BEAM THEORY – NORMAL STRESS

STRUCTURAL MECHANICS

The ERAMCA Project

Environmental Risk Assessment and Mitigation on Cultural Heritage assets in Central Asia

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LECTURER/STUDENTS OBJECTIVES

Lecturer/students objectives

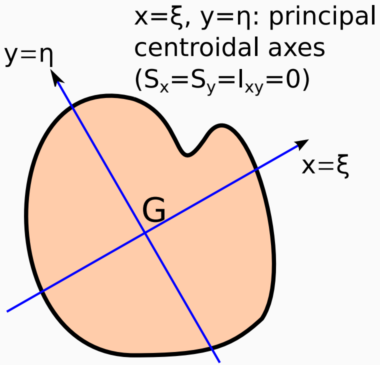
- Present the normal stress calculation for beams.
- Understand the hypotheses, distinguish the different loading condition and apply the proper solutions.

INTRODUCTION

Warning

Throughout the slides...

...the principal centroidal axes, labeled as (x, y) instead of (ξ, η), are used!



Equivalence between stresses and internal forces

$$N_z = \int_A \sigma_z \mathrm{d}A$$

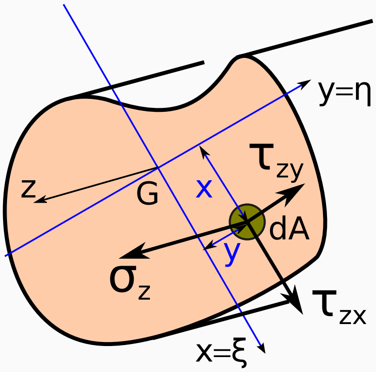
$$T_x = \int_A \tau_{zx} \mathrm{d}A$$

$$T_y = \int_A \tau_{zy} \mathrm{d}A$$

$$M_x = \int_A \sigma_z y \mathrm{d}A$$

$$M_y = -\int_A \sigma_z x \mathrm{d}A$$

$$M_z = \int_A (\tau_{zy} x - \tau_{zx} y) \mathrm{d}A$$



Axial loading

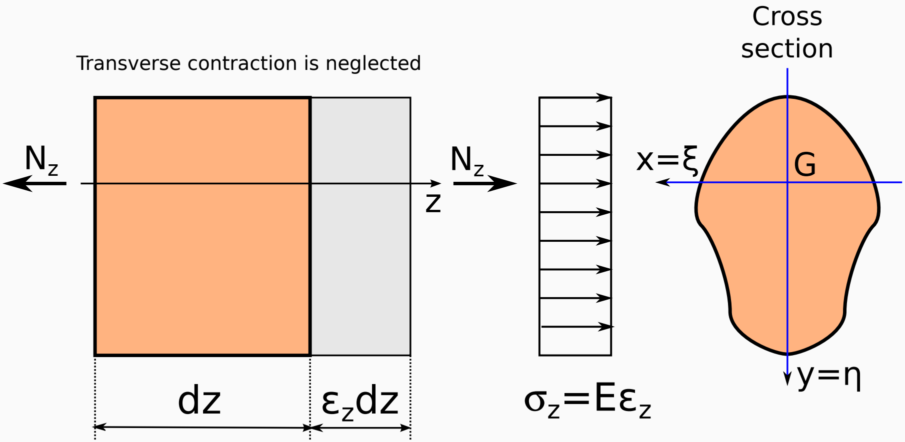
Axial loading – compression

Agrigento temple

AXIAL LOADING – TENSION

Alamillo bridge

Axial loading – model for stresses



Every cross section remains plane and undistorted: ε_z uniform $\Longrightarrow$ σ_z uniform ($\sigma_z = E\varepsilon_z$). It is:

$$\varepsilon_z = \varepsilon_0 = \frac{N_z}{EA}$$

so that:

$$\sigma_z = E\varepsilon_z = E\frac{N_z}{EA} = \frac{N_z}{A}$$

Axes are centroidal ($S_x = S_y = 0$) and principal ($I_{xy} = 0$):

$$M_x = \int_A \sigma_z y dA = \int_A \frac{N_z}{A} y dA = \frac{N_z}{A}\int_A y dA = cS_x = 0$$

