BEAM THEORY – TANGENTIAL STRESS DUE TO TORSION

STRUCTURAL MECHANICS

The ERAMCA Project

Environmental Risk Assessment and Mitigation on Cultural Heritage assets in Central Asia

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Lecturer/students objectives

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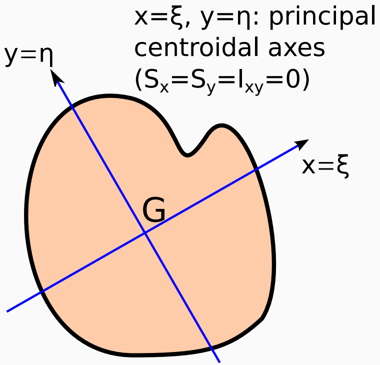
- Present the tangential stress (torsion) calculation for beams.
- Understand the hypotheses, distinguish the different loading condition and apply the proper solutions.

Introduction

Throughout the slides...

...the principal centroidal axes, labeled as (x, y) instead of (ξ, η), are used!



Equivalence between stresses and internal forces

$$N_z = \int_A \sigma_z dA$$

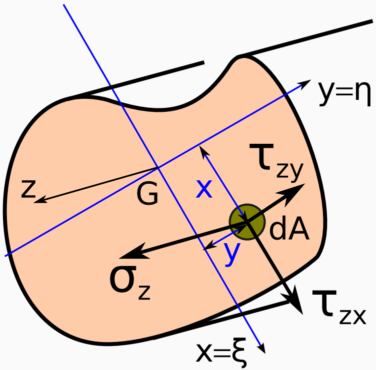
$$T_x = \int_A \tau_{zx} dA$$

$$T_y = \int_A \tau_{zy} dA$$

$$M_x = \int_A \sigma_z y dA$$

$$M_y = -\int_A \sigma_z x dA$$

$$M_z = \int_A (\tau_{zy} x - \tau_{zx} y) dA$$



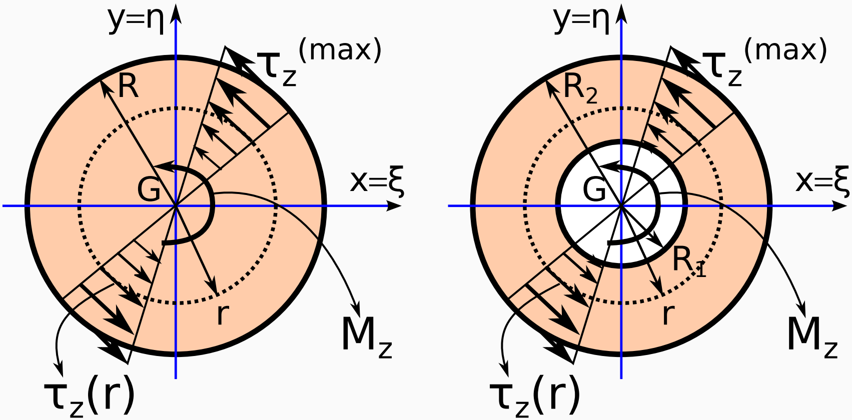
TWISTING

Twisting couple

Bending, shear and twisting couple

TWISTING COUPLE AND BENDING: DRIVE SHAFT

Twisting couple for a circular cross section – model



Circular cross section – stress calculation

Shear stresses are assumed to be:

- perpendicular to r
- proportional to r: $\tau_z = k\,r$

$$M_z = \int_A \tau_z r \mathrm{d}A = \int_A k r^2 \mathrm{d}A = k \int_A r^2 \mathrm{d}A$$

It is defined $I_p = \int_A r^2 \mathrm{d}A$ as polar moment of inertia, hence:

$$M_z = k I_p \implies k = \frac{M_z}{I_p}$$

Circular cross section – stress calculation (2/2)

Hence:

$$\tau_z = \frac{M_z}{I_p} r$$

$$\tau_z^{(max)} = \frac{M_z}{I_p} R$$

$$I_p = \int_A r^2 \mathrm{d}A = \int_A (x^2 + y^2)\, \mathrm{d}A = I_y + I_x = \frac{\pi}{4} R^4 + \frac{\pi}{4} R^4 = \frac{\pi}{2} R^4$$

For hollow circular cross section:

$$I_p = \frac{\pi}{2} \left(R_2^4 - R_1^4 \right)$$

$$\tau_z^{(max)} = \frac{M_z}{I_p} R_2$$

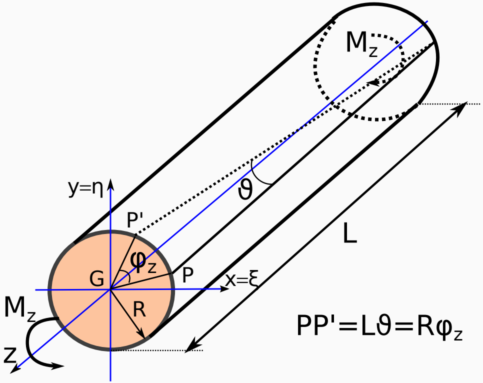
CIRCULAR CROSS SECTION – ANGLE OF TWIST

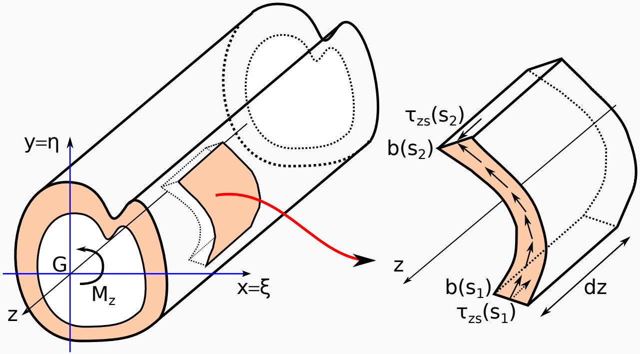
It is, for circular cross section and hollow circular cross section:

$$\vartheta = \frac{M_z}{G\, I_p}$$

where ϑ represents the angle of twist, i.e.:

$$\vartheta = \frac{d\varphi_z}{dz}$$





It is possible to proof that shear flow $\tau_{zs}(s)b(s)$ is constant:

Equilibrium along z: $\tau_{zs}(s_1)\, b(s_1)\mathrm{d}z = \tau_{zs}(s_2)\, b(s_2)\mathrm{d}z$ from which:

$$\tau_{zs}(s)b(s) = \text{constant}$$

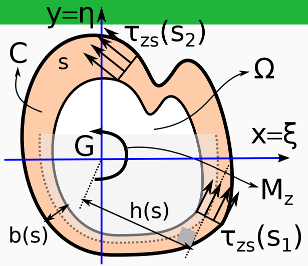
Equivalence between M_z and shear stresses:

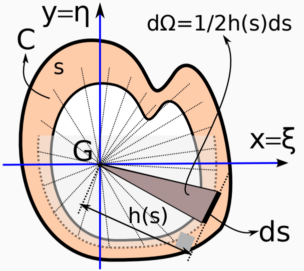
$$M_z = \oint_{\mathcal{C}} \tau_{zs}(s)\, b(s) h(s) ds = \tau_{zs}(s)\, b(s) \oint_{\mathcal{C}} h(s) ds$$

where $\Omega = \frac{1}{2} \oint_{\mathcal{C}} h(s) ds$ is the area bounded by the center line $\mathcal{C}$ of the wall cross section (described by s), hence:

$$M_z = \tau_{zs}(s)\, b(s)\, 2\Omega \quad \text{and:}$$

$$\tau_{zs}(s) = \frac{M_z}{2\,\Omega\, b(s)} \quad \text{Bredt's formula}$$





It can be proven that the angle of twist is:

$$\vartheta = \frac{M_z}{G\, I_t}$$

where:

$$I_t = \frac{4\,\Omega^2}{\oint_C \frac{ds}{b(s)}}$$

Integrals $\oint_C \ldots ds$ are…

… a sum calculated on the closed center line (defined by the curvilinear abscissa s).

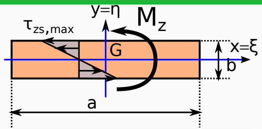
From advanced textbooks, it is possible to obtain the solution for complex cross sections, as for the rectangular cross section.

Shear stresses are parallel to the longer side of the cross section, their distribution is linear with zero on the x axis and the maximum magnitude on the edge $\left(y = \pm\frac{b}{2}\right)$.

For different values of $\frac{a}{b}$:

$$\tau_{zs,max} = \frac{M_z}{c_1\, a\, b^2}$$

$$\vartheta = \frac{M_z}{G(c_2\, a\, b^3)}$$



$\frac{a}{b}$	c_1	c_2
1	0.208	0.141
1.2	0.219	0.166
1.5	0.231	0.196
2	0.246	0.229
2.5	0.248	0.249
3	0.267	0.263
4	0.282	0.281
5	0.291	0.291
10	0.312	0.312
∞	0.333	0.333

Twisting couple – thin-walled open members

For cross section composed by N rectangles with sides equal to a_i and b_i with $\frac{a_i}{b_i} \rightarrow \infty$ (i.e., $c_1 = c_2 = \frac{1}{3}$) it is:

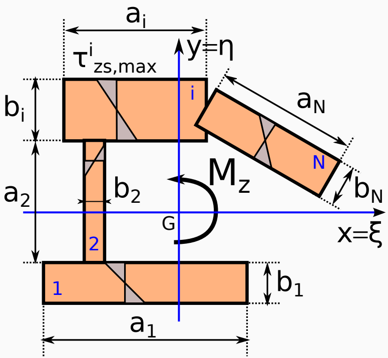
$$\tau^i_{zs,max} = \frac{M_z}{I_t} b_i$$

where:

$$I^i_t = \frac{1}{3} a_i b^3_i \quad \text{and} \quad I_t = \sum_{i=1}^{N} I^i_t$$

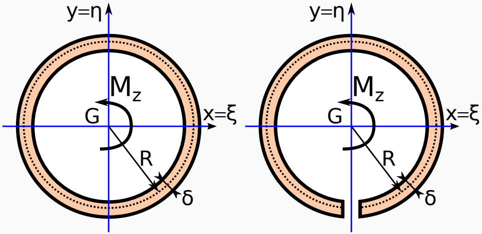
The angle of twist is given by:

$$\vartheta = \frac{M_z}{G\, I_t}$$



Thin-walled hollow members:

$$\tau_{zs} = \frac{M_z}{2\,\Omega\,\delta} = \frac{M_z}{2\,(\pi R^2)\,\delta}$$



Thin-walled open members:

$$\tau_{zs,max} = \frac{M_z}{I_t}\,\delta = \frac{M_z}{\frac{1}{3}\,(2\pi R)\,\delta^3}\,\delta = \frac{3M_z}{2\pi R\delta^2}$$

It is noticed that the maximum stress...

... is attained in the thin-walled open cross section. The thin-walled hollow members is most efficient design against torque

Sign conventions

- A positive sign is assigned to stresses with the same sense of rotation of M_z