ENERGY PRINCIPLES AND VARIATIONAL METHODS IN APPLIED MECHANICS

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CONTENTS

PREFACE

The increasing use of numerical and computational methods in engineering and applied sciences has shed new light on the importance of energy principles and variational methods. The number of engineering courses that make use of energy principles and variational formulations and methods has also grown very rapidly in recent years. In view of the increase in the use of the variational formulations and methods (including the finite element method), there is a need to introduce the concepts of energy principles and variational methods and their use in the formulation and solution of problems of mechanics to both undergraduate and beginning graduate students. This book, an extensively revised version of the author's earlier book *Energy and Variational Methods in Applied Mechanics*, is intended for senior undergraduate students and beginning graduate students in aerospace, civil, and mechanical engineering and applied mechanics, who have had a course in fundamental engineering subjects as well as in ordinary and partial differential equations.

The book is organized into ten chapters and is self-contained as far as the subject matter is concerned. Chapter 1 presents a general introduction to the subject of variational principles. Chapter 2 contains a brief review of the algebra and calculus of vectors and Cartesian tensors. A review of the basic equations of linear solid continuum mechanics is included in Chapter 3. These equations are frequently referred to in subsequent chapters. Much of the material presented in Chapters 1 through 3 can be assigned as reading material, especially in a graduate class.

Chapter 4 deals with the concepts of work, energy, and the basic topics from variational calculus, including Euler equations, the fundamental lemma of calculus of variations, essential and natural boundary conditions, and minimization of functionals without and with equality constraints. Virtual work and energy principles and energy methods of solid and structural mechanics are presented in Chapter 5. Chapter 6 is devoted to a discussion of Hamilton's principle for dynamical systems. Classical variational methods of approximation (e.g., the methods of Ritz, Galerkin, Kantorovich, etc.) are presented in Chapter 7. All of the concepts and methods presented in Chapters 4 through 7 are illustrated using bars and beams, although the methods discussed in Chapter 7 are readily applicable to field problems whose differential equations resemble those of bars and beams. Chapter 8 is dedicated to applications of the energy principles and variational methods developed in earlier chapters to circular and rectangular plates. In the interest of completeness and for use as a reference for approximate solutions, exact solutions are also included. The finite element method is introduced in Chapter 9, with applications to beams and plates. Displacement finite element models of Euler–Bernoulli and Timoshenko beam theories and classical and first-order shear deformation plate theories are presented. A unified approach, more general than that found in most solid mechanics books, is used to introduce the finite element method. As a result, the student can readily extend the method to other subject areas of solid mechanics as well as to other branches of engineering. Lastly, the mixed variational principles of Hellinger and Reissner for elasticity are derived in Chapter 10. Mixed variational formulations, including mixed finite element models of beams and plates, are discussed.

Each chapter of the book contains many example problems and exercises that illustrate, test, and broaden the understanding of the topics covered. A list of references, by no means complete or up-to-date, is also provided at the end of each chapter. Answers to selected problems are included at the end of the book.

The book is suitable as a textbook for a senior undergraduate course or a first-year graduate course on energy principles and variational methods taught in aerospace, civil, and mechanical engineering and applied mechanics departments. To gain the most from the text, the student should have a senior undergraduate or first-year graduate standing in engineering. Some familiarity with basic courses in differential equations, mechanics of materials, and dynamics would also be helpful.

I have benefited by the professional works, encouragement, and support of many colleagues as well as students who have taught me how to explain complicated concepts in simple terms. While it is not possible to name all of them, without their help and support it would not have been possible for me to make some modest contributions to the field of mechanics through teaching, research, and writing.

My sincere thanks are due to my teacher, Professor J. T. Oden (University of Texas at Austin), for the many things I learned from him which have been useful all my life. In the same vein I wish to thank Professor C. W. Bert (University of Oklahoma, Norman) for giving me the opportunity to teach and develop into what I am. I am very grateful for their mentorship, advice, and support.

Deep gratitude is due to my wife for her patience and support while I am occupied with the writing of this book and others, and for her smile when I say every day "I have so much to do."

Most books are not free of errors, especially those with many mathematical equations and numbers. I wish to thank in advance those readers who are willing to draw attention to typos and errors, using the e-mail address: jnreddy@hotmail.com.