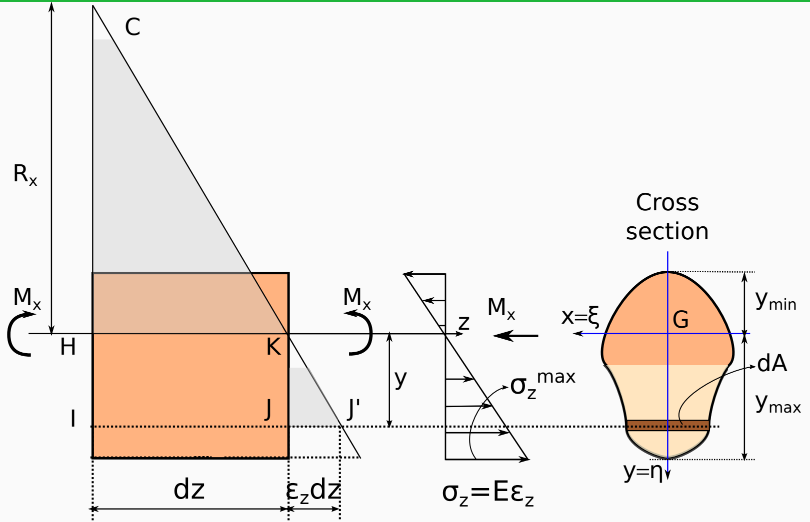
$$-M_y = \int_A \sigma_z x dA = \int_A \frac{N_z}{A} x dA = \frac{N_z}{A}\int_A x dA = cS_y = 0$$

Bending

BENDING AND SHEAR

Rago bridge

SYMMETRIC BENDING – MODEL FOR STRESSES



Cross sections remain plane and not distorted (ε_z linear):

$$\frac{\overline{JJ'}}{\overline{JK}} = \frac{\overline{HK}}{\overline{CH}} \quad \text{i.e.} \quad \frac{\varepsilon_z \, dz}{y} = \frac{dz}{R_x} \qquad \varepsilon_z = \frac{y}{R_x} = \chi_x \, y$$

The curvature is $\chi_x = \frac{1}{R_x} = \frac{M_x}{EI_x}$ so that:

$$\sigma_z = E\varepsilon_z = E\,\chi_x\, y = E\,\frac{M_x}{EI_x} y = \frac{M_x}{I_x} y \qquad \text{(Navier's formula)}$$

In addition...

$$N_z = \int_A \sigma_z dA = \int_A \frac{M_x}{I_x} y \, dA = \frac{M_x}{I_x} \int_A y dA = \frac{M_x}{I_x} S_x = 0$$

$$-M_y = \int_A \sigma_z x dA = \int_A \frac{M_x}{I_x} y \, x dA = \frac{M_x}{I_x} \int_A xy dA = \frac{M_x}{I_x} Ixy = 0$$

SYMMETRIC BENDING – NEUTRAL SURFACE

The stress is $\sigma_z = \frac{M_x}{I_x} y$. The neutral surface is the surface where stress σ_z is zero:

$$\text{Neutral surface: } \sigma_z = \frac{M_x}{I_x} y = 0 \implies y = 0$$

The line $y = 0$ represents the neutral axis for symmetric bending.

Symmetric bending – elastic section moduli

The maximum stress σ_z^{max} occurs at the top and the bottom:

$$\sigma_z^{max} = \frac{M_x}{I_x} y_{max} = \frac{M_x}{W_x}$$

where $W_x = \frac{I_x}{y_{max}}$ represents the elastic section modulus with respect to x and y_{max} the distance between the neutral surface and the point farthest from it. It is a geometric property of the cross section, as area, centroid an moments of area.

For instance, for a rectangular cross section:

$$W_x = \frac{I_x}{y_{max}} = \frac{\frac{bh^3}{12}}{\frac{h}{2}} = \frac{bh^2}{6}, \quad W_y = \frac{I_y}{x_{max}} = \frac{\frac{hb^3}{12}}{\frac{b}{2}} = \frac{hb^2}{6}$$

while, for an I-shaped beam IPE 200 (depth h_{IPE}=200 mm, flange b_{IPE}=100 mm):

$$W_x = \frac{I_x}{\frac{h_{IPE}}{2}} = \frac{1943.0\,\text{cm}^4}{\frac{20.0\,\text{cm}}{2}} = 194.3\,\text{cm}^3, \quad W_y = \frac{I_y}{\frac{b_{IPE}}{2}} = \frac{142.0\,\text{cm}^4}{\frac{10.0\,\text{cm}}{2}} = 28.5\,\text{cm}^3.$$

Symmetric bending – example

A 6 m-long, simply supported IPE 200 steel beam (I-shaped cross section), is to carry its own weight equal to $q_0 = 220\,\mathrm{N/m}$. Determine the magnitude of the maximum normal stress due to bending.

