsymbol{s}$.



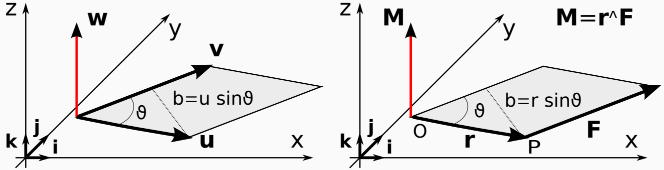
Dot product is used for calculating...

... the work done by a force

- Cross product (symbol "$\wedge$").
  Links two vectors $\boldsymbol{u}$, $\boldsymbol{v}$ to a third vector $\boldsymbol{w} = \boldsymbol{u} \wedge \boldsymbol{v}$, orthogonal to $\boldsymbol{u}$ and $\boldsymbol{v}$ (following right-hand side convention rule), of modulus:

$$||\boldsymbol{w}|| = ||\boldsymbol{u} \wedge \boldsymbol{v}|| = ||\boldsymbol{u}||\ ||\boldsymbol{v}|| \sin\vartheta = v\, u \sin\vartheta = v\, b$$

The angle between $\boldsymbol{u}$ and $\boldsymbol{v}$ is $\sin\vartheta = \frac{||\boldsymbol{w}||}{||\boldsymbol{u}||\ ||\boldsymbol{v}||}$



Notice that...

... $\boldsymbol{u} \wedge \boldsymbol{v} \neq \boldsymbol{v} \wedge \boldsymbol{u}$; it is $\boldsymbol{u} \wedge \boldsymbol{v} = -\boldsymbol{v} \wedge \boldsymbol{u}$

Cross product is used for calculating…

… the moment of a force with respect to pole O (***r*** links O to P), which modulus is :

$$||\boldsymbol{M}|| = M = ||\boldsymbol{r} \wedge \boldsymbol{F}|| = ||\boldsymbol{r}||\,||\boldsymbol{F}|| \sin\vartheta = r\,F \sin\vartheta = F\,b$$

where b is the lever arm, i.e., the distance orthogonal to ***F*** with respect to the pole O

The cross product can also be written as:

$$\boldsymbol{w} = \boldsymbol{u} \wedge \boldsymbol{v} = \det \begin{bmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ u_x & u_y & u_z \\ v_x & v_y & v_z \end{bmatrix}$$

i.e.,

$$\boldsymbol{w} = \boldsymbol{u} \wedge \boldsymbol{v} = \boldsymbol{i}(u_y v_z - u_z v_y) - \boldsymbol{j}(u_x v_z - u_z v_x) + \boldsymbol{k}(u_x v_y - u_y v_x)$$

Find the moment of $\boldsymbol{F} = \{2, 3, 0\}$, applied in P(3,2,0), with respect to the origin (i.e., $\boldsymbol{r} = \{3, 2, 0\}$).

Solution.

$$\boldsymbol{M} = \boldsymbol{r} \wedge \boldsymbol{F} = \det \begin{bmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 3 & 2 & 0 \\ 2 & 3 & 0 \end{bmatrix}$$

so that

$$\boldsymbol{M} = \boldsymbol{i}(2 \times 0 - 0 \times 3) - \boldsymbol{j}(3 \times 0 - 0 \times 2) + \boldsymbol{k}(3 \times 3 - 2 \times 2) = 5\boldsymbol{k}$$

It means that the moment of $\boldsymbol{F}$ is a vector directed to z axis of modulus equal to 5.

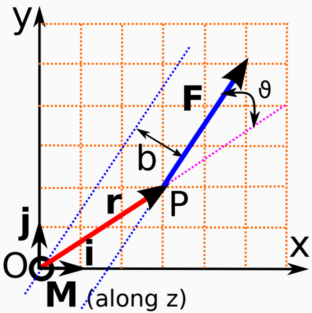
The angle between $\boldsymbol{r}$ and $\boldsymbol{F}$ is:

$$\sin\vartheta = \frac{||\boldsymbol{M}||}{||\boldsymbol{r}||\;||\boldsymbol{F}||} =$$

$$\frac{5}{\sqrt{13}\times\sqrt{13}} = \frac{5}{13} \quad \text{i.e.,} \quad \vartheta \approx 22.6°$$

