BEAM THEORY – VIRTUAL WORK THEOREM

STRUCTURAL MECHANICS

The ERAMCA Project

Environmental Risk Assessment and Mitigation on Cultural Heritage assets in Central Asia

v2022317

Table of Contents

Lecturer/students objectives

Lecturer/students objectives

- Present a formulation to link statics (equilibrium) and kinematics (displacements).
- Understand of the principle and use it for the calculation of displacements and rotations of beams under different loading and boundary conditions.

Introduction

The theorem of virtual work is one of the fundamental pillars of Structural Mechanics. The first ideas on the theorem date to Aristotle. At the end 18th Century, Lagrange published a complete theory on the topic.

In a writing of 1921 G. Colonnetti states:

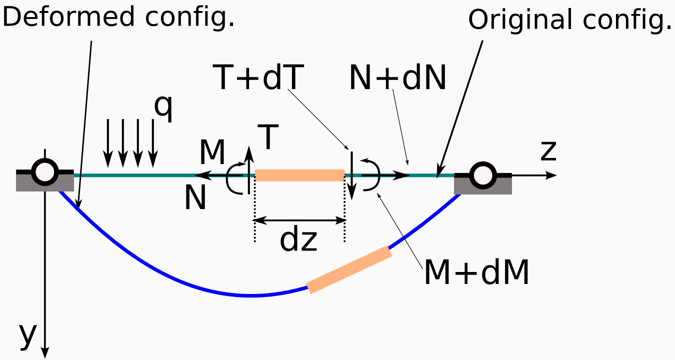
With the principle of virtual we are facing a general expression of equilibrium laws, thus, the most general form of the equilibrium laws.

For the application of the theorem of virtual work, two independent systems a and b are defined:

- a: equilibrated system done by the external forces (loads and support reactions) and internal forces (normal and shear forces, bending moment), i.e., a system that respects the static equation of equilibrium
- b: kinematically admissible system of displacements and strains, i.e., a system that respects the kinematic equations and the conditions of the supports

Equilibrated system a:

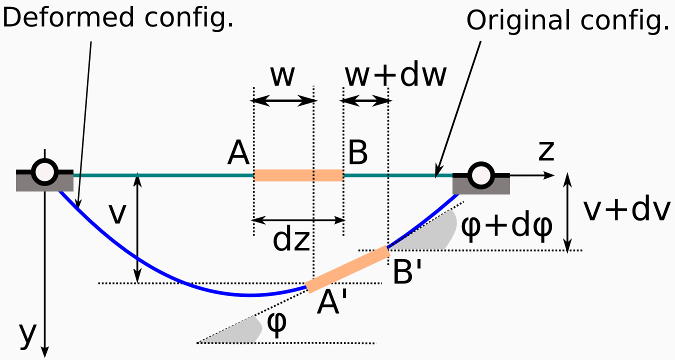
$$\frac{dN_a(z)}{dz} = -p_a(z)$$
$$\frac{dT_a(z)}{dz} = -q_a(z)$$
$$\frac{dM_a(z)}{dz} = T_a(z)$$



Kinematically admissible system b:

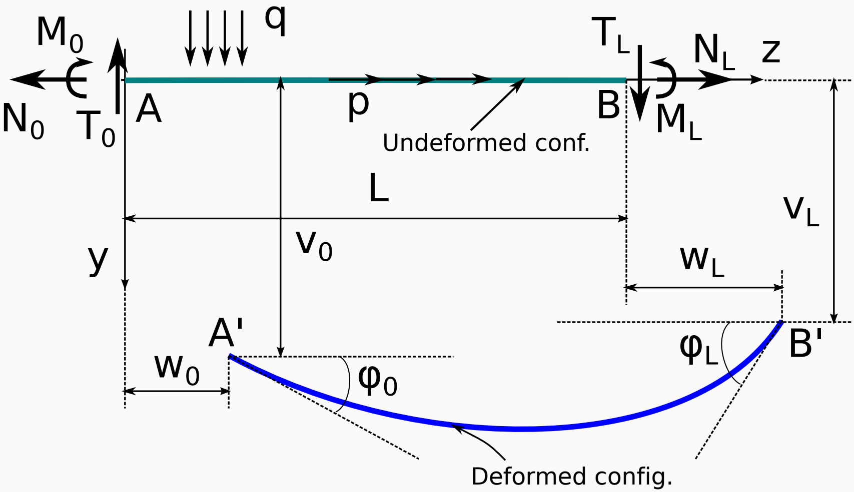
$$\varepsilon_b(z) = \frac{\mathrm{d}w_b(z)}{\mathrm{d}z}$$
$$\gamma_b(z) = \frac{\mathrm{d}v_b(z)}{\mathrm{d}z} + \varphi_b(z)$$
$$\chi_b(z) = \frac{\mathrm{d}\varphi_b(z)}{\mathrm{d}z}$$

where $\varepsilon_b(z) = \varepsilon_{o,b}(z)$



Virtual work

External work



The external virtual work is given by:

