BEAM THEORY – GEOMETRICAL PROPERTIES OF THE CROSS SECTION

STRUCTURAL MECHANICS

The ERAMCA Project

Environmental Risk Assessment and Mitigation on Cultural Heritage assets in Central Asia

v2022317

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Lecturer/students objectives

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- Present the mathematical instrument necessary to evaluate the influence of the cross section on the stress and deformation of beams.
- Learn the procedures necessary to the calculation of the geometrical properties of plane areas (centroid, first and second moments, principal axes). Apply the procedure to plane areas composed by elementary figures.

Introduction

The geometry of plane areas is examined to derive some properties useful for the calculation of deflection and stresses in beams

In the theory of beams…

… the stiffnesses $E\,A$ (axial) and $EI = E\,I_x$ (flexural) are required: the term A represents the area of the cross-section and $I = I_x$ the moment of inertia with respect to a principal centroidal axis (named I_ξ in the following)

Centroid, first and second moments of area

First moments of area and centroid G (1/2)

The first moments of area S_x and S_y are calculated as:

$$A = \int_A \mathrm{d}A$$

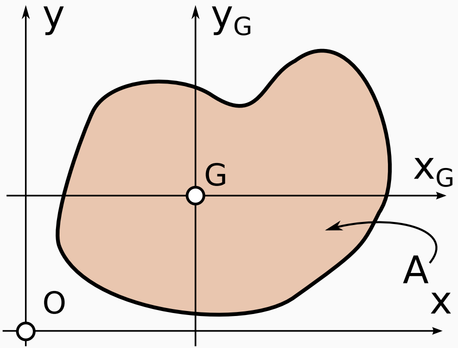
$$S_x = \int_A y\,\mathrm{d}A$$

$$S_y = \int_A x\,\mathrm{d}A$$

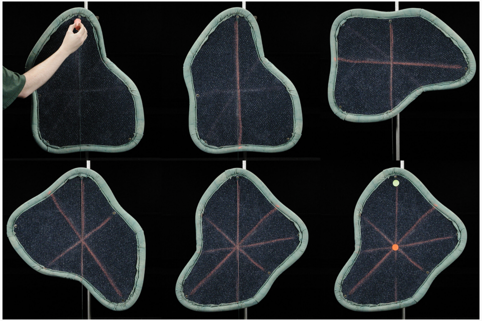
while the centroid G as:

$$x_G = \frac{S_y}{A} = \frac{\int_A x\mathrm{d}A}{A}$$

$$y_G = \frac{S_x}{A} = \frac{\int_A y\mathrm{d}A}{A}$$



First moments of the area and centroid G (2/2)



- The centroid is the point of an area to the respect of which both first moments of area are null.
- The first moments of the area are null for any pair of centroidal axes
- The first moments may be positive, negative or null; the area A is always positive

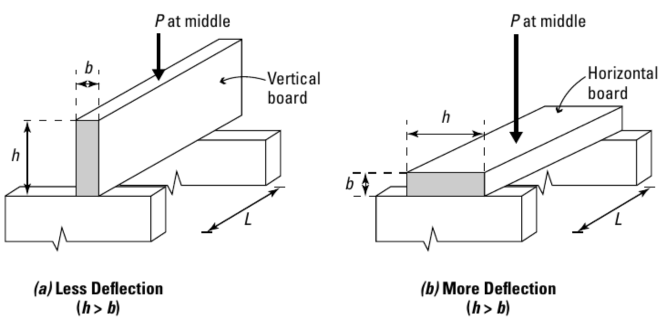
Second moments of the area

The second moments of the area or moments of inertia are

$$I_{xx} = \int_A y^2 \mathrm{d}A$$

$$I_{yy} = \int_A x^2 \mathrm{d}A$$

$$I_{xy} = \int_A xy \mathrm{d}A$$



- I_{xx} and I_{yy} are always greater than zero
- I_{xy} (mixed moment of inertia) can be positive, negative or null

Parallel-axis theorems

Parallel-axis theorems

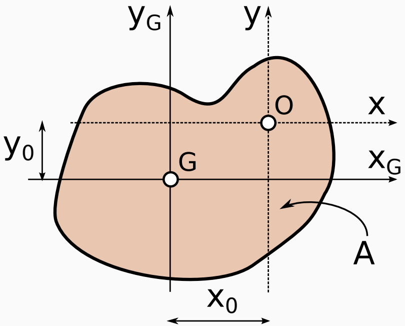
The moment of inertia of an area with respect to a given axis is given by:

$$I_{xx} = I_{x_G x_G} + A\, y_O{}^2$$

$$I_{yy} = I_{y_G y_G} + A\, x_O{}^2$$

$$I_{xy} = I_{x_G y_G} + A\, x_O y_O$$

(Formulae by Huygens)