Answer The maximum bending moment, in the middle of the beam, is:

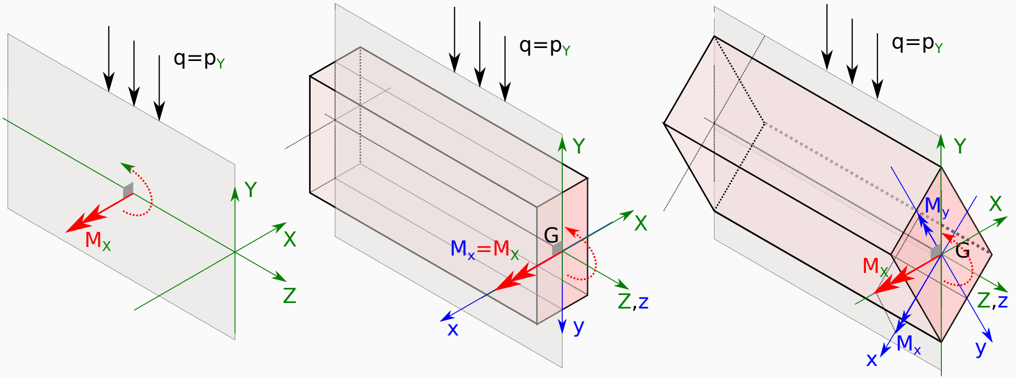
$$M_{max} = \frac{1}{8} q_0 L^2 = \frac{1}{8} \times (0.22\,\mathrm{N/mm}) \times (6000\,\mathrm{mm})^2 = 9.9 \times 10^5\,\mathrm{Nmm}$$

the maximum normal stress in the middle of the beam:

$$\sigma_z^{max} = \frac{M_x}{I_x} y_{max} = \frac{M_x}{W_x} = \frac{9.9 \times 10^5\,\mathrm{Nmm}}{1.943 \times 10^5\,\mathrm{mm}^3} = 5.1\,\mathrm{MPa}$$

If the allowable stress is $f_{yd} = 220\,\mathrm{MPa}$, the design is acceptable. The self-weight gets about 2% of the allowable stress.

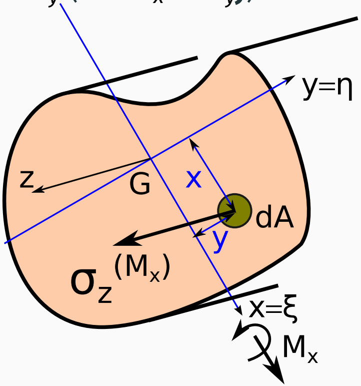
Unsymmetric bending – model

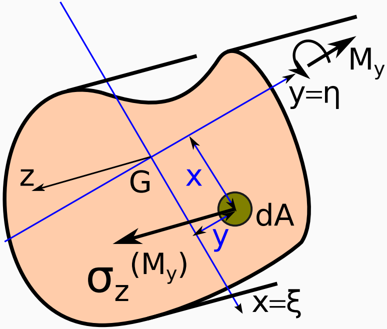


Load plane (vertical)	Load plane (vertical)	Load plane (vertical)
Y vertical	Y and y both vertical	Y vertical but y not aligned

Rotated rectangular section; take care to the reference frame (see the L-shaped section)

Unsymmetric bending: the moment **M** is not aligned with one of the principal centroidal axis. The stress σ_z is obtained by superposition, adding the stress due to M_x and M_y ($\mathbf{M} = M_x \boldsymbol{i} + M_y \boldsymbol{j}$).





Superposition: $\sigma_z = \sigma_z^{(M_x)} + \sigma_z^{(M_y)}$

$$\sigma_z^{(M_x)} = \frac{M_x}{I_x} y, \qquad \sigma_z^{(M_y)} = -\frac{M_y}{I_y} x$$

The negative sign in the second equation...

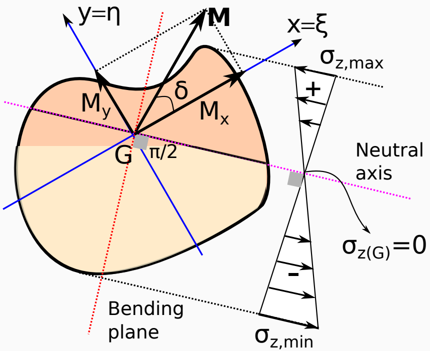
... if due to the fact that M_y positive causes tension in the part of the cross section where x is negative

$$\sigma_z = \frac{M_x}{I_x} y - \frac{M_y}{I_y} x \quad (M_x = M \cos\delta, \quad M_y = M \sin\delta)$$

Neutral axis ($\sigma_z = 0$) : $\quad \dfrac{M_x}{I_x} y - \dfrac{M_y}{I_y} x = 0$

Unsymmetric bending – stresses and neutral surface

- The neutral axis is represented by a line passing through the centroid G, the equation is $\frac{M_x}{I_x} y - \frac{M_y}{I_y} x = 0$
- The magnitude of stress is proportional to the distance from the neutral axis
- The most stressed points are those farthest from the neutral axis



An I-shaped beam IPE 200, whose flange forms an angle equal to 25° from the horizontal, is subjected to a couple of 10 kNm acting in a vertical plane. Find the stress inside the beam.