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INTRODUCTION

1.1 PRELIMINARY COMMENTS

The phrase "energy methods" in the present study refers to methods that make use of the total potential energy (i.e., strain energy and potential energy due to applied loads) of a system to obtain values of an unknown displacement or force, at a specific point of the system. These include Castigliano's theorems, unit-dummy-load and unit-dummy-displacement methods, and Betti's and Maxwell's theorems. These methods are often limited to the (exact) determination of generalized displacements or forces at fixed points in the structure; in most cases, they cannot be used to determine the complete solution (i.e., displacements and/or forces) as a function of position in the structure. The phrase "variational methods," on the other hand, refers to methods that make use of the variational principles, such as the principles of virtual work and the principle of minimum total potential energy, to determine approximate solutions as continuous functions of position in a body. In the classical sense, a *variational principle* has to do with the minimization or finding stationary values of a functional with respect to a set of undetermined parameters introduced in the assumed solution. The functional represents the total energy of the system in solid and structural mechanics problems, and in other problems it is simply an integral representation of the governing equations. In all cases, the functional includes all the intrinsic features of the problem, such as the governing equations, boundary and/or initial conditions, and constraint conditions.

1.2 THE ROLE OF ENERGY METHODS AND VARIATIONAL PRINCIPLES

Variational principles have always played an important role in mechanics. Variational formulations can be useful in three related ways. First, many problems of mechanics are posed in terms of finding the extremum (i.e., minima or maxima) and thus, by their nature, can be formulated in terms of variational statements. Second, there are problems that can be formulated by other means, such as by vector mechanics (e.g., Newton's laws), but these can also be formulated by means of variational principles. Third, variational formulations form a powerful basis for obtaining approximate solutions to practical problems, many of which are intractable otherwise. The principle of minimum total potential energy, for example, can be regarded as a substitute for the equations of equilibrium of an elastic body, as well as a basis for the development of displacement finite element models that can be used to determine approximate displacement and stress fields in the body. Variational formulations can also serve to unify diverse fields, suggest new theories, and provide a powerful means for studying the existence and uniqueness of solutions to problems. In many cases they can also be used to establish upper and/or lower bounds on approximate solutions.

1.3 SOME HISTORICAL COMMENTS

In modern times, the term "variational formulation" applies to a wide spectrum of concepts having to do with weak, generalized, or direct variational formulations of boundary- and initial-value problems. Still, many of the essential features of variational methods remain the same as they were over 200 years ago when the first notions of variational calculus began to be formulated.

Although Archimedes (287–212 B.C.) is generally credited with the first to use work arguments in his study of levers, the most primitive ideas of variational theory (the minimum hypothesis) are present in the writings of the Greek philosopher Aristotle (384–322 B.C.), to be revived again by the Italian mathematician/engineer Galileo (1564–1642), and finally formulated into a Principle of Least Time by the French mathematician Fermat (1601–1665). The phrase *virtual velocities* was used by Jean Bernoulli in 1717 in his letter to Varignon (1654–1722). The development of early variational calculus, by which we mean the classical problems associated with minimizing certain functionals, had to await the works of Newton (1642–1727) and Leibniz (1646–1716). The earliest applications of such variational ideas included the classical *isoperimetric problem* of finding among closed curves of given length the one that encloses the greatest area, and Newton's problem of determining the solid of revolution of "minimum resistance." In 1696, Jean Bernoulli proposed the problem of the *brachistochrone*: among all curves connecting two points, find the curve traversed in the shortest time by a particle under the influence of gravity. It stood as a challenge to the mathematicians of their day to solve the problem using the rudimentary tools of analysis then available to them or whatever new ones they were capable of developing. Solutions to this problem were presented by some of the greatest mathematicians of the time: Leibniz, Jean Bernoulli's older brother Jacques Bernoulli, L'Hôpital, and Newton.

The first step toward developing a general method for solving variational problems was given by the Swiss genius Leonhard Euler (1707–1783) in 1732 when he presented a "general solution of the isoperimetric problem," although Maupertuis is credited with having put forward a law of minimal property of potential energy for stable equilibrium in his *Mémoires de l'Académie des Sciences* in 1740. It was in Euler's 1732 work and subsequent publication of the principle of least action (in his book *Methodus inveniendi lineas curvas* ...) in 1744 by Euler that variational concepts found a welcome and permanent home in mechanics. He developed all ideas surrounding the principle of minimum potential energy in his work on the *Elastica*, and he demonstrated the relationship between his variational equations and those governing the flexure and buckling of thin rods.

A great impetus to the development of variational mechanics began in the writings of Lagrange (1736–1813), first in his correspondence with Euler. Euler worked intensely in developing Lagrange's method, but delayed publishing his results until Lagrange's works were published in 1760 and 1761. Lagrange used d'Alembert's principle to convert dynamics to statics and then used the principle of virtual displacements to derive his famous equations governing the laws of dynamics in terms of kinetic and potential energy. Euler's work, together with Lagrange's *Mécanique analytique* of 1788, laid down the basis for the variational theory of dynamical systems. Further generalizations appeared in the fundamental work of Hamilton in 1834. Collectively, all these works have had a monumental impact on virtually every branch of mechanics.