The arm of $\boldsymbol{F}$ with respect to the origin of axes is:

$$b = \sqrt{13}\,\sin 22.6° = 1.39$$



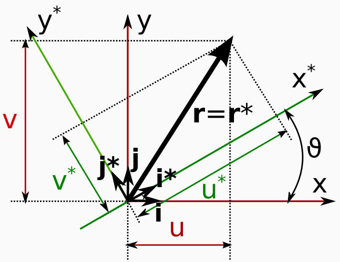
ROTATION OF THE REFERENCE SYSTEM

A rotation of the reference system turns the components $\{u, v\}$ of a vector denoted as $\boldsymbol{r}$ in the reference system (x, y) into $\{u^*, v^*\}$, i.e., the components of the same vector in the rotated reference system(x^*, y^*) (denoted as $\boldsymbol{r}^*$):

$$\boldsymbol{r}^* = \begin{bmatrix} u^* \\ v^* \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}$$

that is $\boldsymbol{r}^* = \boldsymbol{N}\boldsymbol{r}$, where $\boldsymbol{N}$ is an orthogonal matrix ($\boldsymbol{N}^T = \boldsymbol{N}^{-1}$) that does not change the length of the vector:

$$||\boldsymbol{r}^*|| = \sqrt{(u^*)^2 + (v^*)^2} = ||\boldsymbol{r}|| = \sqrt{u^2 + v^2}$$



Example: rotation of the reference system

Find the components of vector $\boldsymbol{a} = \{3, -1\}$ in a new frame rotated of $\frac{\pi}{6}$ counterclockwise.

Solution. Matrix $\boldsymbol{N}$ is given by:

$$\boldsymbol{N} = \begin{bmatrix} \cos\frac{\pi}{6} & \sin\frac{\pi}{6} \\ -\sin\frac{\pi}{6} & \cos\frac{\pi}{6} \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}$$

so that:

$$\boldsymbol{a}^* = \boldsymbol{N}\boldsymbol{a} = \begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix} \begin{bmatrix} 3 \\ -1 \end{bmatrix} = \begin{bmatrix} \frac{3\sqrt{3}-1}{2} \\ -\frac{3+\sqrt{3}}{2} \end{bmatrix} \approx \begin{bmatrix} +2.1 \\ -2.4 \end{bmatrix}$$

Differential equations

A differential equation is a mathematical equation that relates some function with its derivatives.

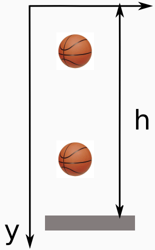
As example, the vertical motion of an object falling (uniform gravitational field without air resistance) is examined:

$$a(t) = \frac{\mathrm{d}v(t)}{\mathrm{d}t} = +g$$

$$\mathrm{d}v(t) = +g\,\mathrm{d}t$$

$$\int \mathrm{d}v(t) = \int g\,\mathrm{d}t + C_1 = g \int \mathrm{d}t + C_1$$

$$v(t) = g\,t + C_1$$



The integration is based on variable separation method.

It is also:

$$v(t) = \frac{\mathrm{d}y(t)}{\mathrm{d}t} = g\,t + C_1$$
$$\mathrm{d}y(t) = (g\,t + C_1)\,\mathrm{d}t$$
$$\int \mathrm{d}y(t) = \int (g\,t + C_1)\,\mathrm{d}t + C_2$$
$$y(t) = \frac{1}{2} g\,t^2 + C_1 t + C_2$$

where g is the acceleration due to gravity (9.81 $\mathrm{m/s^2}$ near the surface of the earth.

The boundary conditions are $v(0) = 0$ and $y(0) = 0$ so that $C_1 = C_2 = 0$. Hence:

$$v(t) = g\,t$$
$$y(t) = \frac{1}{2} g\,t^2$$

The velocity of an object after a fall from a height h is:

$$v(h) = \sqrt{2\,g\,h}$$

while the time t_h to travel the distance h:

$$t_h = \sqrt{\frac{2\,h}{g}}$$