Answer The bending moments M_x and M_y are:

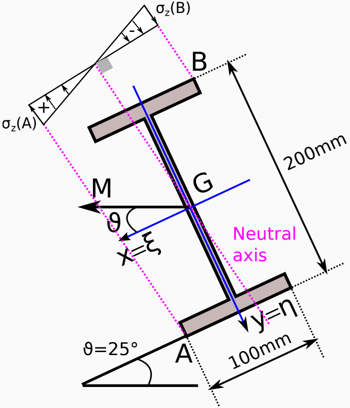
$$M_x = +M\cos 25° = +(10\,\text{kNm})(\cos 25°) = +9.06\,\text{kNm}$$

$$M_y = -M\sin 25° = -(10\,\text{kNm})(\sin 25°) = -4.23\,\text{kNm}$$

Stress σ_z is given by:

$$\sigma_z = \frac{+9.06\times10^6\,\text{Nmm}}{1.943\times10^7\,\text{mm}^4}y - \frac{-4.23\times10^6\,\text{Nmm}}{1.420\times10^6\,\text{mm}^4}x = (0.466\,\text{N/mm}^3)y + (2.98\,\text{N/mm}^3)x$$

The neutral axis equation is:



The stresses at points A and B (the farthest form neutral axis) are:

$$\sigma_z^{(A)} = \left(0.466\,\mathrm{N/mm^3}\right)\left(\frac{+200\,\mathrm{mm}}{2}\right) + \left(2.98\,\mathrm{N/mm^3}\right)\left(\frac{+100\,\mathrm{mm}}{2}\right) = +196\,\mathrm{MPa}$$

$$\sigma_z^{(B)} = \left(0.466\,\mathrm{N/mm^3}\right)\left(\frac{-200\,\mathrm{mm}}{2}\right) + \left(2.98\,\mathrm{N/mm^3}\right)\left(\frac{-100\,\mathrm{mm}}{2}\right) = -196\,\mathrm{MPa}$$

With an angle equal to 0° (symmetric bending):

$$\sigma_z^{(A)} = \frac{+10 \times 10^6\,\mathrm{Nmm}}{1.943 \times 10^7\,\mathrm{mm^4}}\left(\frac{+200\,\mathrm{mm}}{2}\right) = +52\,\mathrm{MPa}$$

$$\sigma_z^{(B)} = \frac{+10 \times 10^6\,\mathrm{Nmm}}{1.943 \times 10^7\,\mathrm{mm^4}}\left(\frac{-200\,\mathrm{mm}}{2}\right) = -52\,\mathrm{MPa}$$

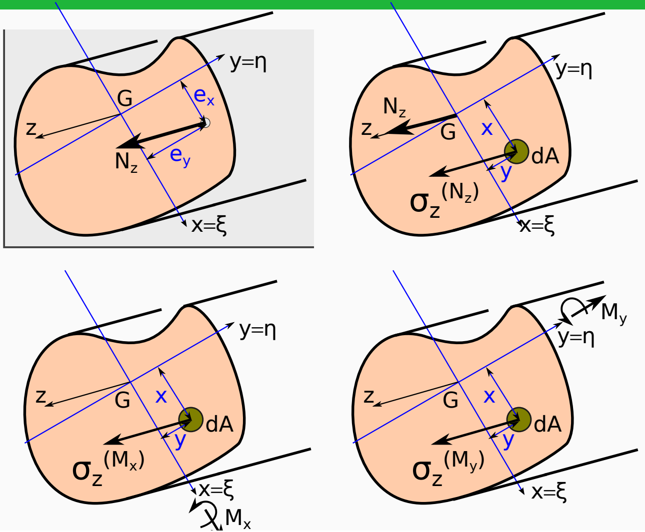
a stress $\approx 1/4$ with respect to the value for unsymmetric bending is found!