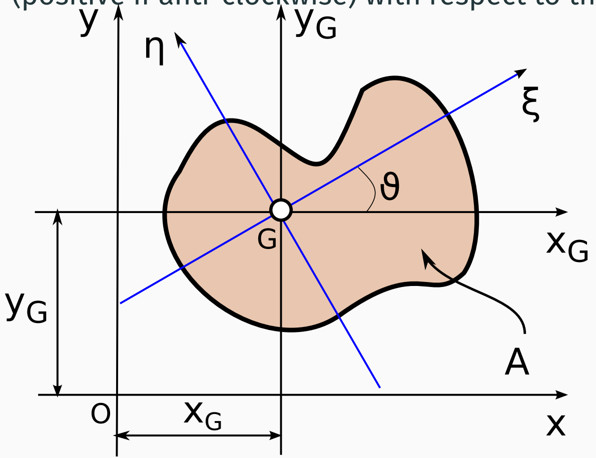


The moments of inertia $I_{x_G x_G}$ and $I_{y_G y_G}$...

...are the smallest within all the possible pairs of axes parallels to the original ones

Principal axes

It is possible to find a pair of axes into which $I_{\xi\eta} = 0$ (principal axes), that are rotated of about ϑ (positive if anti-clockwise) with respect to the centroidal axes



- Orientation of the principal axes (ξ, η)

$$\vartheta = \frac{1}{2} \arctan \frac{2\, I_{x_G y_G}}{I_{y_G y_G} - I_{x_G x_G}}$$

- Moments of inertia with respect to the principal axes (I_ξ,I_η):

$$I_\xi, I_\eta = \frac{1}{2}(I_{x_G x_G} + I_{y_G y_G}) \pm \frac{1}{2}\sqrt{(I_{y_G y_G} - I_{x_G x_G})^2 + 4\, {I_{x_G y_G}}^2}$$

choosing I_ξ, I_η in such a way that:

- $I_\xi > I_\eta$ if $I_{x_G x_G} > I_{y_G y_G}$
- $I_\xi < I_\eta$ if $I_{x_G x_G} < I_{y_G y_G}$

- If $I_{x_G x_G} = I_{y_G y_G}$ it is:

$$\vartheta = +\frac{\pi}{4} \quad \text{and} \quad I_\xi = I_{x_G x_G} - I_{x_G y_G}, \quad I_\eta = I_{x_G x_G} + I_{x_G y_G}$$

Note

- In the principal axes $I_{\xi\eta} = 0$
- I_ξ and I_η are always greater than zero
- Axes both centroidal and principal: central

Simple sections

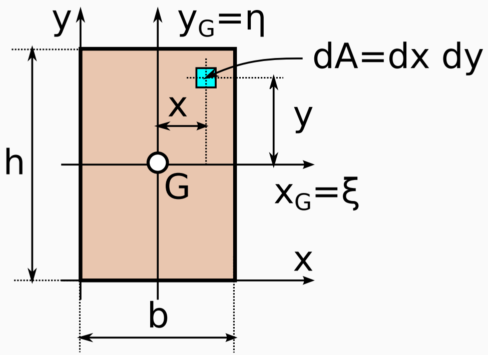
$$A = \int_A dA = \int_0^b \int_0^h dxdy = b\,h$$