A more solid mathematical basis for variational theory began to be developed in the eighteenth and early nineteenth century. Necessary conditions for the existence of "minimizing curves" of certain functionals were studied during this period, and we find among contributors of that era the familiar names of Legendre, Jacobi, and Weierstrass. Legendre gave criteria for distinguishing between maxima and minima in 1786, without considering criteria for existence, and Jacobi gave sufficient conditions for existence of extrema in 1837. A more rigorous theory of existence of extrema was put together by Weierstrass, who, with Erdmann, established in 1865 conditions on extrema for variational problems involving corner behavior.

During the last half of the nineteenth century, the use of variational ideas was widespread among leaders in theoretical mechanics. We mention the works of Kirchhoff on plate theory, Lamé, Green, and Kelvin on elasticity, and the works of Betti, Maxwell, Castigliano, Menabrea, and Engesser for discrete structural systems. Lamé was the first in 1852 to prove a work equation, named after his colleague Clapeyron, for deformable bodies. Lamé's equation was used by Maxwell [1] for the solution of redundant frame works using the unit-dummy-load technique. In 1875 Castigliano published an extremum version of this technique, but attributed the idea to Menabrea. A generalization of Castigliano's work is due to Engesser [2].

Among prominent contributors to the subject near the end of the nineteenth century and in the early years of the twentieth century, particularly in the area of variational methods of approximation and their applications to physical problems, were Rayleigh [3], Ritz [4], and Galerkin [5]. Modern variational principles began in the 1950s with the works of Hellinger [6] and Reissner [7,8] on mixed variational principles for elasticity problems. A variety of generalizations of classical variational principles have appeared, and we shall not describe them here.

In closing this section, we note that a short historical account of early variational methods in mechanics can be found in the book of Lanczos [9] and a brief review of certain aspects of the subject as it stood in the early 1950s can be found in the book of Truesdell and Toupin [10]; additional information can be found in Smith's history of mathematics [11] and in the historical treatises on mechanics by Mach [12], Dugas [13], and Timoshenko [14]. Reference to much of the relevant contemporary literature can be found in the books by Washizu [15] and Oden and Reddy [16]. Additional historical papers and textbooks on variational methods are listed at the end of this chapter (see [17–56]).

1.4 PRESENT STUDY

The objective of the present study is to introduce energy methods and variational principles of solid and structural mechanics and to illustrate their use in the derivation and solution of the equations of applied mechanics, including plane elasticity, beams, frames, and plates. Of course, variational formulations and methods presented in this book are also applicable to problems outside solid mechanics. To equip the reader with the necessary mathematical tools and background from the theory of elasticity that are useful in the sequel, a review of vectors, matrices, tensors, and governing equations of elasticity are provided in the next two chapters. To keep the scope of the book within reasonable limits, only linear problems are considered. Although stability and vibration problems are introduced via examples and exercises, a detailed study of these topics is omitted.

In the following chapter we summarize the algebra and calculus of vectors and tensors. In Chapter 3 we give a brief review of the equations of solid mechanics, and in Chapter 4 we present the concepts of work and energy, energy principles, and Castigliano's theorems of structural mechanics. In Chapter 5 we present principles of virtual work, potential energy, and complementary energy. Chapter 6 is dedicated to Hamilton's principle for dynamical systems, and in Chapter 7 we introduce the Ritz, Galerkin, and weighted-residual methods. In Chapter 8, applications of variational methods to the formulation of plate bending theories and their solution by variational methods are presented. For the sake of completeness and comparison, analytical solutions of bending, vibration, and buckling of circular and rectangular plates are also presented. An introduction to the finite element method and its application to displacement finite element models of beams and plates is discussed in Chapter 9. The final chapter, Chapter 10, is devoted to the discussion of mixed variational principles, and mixed finite element models of beams and plates. To keep the scope of the book within reasonable limits, theory and analysis of shells is not included.

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2

MATHEMATICAL PRELIMINARIES

2.1 INTRODUCTION

Our approach in this book is evolutionary; that is, we wish to begin with concepts that are simple and intuitive, and then generalize these concepts step by step into a broader and more abstract body of analysis. This is a natural inductive approach, more or less in accord with the development of the subject of variational methods.

In analyzing physical phenomena, we set up relations between various quantities that characterize the phenomena (such as Newton's laws, energy conservation, etc.). As a means of expressing a natural law, a coordinate system in a chosen frame of reference can be introduced, and the various physical quantities involved can be expressed in terms of measurements made in that system. The mathematical form of the law thus depends upon the chosen coordinate system and may appear different in another type of coordinate system. The laws of nature, however, should be independent of the artificial choice of a coordinate system, and we may seek to represent the law in a manner independent of a particular coordinate system. A way of doing this is provided by vector and tensor analysis. When vector notation is used, a particular coordinate system need not be introduced. Consequently, the use of vector notation in formulating natural laws leaves them *invariant* to coordinate transformations. A study of physical phenomena by means of vector equations often leads to a deeper understanding of the problem in addition to bringing simplicity and versatility into the analysis.

