

BEAM THEORY – TANGENTIAL STRESS DUE TO TORSION

STRUCTURAL MECHANICS

The ERAMCA Project

Environmental Risk Assessment and Mitigation on Cultural Heritage assets in Central Asia

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LECTURER/STUDENTS OBJECTIVES

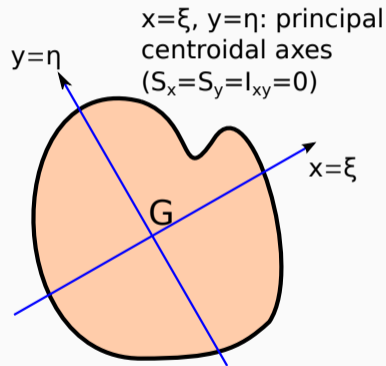
-  Present the tangential stress (torsion) calculation for beams.
-  Understand the hypotheses, distinguish the different loading condition and apply the proper solutions.

INTRODUCTION

WARNING

Throughout the slides...

...the **principal centroidal axes**, labeled as (x, y) instead of (ξ, η) , are used!



EQUIVALENCE BETWEEN STRESSES AND INTERNAL FORCES

$$N_z = \int_A \sigma_z dA$$

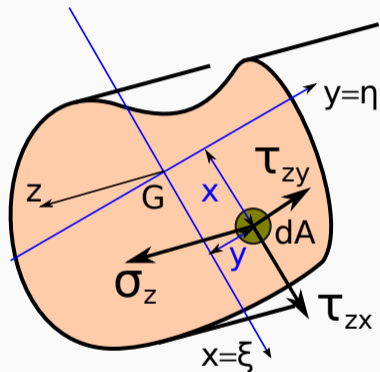
$$T_x = \int_A \tau_{zx} dA$$

$$T_y = \int_A \tau_{zy} dA$$

$$M_x = \int_A \sigma_z y dA$$

$$M_y = - \int_A \sigma_z x dA$$

$$M_z = \int_A (\tau_{zy} x - \tau_{zx} y) dA$$



TWISTING

TWISTING COUPLE



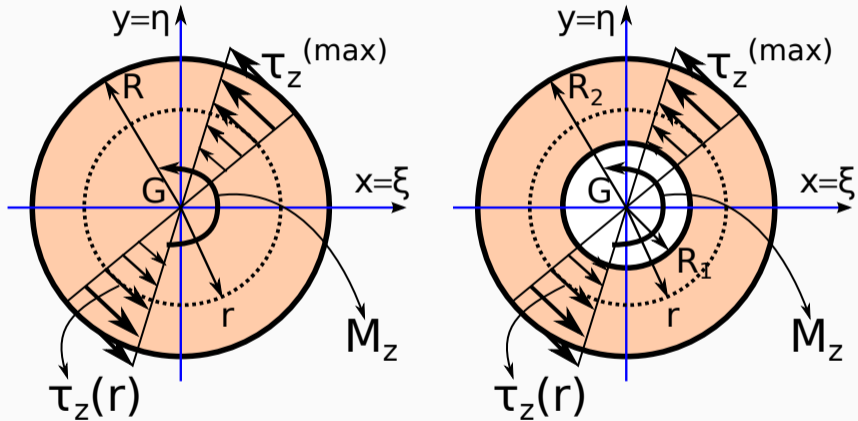
BENDING, SHEAR AND TWISTING COUPLE



TWISTING COUPLE AND BENDING: DRIVE SHAFT



TWISTING COUPLE FOR A CIRCULAR CROSS SECTION – MODEL



Shear stresses are assumed to be:

- perpendicular to r
- proportional to r : $\tau_z = k r$

$$M_z = \int_A \tau_z r dA = \int_A k r^2 dA = k \int_A r^2 dA$$

It is defined $I_p = \int_A r^2 dA$ as **polar moment of inertia**, hence:

$$M_z = k I_p \implies k = \frac{M_z}{I_p}$$

Hence:

$$\tau_z = \frac{M_z}{I_p} r$$

$$\tau_z^{(max)} = \frac{M_z}{I_p} R$$

$$I_p = \int_A r^2 dA = \int_A (x^2 + y^2) dA = I_y + I_x = \frac{\pi}{4} R^4 + \frac{\pi}{4} R^4 = \frac{\pi}{2} R^4$$

For hollow circular cross section:

$$I_p = \frac{\pi}{2} (R_2^4 - R_1^4)$$

$$\tau_z^{(max)} = \frac{M_z}{I_p} R_2$$

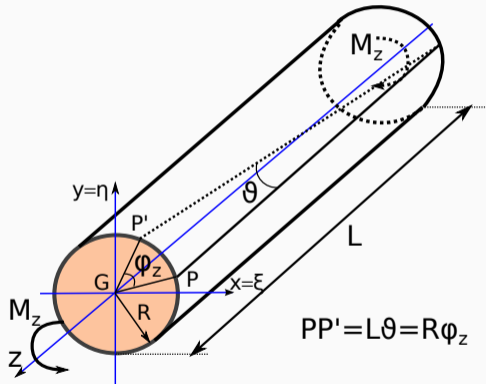
CIRCULAR CROSS SECTION – ANGLE OF TWIST

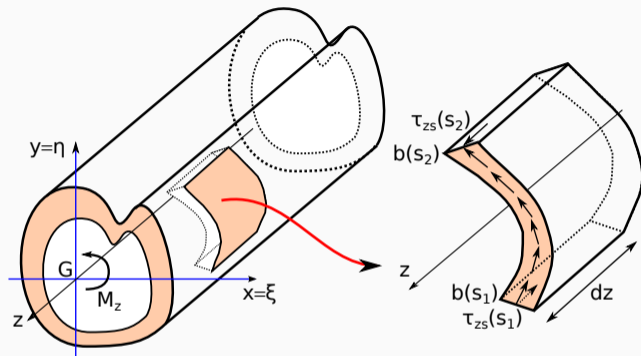
It is, for circular cross section and hollow circular cross section:

$$\vartheta = \frac{M_z}{G I_p}$$

where ϑ represents the **angle of twist**, i.e.:

$$\vartheta = \frac{d\varphi_z}{dz}$$





It is possible to prove that **shear flow** $\tau_{zs}(s)b(s)$ is constant:

Equilibrium along z : $\tau_{zs}(s_1) b(s_1)dz = \tau_{zs}(s_2) b(s_2)dz$ from which:

$$\tau_{zs}(s)b(s) = \text{constant}$$

Equivalence between M_z and shear stresses:

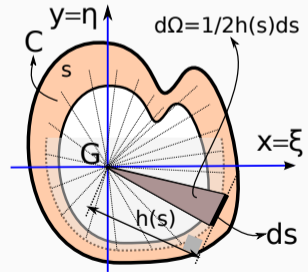
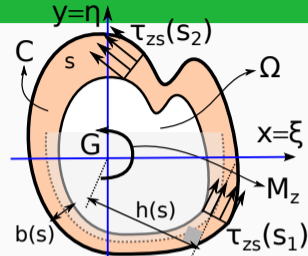
$$M_z = \oint_C \tau_{zs}(s) b(s) h(s) ds =$$

$$\tau_{zs}(s) b(s) \oint_C h(s) ds$$

where $\Omega = \frac{1}{2} \oint_C h(s) ds$ is the area bounded by the center line C of the wall cross section (described by s), hence:

$$M_z = \tau_{zs}(s) b(s) 2\Omega \quad \text{and:}$$

$$\tau_{zs}(s) = \frac{M_z}{2\Omega b(s)} \quad \text{Bredt's formula}$$



It can be proven that the angle of twist is:

$$\vartheta = \frac{M_z}{G I_t}$$

where:

$$I_t = \frac{4 \Omega^2}{\oint_C \frac{ds}{b(s)}}$$

Integrals $\oint_C \dots ds$ are...

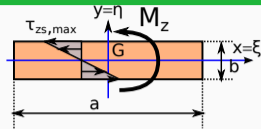
... a sum calculated on the closed center line (defined by the curvilinear abscissa s).

TWISTING COUPLE – RECTANGULAR CROSS SECTION

From advanced textbooks, it is possible to obtain the solution for complex cross sections, as for the **rectangular cross section**.

Shear stresses are **parallel** to the longer side of the cross section, their distribution is **linear** with zero on the x axis and the maximum magnitude on the edge ($y = \pm \frac{b}{2}$).
For different values of $\frac{a}{b}$:

$$\tau_{zs,max} = \frac{M_z}{c_1 a b^2}$$
$$\vartheta = \frac{M_z}{G(c_2 a b^3)}$$



$\frac{a}{b}$	c_1	c_2
1	0.208	0.141
1.2	0.219	0.166
1.5	0.231	0.196
2	0.246	0.229
2.5	0.248	0.249
3	0.267	0.263
4	0.282	0.281
5	0.291	0.291
10	0.312	0.312
∞	0.333	0.333

TWISTING COUPLE – THIN-WALLED OPEN MEMBERS

For cross section composed by N rectangles with sides equal to a_i and b_i with $\frac{a_i}{b_i} \rightarrow \infty$ (i.e., $c_1 = c_2 = \frac{1}{3}$) it is:

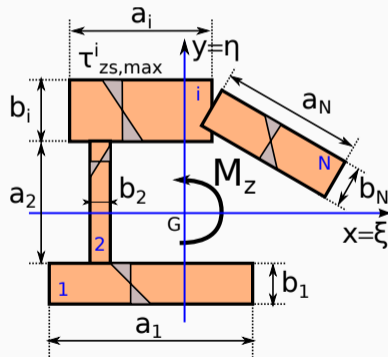
$$\tau_{zs,max}^i = \frac{M_z}{I_t} b_i$$

where:

$$I_t^i = \frac{1}{3} a_i b_i^3 \quad \text{and} \quad I_t = \sum_{i=1}^N I_t^i$$

The angle of twist is given by:

$$\vartheta = \frac{M_z}{G I_t}$$



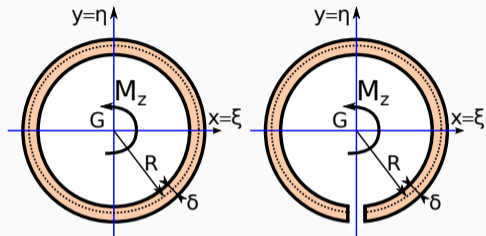
THIN-WALLED HOLLOW AND THIN-WALLED OPEN MEMBERS

Thin-walled hollow members:

$$\tau_{zs} = \frac{M_z}{2 \Omega \delta} = \frac{M_z}{2 (\pi R^2) \delta}$$

Thin-walled open members:

$$\tau_{zs,max} = \frac{M_z}{I_t} \delta = \frac{M_z}{\frac{1}{3} (2\pi R) \delta^3} \delta = \frac{3M_z}{2\pi R \delta^2}$$



It is noticed that the maximum stress...

... is attained in the thin-walled open cross section. The thin-walled hollow members is most efficient design against torque

SIGN CONVENTIONS

- A positive sign is assigned to stresses with the same sense of rotation of M_z