ECCENTRIC AXIAL LOADING

ECCENTRIC AXIAL LOADING

Pisa tower

ECCENTRIC AXIAL LOADING AND SHEAR


y=η
G
e_x
z
e_y
N_z
x=ξ
y=η
N_z
z
G
x
dA
y
σ_z^(N_z)
x=ξ
y=η
z
G
x
dA
y
σ_z^(M_x)
x=ξ
M_x
M_y
y=η
z
G
x
dA
y
σ_z^(M_y)
x=ξ

The eccentric axial loading N_z applied in a point of coordinates (e_x, e_y) is equivalent to a system given by the eccentric axial loading applied in G and two moments $M_x = N_z\, e_y$, $M_y = -N_z\, e_x$

Superposition: $\sigma_z = \sigma_z^{(N_z)} + \sigma_z^{(M_x)} + \sigma_z^{(M_y)}$

$$\sigma_z^{(N_z)} = \frac{N_z}{A}, \qquad \sigma_z^{(M_x)} = \frac{M_x}{I_x} y, \qquad \sigma_z^{(M_y)} = -\frac{M_y}{I_y} x$$

$$\sigma_z = \frac{N_z}{A} + \frac{M_x}{I_x} y - \frac{M_y}{I_y} x$$

The negative sign in the third equation...

... if due to the fact that M_y positive causes tension in the part of the cross section where x is negative

By grouping N_z/A it is obtained:

$$\sigma_z = \frac{N_z}{A}\left(1 + \frac{Ae_y}{I_x}y + \frac{Ae_x}{I_y}x\right) = \frac{N_z}{A}\left(1 + \frac{e_y}{\varrho_x^{\,2}}y + \frac{e_x}{\varrho_y^{\,2}}x\right)$$

where ϱ_x and ϱ_y represent the radius of gyration:

$$\varrho_x^{\,2} = \frac{I_x}{A}, \qquad \varrho_y^{\,2} = \frac{I_y}{A}$$

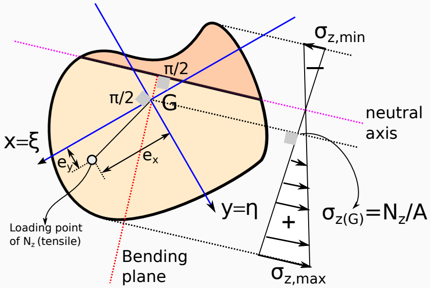
The neutral surface is obtained setting $\sigma_z = 0$:

$$1 + \frac{e_y}{\varrho_x^{\,2}}y + \frac{e_x}{\varrho_y^{\,2}}x = 0$$

The neutral axis (equation of straight line) is independent of the magnitude of N_z but it is dependent upon its position and upon the geometrical properties of the cross section

ECCENTRIC AXIAL LOADING – NEUTRAL SURFACE

- The neutral axis is represented by a straight line that does not coincide with the centroidal axis of the section; its equation is $1 + \frac{e_y}{\varrho_x^2} y + \frac{e_x}{\varrho_y^2} x = 0$
- The magnitude of stress is proportional to the distance from the neutral axis, the value corresponding to the centroid equal to $\frac{N_z}{A}$
- The most stressed points are those farthest from the neutral axis



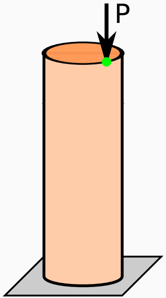
Eccentric axial loading – example

A bar with circular cross section (radius R) is subjected to a compressive normal forces. Find the maximum and the minimum stress.

Answer The load produces $M_x = N_z\, e_y = (-P)(+R) = -PR$ and $M_y = -N_z\, e_x = -(-P)(0) = 0$ (P absolute value):

$$\sigma_z = \frac{N_z}{A} + \frac{M_x}{I_x} y - \frac{M_y}{I_y} x = \frac{-P}{\pi R^2} + \frac{-PR}{\frac{\pi R^4}{4}} y =$$

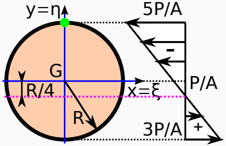
$$-\frac{P}{\pi R^2}\left(1 + \frac{4}{R} y\right)$$



The neutral axis is $y = -\frac{R}{4}$ while the stresses are:

$$\sigma_z(+R) = -\frac{P}{\pi R^2}\left[1 + \frac{4}{R}(+R)\right] = -5\frac{P}{\pi R^2} = -5\frac{P}{A}$$

$$\sigma_z(-R) = -\frac{P}{\pi R^2}\left[1 + \frac{4}{R}(-R)\right] = +3\frac{P}{\pi R^2} = +3\frac{P}{A}$$



Sign conventions

- The normal force N_z is positive if tension; the moments M_j acting on a positive face (positive "face" is perpendicular to the positive axis direction) if along the corresponding axis (j means the axis x or y).
- The normal stress σ_z is positive if it produces tension.