The term *vector* is used often to imply a *physical* vector that has "magnitude and direction" and obeys certain rules of vector addition and scalar multiplication. In the sequel we consider more general, abstract objects than physical vectors, which are also called vectors. It transpires that the physical vector is a special case of what is known as a "vector from a linear vector space." Then the notion of vectors in modern mathematical analysis is an abstraction of the elementary notion of a physical vector. While the definition of a vector in abstract analysis does not require the vector to have a magnitude, in nearly all cases of practical interest the vector is endowed with a magnitude, in which case the vector is said to belong to a normed vector space.

Like physical vectors, which have direction and magnitude and satisfy the parallelogram law of addition, *tensors* are more general objects that are endowed with a magnitude and multiple direction(s) and satisfy rules of tensor addition and scalar multiplication. In fact, vectors are often termed the first-order tensors. As will be shown shortly, the stress (i.e., force per unit area) requires a magnitude and two directions—one normal to the plane on which the stress is measured and the other is the direction of the force—to specify it uniquely.

In this chapter we review basic elements of vector and tensor analysis that are of use in the sequel. For additional reading, the books listed at the end of the chapter [1–27] may be consulted.

2.2 VECTORS

2.2.1 Definition of a Vector

In the analysis of physical phenomena we are concerned with quantities that may be classified according to the information needed to specify them completely. Consider the following two groups:

Scalars	Nonscalars
Mass	Force
Temperature	Moment
Density	Stress
Volume	Acceleration
Time	Displacement

After units have been selected, the scalars are given by a single number. Nonscalars need not only a magnitude specified, but also additional information, such as direction. Nonscalars that obey certain rules (such as the parallelogram law of addition) are called *vectors*. Not all nonscalar quantities are vectors. The specification of a stress requires not only a force, which is a vector, but also an area upon which the force acts. A stress is a second-order tensor, as will be shown shortly.

In written or typed material, it is customary to place an arrow or a bar over the letter denoting the vector, such as $\vec{A}$. Sometimes the typesetter's mark of a tilde under the letter is used. In printed material the vector letter is denoted by a boldface letter, **A**, such as used in this book. The magnitude of the vector **A** is given by $|\mathbf{A}|$ or just A.

Two vectors **A** and **B** are equal if their magnitudes are equal, $|\mathbf{A}| = |\mathbf{B}|$, and if their directions and sense are equal. Consequently a vector is not changed if it is moved parallel to itself. This means that the position of a vector in space may be

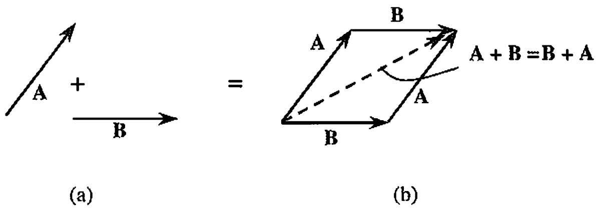


Figure 2.1 (a) Addition of vectors. (b) Parallelogram law of addition.

chosen arbitrarily. In certain applications, however, the actual point of location of a vector may be important (for instance, a moment or a force acting on a body). A vector associated with a given point is known as a localized or bound vector.

Let **A** and **B** be any two vectors. Then we can add them as shown in Fig. 2.1a. The combination of the two diagrams in Fig. 2.1a gives the parallelogram shown in Fig. 2.1b. Thus we say the vectors add according to the *parallelogram law* of addition so that

$$\mathbf{C} = \mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}. \tag{2.1}$$

We thus see that vector addition is *commutative*.

Subtraction of vectors is carried out along the same lines. To form the difference **A** − **B**, we write

$$\mathbf{A} - \mathbf{B} = \mathbf{A} + (-\mathbf{B}) \tag{2.2}$$

and subtraction reduces to the operation of addition. The negative vector −**B** has the same magnitude as **B**, but has the opposite *sense*.

With the rules of addition in place, we can define a (geometric) vector. *A vector is a quantity that possesses both magnitude and direction and obeys the parallelogram law of addition.* Obeying the law is important because there are quantities having both magnitude and direction that do not obey this law. A finite rotation of a rigid body is not a vector, although infinitesimal rotations are. The definition given above is a *geometrical* definition. That vectors can be represented graphically is an *incidental* rather than a fundamental feature of the vector concept.

A vector of unit length is called a *unit vector*. The unit vector may be defined as follows:

$$\hat{\mathbf{e}}_A = \frac{\mathbf{A}}{A}. \tag{2.3}$$

We may now write

$$\mathbf{A} = A\hat{\mathbf{e}}_A. \tag{2.4}$$

Thus *any vector may be represented as a product of its magnitude and a unit vector.* A unit vector is used to designate direction. It does not have any physical dimensions. We denote a unit vector by a "hat" (caret) above the boldface letter.