- the loads along the beam (for an element of length dz):
  - $\mathrm{d}\mathcal{L}_p = (p\mathrm{d}z)\, w = pw\mathrm{d}z$
  - $\mathrm{d}\mathcal{L}_q = (q\mathrm{d}z)\, v = qv\mathrm{d}z$
- the forces at the ends:
  - $z = 0$:
    - $\mathcal{L}_{N_0} = -N_0\, w_0$
    - $\mathcal{L}_{T_0} = -T_0\, v_0$
    - $\mathcal{L}_{M_0} = -M_0\, \varphi_0$
  - $z = L$:
    - $\mathcal{L}_{N_L} = N_L\, w_L$
    - $\mathcal{L}_{T_L} = T_L\, v_L$
    - $\mathcal{L}_{M_L} = M_L\, \varphi_L$

Theorem of virtual work (for deformable bodies)

The necessary and sufficient condition for the equilibrium of a supported body (no friction at the supports) is that the sum of the virtual works of the applied forces (external work) has to be equal to the sum of the works of the internal forces (internal work) for all the possibile systems of virtual displacements, i.e., small displacements that are kinematically admissible with respect to the supports and the structural integrity

- External virtual work:

$$\mathcal{L}_{ve} = \int_0^L (p_a\, w_b + q_a\, v_b)\, \mathrm{d}z +$$
$$+ (N_a\, w_b + T_a\, v_b + M_a\, \varphi_b)\,|_L + (-N_a\, w_b - T_a\, v_b - M_a\, \varphi_b)\,|_0 =$$
$$= \int_0^L (p_a\, w_b + q_a\, v_b)\, \mathrm{d}z + (N_a\, w_b + T_a\, v_b + M_a\, \varphi_b)\,|_0^L$$

- Internal virtual work:

$$\mathcal{L}_{vi} = \int_0^L (N_a\, \varepsilon_b + T_a\, \gamma_b + M_a\, \chi_b)\, \mathrm{d}z$$

- Equality of the external and internal virtual work:

$$\mathcal{L}_{vi} = \mathcal{L}_{ve}$$

The external work is:

$$\mathcal{L}_{ve} = \int_0^L (p_a\, w_b + q_a\, v_b)\, \mathrm{d}z + (N_a\, w_b + T_a\, v_b + M_a\, \varphi_b)\,|_0^L =$$

$$= \int_0^L \left[(p_a\, w_b + q_a\, v_b) + \frac{\mathrm{d}}{\mathrm{d}z}(N_a\, w_b + T_a\, v_b + M_a\, \varphi_b) \right] \mathrm{d}z$$

Differentiating and simplifying:

$$\mathcal{L}_{ve} = \int_0^L \left[\underbrace{\left(\frac{\mathrm{d}N_a}{\mathrm{d}z} + p_a \right)}_{0} w_b + \underbrace{\left(\frac{\mathrm{d}T_a}{\mathrm{d}z} + q_a \right)}_{0} v_b + \underbrace{\left(\frac{\mathrm{d}M_a}{\mathrm{d}z} \right)}_{T_a} \varphi_b \right] \mathrm{d}z +$$

$$+ \int_0^L \left[N_a \underbrace{\frac{\mathrm{d}w_b}{\mathrm{d}z}}_{\varepsilon_b} + T_a \frac{\mathrm{d}v_b}{\mathrm{d}z} + M_a \underbrace{\frac{\mathrm{d}\varphi_b}{\mathrm{d}z}}_{\chi_b} \right] \mathrm{d}z$$

Since the system a is in equilibrium and system b is kinematically admissible (compatible):

$$\mathcal{L}_{ve} = \int_0^L (N_a\,\varepsilon_b + T_a\,\gamma_b + M_a\,\chi_b)\,\mathrm{dz}$$

that is equal to the expression previously defined for the internal virtual work $\mathcal{L}_{vi}$.

Virtual work? (1/2)

- The theorem of virtual work is valid if the system a is in equilibrium and the system b is kinematically admissible. The two systems are independent.

- The system of displacements induced by the external loads a particular system of virtual displacements. An infinite quantity of alternative systems can be found.

- There are no hypotheses on the behavior of the material. The theorem is valid for whichever constitutive law (linear elastic, plastic, viscous, elasto-plastic...).
- From the theorem of virtual work the following applications are possible:
  - determine the displacements in elastic structures
  - determine the support reactions in statically determinate structures
  - determine the support reactions in statically indeterminate structures

Rigid bodies

If the body is rigid ($\varepsilon_b \equiv 0$, $\gamma_b \equiv 0$ and $\chi_b \equiv 0$ in any point) follows that $\mathcal{L}_{vi} = 0$ and, thus, $\mathcal{L}_{ve} = 0$:

$$\mathcal{L}_{ve} = \int_0^L (p_a\, w_b + q_a\, v_b)\, \mathrm{d}z + (N_a\, w_b + T_a\, v_b + M_a\, \varphi_b)\,|_0^L = 0$$

Thus on a rigid body…

…the virtual work of all the forces acting on it is zero!

Theorem of virtual work (for rigid bodies)

The necessary and sufficient condition for the equilibrium of a supported body (no friction at the supports) is that the sum of the virtual works of the applied forces has to be null for all the possible systems of virtual displacements, i.e., small displacements that are kinematically admissible with respect to the supports and the structural integrity