$$S_x = \int_A y\,dA = \int_0^b \int_0^h y\,dxdy = +\frac{b\,h^2}{2}$$

$$S_y = \int_A x\,dA = \int_0^b \int_0^h x\,dxdy = +\frac{h\,b^2}{2}$$

$$x_G = \frac{S_y}{A} = \frac{+\frac{h\,b^2}{2}}{b\,h} = +\frac{b}{2}$$

$$y_G = \frac{S_x}{A} = \frac{+\frac{b\,h^2}{2}}{b\,h} = +\frac{h}{2}$$



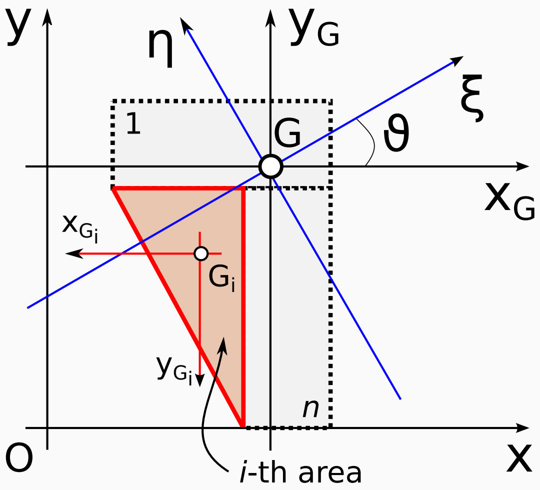
$$I_{xx} = \int_A y^2 dA = \int_0^b \int_0^h y^2 dxdy = \frac{b\,h^3}{3}$$
$$I_{yy} = \int_A x^2 dA = \int_0^b \int_0^h x^2 dxdy = \frac{h\,b^3}{3}$$
$$I_{xy} = \int_A xy dA = \int_0^b \int_0^h xy dxdy = +\frac{b^2\,h^2}{4}$$

$$I_{x_G x_G} = I_{xx} - A\,{y_G}^2 = \frac{b\,h^3}{3} - (bh)\left(+\frac{h}{2}\right)^2 = \frac{b\,h^3}{12}$$
$$I_{y_G y_G} = I_{yy} - A\,{x_G}^2 = \frac{h\,b^3}{3} - (bh)\left(+\frac{b}{2}\right)^2 = \frac{h\,b^3}{12}$$
$$I_{x_G y_G} = I_{xy} - A\,x_G y_G = +\frac{b^2\,h^2}{4} - (bh)\left(+\frac{b}{2}\right)\left(+\frac{h}{2}\right) = 0$$

The rectangular cross-section is doubly symmetrical

Hence, $x_G \equiv \xi$, $y_G \equiv \eta$, ϑ=0 and $I_{x_G y_G} \equiv 0$!

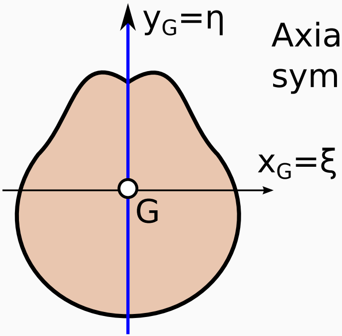
Composite areas

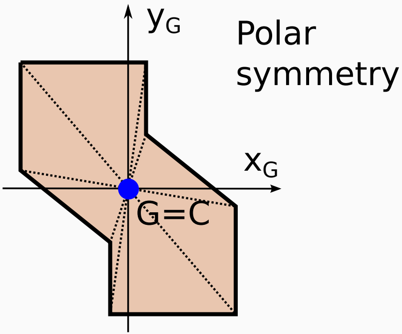

y
η
y_G
ξ
1
G
ϑ
x_G
x_{G_i}
G_i
y_{G_i}
n
O
i-th area
x

- Divide the area in n simple shapes (which geometrical properties are known, from tables)
- Compute the total area (summing the n elementary areas)
- Compute the total first moments (summing the first moments of the n elementary areas, keeping in mind the signs) with respect to an arbitrary set of axes (x, y)
- Determine the position of the centroid

- Compute the total moments of inertia (summing the moments of inertia of the *n* elementary areas) remembering that:
  - the parallel-axis theorem has to be used in order to refer all the moments of inertia to the global centroidal axes (x_G,y_G); in tables the m.o.i. are related to the local centroids of the elementary areas (x_{G_i},y_{G_i})
  - the mixed inertia moment of the *i*-th area depends on the direction of the axes. It is thus necessary to control the directions of the local axes x_{G_i},y_{G_i} with respect to the global ones x_G,y_G
- Compute the orientation of the principal axes (ϑ) and the moments of inertia related to the principal axes (I_ξ e I_η)

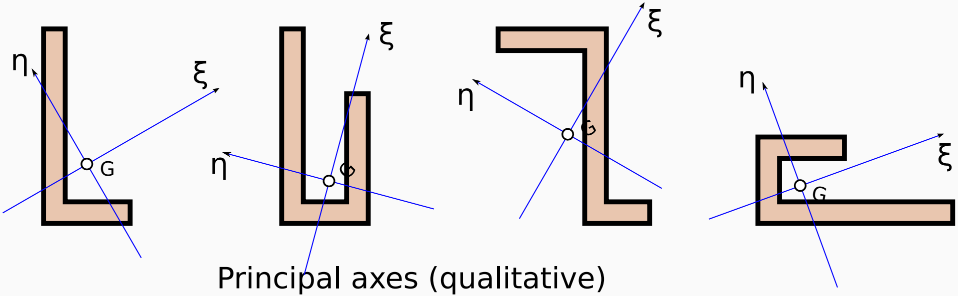
Particular symmetries





- Axial symmetry: the centroid is on the symmetry axis and one of the principal axes coincides with the symmetry axis. The other one is perpendicular
- Polar symmetry: the centroid G coincides with the center of the area C

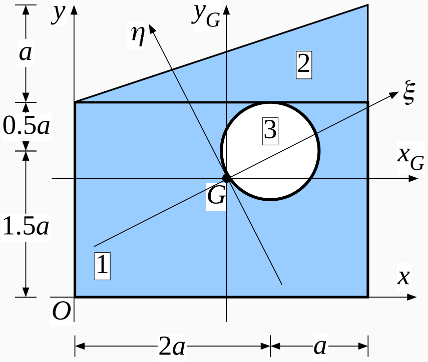
Be careful...