A vector of zero magnitude is called a *zero vector* or a *null vector*. All null vectors are considered equal to each other without consideration as to direction:

$$\mathbf{A} + \mathbf{0} = \mathbf{A} \quad \text{and} \quad 0\mathbf{A} = \mathbf{0}. \tag{2.5}$$

The laws that govern addition, subtraction, and scalar multiplication of vectors are identical with those governing the operations of scalar algebra.

2.2.2 Scalar and Vector Products

Besides addition, subtraction, and multiplication by a scalar, we must consider multiplication of two vectors. There are several ways the product of two vectors can be defined. We consider first the so-called scalar product. Let us recall the concept of work. When a force $\mathbf{F}$ acts on a mass point and moves through an infinitesimal displacement vector $\mathbf{ds}$, the work done by the force vector is defined by the *projection* of the force in the direction of the displacement times the magnitude of the displacement (see Fig. 2.2). Such an operation may be defined for any two vectors. Since the result of the product is a scalar, it is called the *scalar product*. We denote this product as follows:

$$\mathbf{F} \cdot \mathbf{ds} = F ds \cos\theta, \qquad 0 \leq \theta \leq \pi. \tag{2.6}$$

The scalar product is also known as the *dot product* or *inner product*.

To understand the vector product, consider the concept of the *moment* due to a force. Let us describe the moment about a point O of a force $\mathbf{F}$ acting at a point P, such as shown in Fig. 2.3a. By definition, the magnitude of the moment is given by

$$M = Fl, \qquad F = |\mathbf{F}|, \tag{2.7}$$

where l is the lever arm for the force about the point O. If $\mathbf{r}$ denotes the vector $\mathbf{OP}$ and θ the angle between $\mathbf{r}$ and $\mathbf{F}$ as shown, such that $0 \leq \theta \leq \pi$, we have $l = r\sin\theta$, and thus

$$M = Fr\sin\theta. \tag{2.8}$$

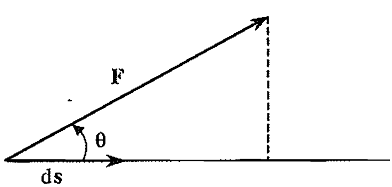


Figure 2.2 Representation of work.

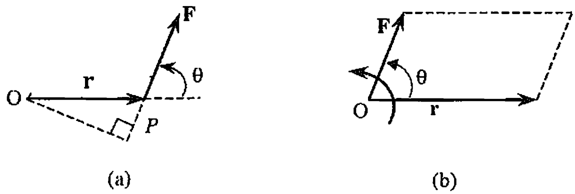


Figure 2.3 (a) Representation of a moment. (b) Direction of rotation.

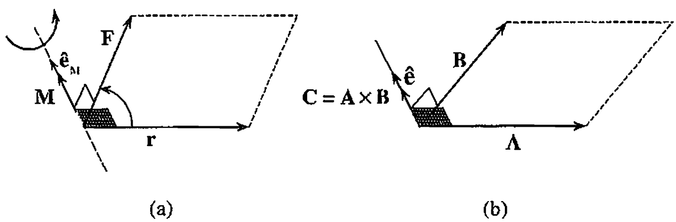


Figure 2.4 (a) Axis of rotation. (b) Representation of the vector.

A direction can now be assigned to the moment. Drawing the vectors **F** and **r** from the common origin O, we note that the rotation due to **F** tends to bring **r** into **F** (see Fig. 2.3b). We now set up an axis of rotation perpendicular to the plane formed by **F** and **r**. Along this axis of rotation we set up a preferred direction as that in which a right-handed screw would advance when turned in the direction of rotation due to the moment (see Fig. 2.4a). Along this axis of rotation we draw a unit vector $\hat{\mathbf{e}}_M$ and agree that it represents the direction of the moment **M**. Thus we have

$$\mathbf{M} = Fr\sin\theta\,\hat{\mathbf{e}}_M$$
$$= \mathbf{r} \times \mathbf{F}. \tag{2.9}$$

According to this expression, **M** may be looked upon as resulting from a special operation between the two vectors **F** and **r**. It is thus the basis for defining a product between any two vectors. Since the result of such a product is a vector, it may be called the *vector product*.

The vector product of two vectors **A** and **B** is a vector **C** whose magnitude is equal to the product of the magnitude of **A** and **B** times the sine of the angle measured from **A** to **B** such that $0 \leq \theta \leq \pi$, and whose direction is specified by the condition that **C** be perpendicular to the plane of the vectors **A** and **B** and points in the direction which a right-handed screw advances when turned so as to bring **A** into **B**.

The vector product is usually denoted by

$$\mathbf{C} = \mathbf{A} \times \mathbf{B} = AB\sin(\mathbf{A}, \mathbf{B})\,\hat{\mathbf{e}}, \tag{2.10}$$

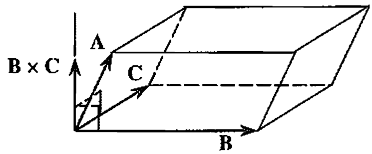


Figure 2.5 Scalar triple product as the volume of a parallelepiped.

where sin$(\mathbf{A}, \mathbf{B})$ denotes the sine of the angle between vectors **A** and **B**. This product is called the *cross product*, *skew product*, and also *outer product*, as well as the vector product (see Fig. 2.4b).

Now consider the various products of three vectors:

$$\mathbf{A}(\mathbf{B} \cdot \mathbf{C}), \qquad \mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}), \qquad \mathbf{A} \times (\mathbf{B} \times \mathbf{C}). \tag{2.11}$$

The product $\mathbf{A}(\mathbf{B} \cdot \mathbf{C})$ is merely a multiplication of the vector **A** by the scalar $\mathbf{B} \cdot \mathbf{C}$. The product $\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C})$ is a scalar. It can be seen that the product $\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C})$, except for the algebraic sign, is the volume of the parallelepiped formed by the vectors **A**, **B**, and **C**, as shown in Fig. 2.5.

We also note the following properties:

1. The dot and cross can be interchanged without changing the value:

$$\mathbf{A} \cdot \mathbf{B} \times \mathbf{C} = \mathbf{A} \times \mathbf{B} \cdot \mathbf{C} \equiv [\mathbf{ABC}]. \tag{2.12}$$

2. A cyclical permutation of the order of the vectors leaves the result unchanged:

$$\mathbf{A} \cdot \mathbf{B} \times \mathbf{C} = \mathbf{C} \cdot \mathbf{A} \times \mathbf{B} = \mathbf{B} \cdot \mathbf{C} \times \mathbf{A} \equiv [\mathbf{ABC}]. \tag{2.13}$$

3. If the cyclic order is changed, the sign changes:

$$\mathbf{A} \cdot \mathbf{B} \times \mathbf{C} = -\mathbf{A} \cdot \mathbf{C} \times \mathbf{B} = -\mathbf{C} \cdot \mathbf{B} \times \mathbf{A} = -\mathbf{B} \cdot \mathbf{A} \times \mathbf{C}. \tag{2.14}$$

4. A necessary and sufficient condition for any three vectors, **A**, **B**, **C** to be coplanar is that $\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = 0$. Note also that the scalar triple product is zero when any two vectors are the same.

The product $\mathbf{A} \times (\mathbf{B} \times \mathbf{C})$ is a vector normal to the plane formed by **A** and $(\mathbf{B} \times \mathbf{C})$. The vector $(\mathbf{B} \times \mathbf{C})$, however, is perpendicular to the plane formed by **B** and **C**. This means that $\mathbf{A} \times (\mathbf{B} \times \mathbf{C})$ lies in the plane formed by **B** and **C** and is perpendicular to **A** (see Fig. 2.6). Thus $\mathbf{A} \times (\mathbf{B} \times \mathbf{C})$ can be expressed as a linear combination of **B** and **C**:

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = m_1\mathbf{B} + n_1\mathbf{C}. \tag{2.15}$$

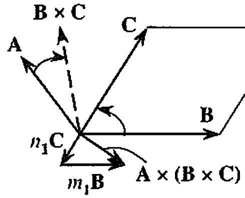


Figure 2.6 The vector triple product.

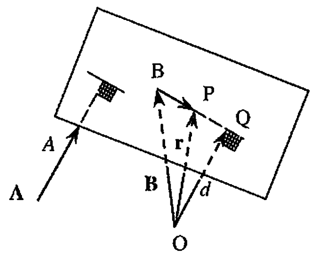


Figure 2.7 Plane perpendicular to A, and passing through the terminal point of **B**.