Principal axes (qualitative)

- Some simple shapes may result misleading...

Examples



No.	A^i	x_{G_i}	y_{G_i}	S_y^i	S_x^i	$x_G - x_{G_i}$	$y_G - y_{G_i}$	$(x_G - x_{G_i})\times$	$I_{x_Gx_G}$		$I_{y_Gy_G}$		$I_{x_Gy_G}$	
				$(A^ix_{G_i})$	$(A^iy_{G_i})$			$(y_G - y_{G_i})$	Local	Transfer	Local	Transfer	Local	Transfer
1	$6.000a^2$	$1.500a$	$1.000a$	$9.000a^3$	$6.000a^3$	$+0.053a$	$+0.239a$	$0.013a^2$	$2.000a^4$	$0.343a^4$	$4.500a^4$	$0.017a^4$	$0.000a^4$	$0.076a^4$
2	$1.500a^2$	$2.000a$	$2.333a$	$3.000a^3$	$3.500a^3$	$-0.447a$	$-1.094a$	$0.489a^2$	$0.083a^4$	$1.795a^4$	$0.750a^4$	$0.300a^4$	$+0.125a^4$	$0.734a^4$
3	$0.785a^2$	$2.000a$	$1.500a$	$1.570a^3$	$1.178a^3$	$-0.447a$	$-0.261a$	$0.117a^2$	$0.049a^4$	$0.053a^4$	$0.049a^4$	$0.157a^4$	$0.000a^4$	$0.092a^4$
Tot.	$6.715a^2$	–	–	$10.430a^3$	$8.323a^3$	–	–	–	$2.034a^4$	$2.085a^4$	$5.201a^4$	$0.160a^4$	$0.125a^4$	$0.718a^4$

Solved example

$$x_G = \frac{S_y}{A} = \frac{\sum_{i=1}^{n} S_y^i}{\sum_{i=1}^{n} A^i} = \frac{\sum_{i=1}^{n} A^i\, x_{G_i}}{\sum_{i=1}^{n} A^i} = \frac{10.430a^3}{6.715a^2} = +1.553a$$

$$y_G = \frac{S_x}{A} = \frac{\sum_{i=1}^{n} S_x^i}{\sum_{i=1}^{n} A^i} = \frac{\sum_{i=1}^{n} A^i\, y_{G_i}}{\sum_{i=1}^{n} A^i} = \frac{8.323a^3}{6.715a^2} = +1,239a$$

$$I_{x_G x_G} = 2.034a^4 + 2.085a^4 = 4.119a^4$$

$$I_{y_G y_G} = 5.201a^4 + 0.160a^4 = 5.361a^4$$

$$I_{x_G y_G} = 0.125a^4 + 0.718a^4 = +0.843a^4$$

$$\vartheta = \frac{1}{2}\arctan\frac{2\, I_{x_G y_G}}{I_{y_G y_G} - I_{x_G x_G}} = \frac{1}{2}\arctan\frac{2\times(+0.843a^4)}{5.361a^4 - 4.119a^4} = +26.8^\circ \quad \text{(counterclockwise)}$$

$$I_\xi, I_\eta = \frac{1}{2}(I_{x_G x_G} + I_{y_G y_G}) \pm \frac{1}{2}\sqrt{(I_{y_G y_G} - I_{x_G x_G})^2 + 4\, I_{x_G y_G}{}^2} = \ldots = 4.740a^4 \pm 1.047a^4$$

$$\text{so that } \begin{cases} I_\xi = 3.693a^4 \\ I_\eta = 5.787a^4 \end{cases}$$

Physical dimensions

Parameter	Physical dimension	SI unit
x_G, y_G	L	m
A	L^2	m^2
S_x, S_y	L^3	m^3
I_{xx}, I_{yy}, I_{xy}	L^4	m^4
$I_{x_G x_G}$, $I_{y_G y_G}$, $I_{x_G y_G}$, I_ξ, I_η	L^4	m^4