Likewise, we would find that

$$(\mathbf{A} \times \mathbf{B}) \times \mathbf{C} = m_2\mathbf{A} + n_2\mathbf{B}. \tag{2.16}$$

Thus the parentheses *cannot* be interchanged or removed. It can be shown that

$$m_1 = \mathbf{A} \cdot \mathbf{C}, \qquad n_1 = -\mathbf{A} \cdot \mathbf{B},$$

and hence that

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = (\mathbf{A} \cdot \mathbf{C})\mathbf{B} - (\mathbf{A} \cdot \mathbf{B})\mathbf{C}. \tag{2.17}$$

Example 2.1 The equation of a plane perpendicular to a vector **A** and passing through the terminal point of vector **B** can be obtained without the use of any coordinate system (see Fig. 2.7). Let O be the origin and B the terminal point of vector **B**. Draw a directed line segment from O to Q, such that **OQ** is parallel to **A** and Q is in the plane. Then $\mathbf{OQ} = \alpha\mathbf{A}$, where α is a scalar. Let P be an arbitrary point on the line BQ. If the position vector of the point P is **r**, then

$$\mathbf{BP} = \mathbf{OP} - \mathbf{OB} = \mathbf{r} - \mathbf{B}.$$

Since **BP** is perpendicular to $\mathbf{OQ} = \alpha\mathbf{A}$, we must have

$$\mathbf{BP} \cdot \mathbf{OQ} = 0 \qquad \text{or} \qquad (\mathbf{r} - \mathbf{B}) \cdot \mathbf{A} = 0,$$

which is the equation of the plane in question.

The perpendicular distance from point O to the plane is the magnitude of **OQ**. However, we do not know its magnitude (or, α is not known). The distance is also given by the projection of vector **B** along **OQ**:

$$d = \mathbf{B} \cdot \frac{\mathbf{OQ}}{|\mathbf{OQ}|} = \mathbf{B} \cdot \hat{\mathbf{e}}_A,$$

where $\hat{\mathbf{e}}_A$ is the unit vector along **A**, $\hat{\mathbf{e}}_A = \mathbf{A}/A$.

Example 2.2 Let **A** and **B** be any two vectors in space. Then vector **A** can be expressed in terms of components along (i.e., parallel) and perpendicular to **B**: the component of **A** along **B** is given by $(\mathbf{A} \cdot \hat{\mathbf{e}}_B)$, where $\hat{\mathbf{e}}_B = \mathbf{B}/B$. The component of **A** perpendicular to **B** and in the plane of **A** and **B** is given by the vector triple product $\hat{\mathbf{e}}_B \times (\mathbf{A} \times \hat{\mathbf{e}}_B)$. Thus,

$$\mathbf{A} = (\mathbf{A} \cdot \hat{\mathbf{e}}_B)\hat{\mathbf{e}}_B + \hat{\mathbf{e}}_B \times (\mathbf{A} \times \hat{\mathbf{e}}_B). \tag{a}$$

Alternately, using Eq. (a) with $\mathbf{A} = \mathbf{C} = \hat{\mathbf{e}}_B$ and $\mathbf{B} = \mathbf{A}$, we obtain

$$\hat{\mathbf{e}}_B \times (\mathbf{A} \times \hat{\mathbf{e}}_B) = \mathbf{A} - (\hat{\mathbf{e}}_B \cdot \mathbf{A})\hat{\mathbf{e}}_B.$$

2.2.3 Components of a Vector

So far we have proceeded on a geometrical description of a vector as a directed line segment. We now embark on an analytical description of a vector and some of the operations associated with this description. Such a description yields a connection between vectors and ordinary numbers and relates operation on vectors with those on numbers. The analytical description is based on the notion of components of a vector.

In what follows, we shall consider a three-dimensional space, and the extensions to n dimensions will be evident (except for a few exceptions). A set of n vectors is said to be linearly dependent if a set of n numbers $\beta_1, \beta_2, \ldots, \beta_n$ can be found such that

$$\beta_1\mathbf{A}_1 + \beta_2\mathbf{A}_2 + \cdots + \beta_n\mathbf{A}_n = \mathbf{0}, \tag{2.18}$$

where $\beta_1, \beta_2, \ldots, \beta_n$ cannot all be zero. If this expression cannot be satisfied, the vectors are said to be *linearly independent*.

In a three-dimensional space a set of no more than three linearly independent vectors can be found. Let us choose any set and denote it as follows:

$$\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3. \tag{2.19}$$

This set is called a *basis* (or a base system).

It is clear from the concept of linear dependence that we can represent any vector in three-dimensional space as a linear combination of the basis vectors (see Fig. 2.8):

$$\mathbf{A} = A_1\mathbf{e}_1 + A_2\mathbf{e}_2 + A_3\mathbf{e}_3. \tag{2.20}$$

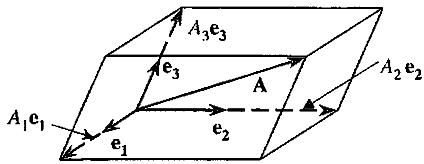


Figure 2.8 Components of a vector.

The vectors $A_1\mathbf{e}_1$, $A_2\mathbf{e}_2$, and $A_3\mathbf{e}_3$ are called the *vector components* of **A**, and A_1, A_2, and A_3 are called *scalar components* or *measure numbers* of **A** associated with the basis $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$. Also, we use the notation $\mathbf{A} = (A_1, A_2, A_3)$ to denote a vector by its components.

2.2.4 Summation Convention

It is useful to abbreviate a summation of terms by understanding that a repeated index means summation over all values of that index. Thus the summation

$$\mathbf{A} = \sum_{i=1}^{3} A_i \mathbf{e}_i \tag{2.21}$$

can be shortened to

$$\mathbf{A} = A_i \mathbf{e}_i . \tag{2.22}$$

The repeated index is a *dummy index* and thus can be replaced by *any other symbol that has not already been used.* Thus we can also write

$$\mathbf{A} = A_i \mathbf{e}_i = A_m \mathbf{e}_m , \tag{2.23}$$

and so on.

When a basis is unit and orthogonal, that is, orthonormal, we have

$$[\mathbf{e}_1 \mathbf{e}_2 \mathbf{e}_3] = 1. \tag{2.24}$$

In many situations an *orthonormal basis* simplifies calculations.

For an orthonormal basis the vectors **A** and **B** can be written as

$$\mathbf{A} = A_1\hat{\mathbf{e}}_1 + A_2\hat{\mathbf{e}}_2 + A_3\hat{\mathbf{e}}_3 = A_i\hat{\mathbf{e}}_i$$
$$\mathbf{B} = B_1\hat{\mathbf{e}}_1 + B_2\hat{\mathbf{e}}_2 + B_3\hat{\mathbf{e}}_3 = B_i\hat{\mathbf{e}}_i ,$$

where $\mathbf{e}_i \equiv \hat{\mathbf{e}}_i$ $(i = 1, 2, 3)$ is the orthonormal basis, and A_i and B_i are the corresponding *physical components* (i.e., the components have the same physical dimensions as the vector).

It is convenient at this time to introduce the alternating symbol ε_{ijk} for representing the cross product of two orthonormal vectors in a right-handed basis system. We define the cross product $\hat{\mathbf{e}}_i \times \hat{\mathbf{e}}_j$ for a right-handed system as

$$\hat{\mathbf{e}}_i \times \hat{\mathbf{e}}_j \equiv \varepsilon_{ijk}\hat{\mathbf{e}}_k, \tag{2.25}$$

where

$$\varepsilon_{ijk} = \begin{cases} 1, & \text{if } i, j, k \text{ are in cyclic order and not repeated } (i \neq j \neq k), \\ -1, & \text{if } i, j, k \text{ are not in cyclic order and not repeated } (i \neq j \neq k), \\ 0, & \text{if any of } i, j, k \text{ are repeated.} \end{cases} \tag{2.26}$$

The symbol ε_{ijk} is called the *alternating symbol* or *permutation symbol.*

In an orthonormal basis the scalar and vector products can be expressed in the index form using the Kronecker delta $\delta_{ij} \equiv \hat{\mathbf{e}}_i \cdot \hat{\mathbf{e}}_j$, and the alternating symbols:

$$\begin{aligned} \mathbf{A} \cdot \mathbf{B} &= (A_i\hat{\mathbf{e}}_i) \cdot (B_j\hat{\mathbf{e}}_j) = A_i B_j \delta_{ij} = A_i B_i, \\ \mathbf{A} \times \mathbf{B} &= (A_i\hat{\mathbf{e}}_i) \times (B_j\hat{\mathbf{e}}_j) = A_i B_j \varepsilon_{ijk}\hat{\mathbf{e}}_k. \end{aligned} \tag{2.27}$$

Further, the Kronecker delta and the permutation symbol are related by the identity, known as the *ε–δ identity,*

$$\varepsilon_{ijk}\varepsilon_{imn} = \delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km}. \tag{2.28}$$

The permutation symbol and the Kronecker delta prove to be very useful in proving vector identities. Since a vector form of any identity is invariant (i.e., valid in any coordinate system), it suffices to prove it in one coordinate system. In particular, an orthonormal system is very convenient because of the permutation symbol and the Kronecker delta. The following example illustrates some of the uses of δ_{ij} and ε_{ijk}.

Example 2.3 We wish to express the vector operation $(\mathbf{A} \times \mathbf{B}) \cdot (\mathbf{C} \times \mathbf{D})$ in an alternate vector form (to establish a vector identity):

$$\begin{aligned} (\mathbf{A} \times \mathbf{B}) \cdot (\mathbf{C} \times \mathbf{D}) &= (A_i B_j \varepsilon_{ijk}\hat{\mathbf{e}}_k) \cdot (C_m D_n \varepsilon_{mnp}\hat{\mathbf{e}}_p) \\ &= A_i B_j C_m D_n \varepsilon_{ijk}\varepsilon_{mnp}\delta_{kp} \\ &= A_i B_j C_m D_n \varepsilon_{ijk}\varepsilon_{mnk} \\ &= A_i B_j C_m D_n (\delta_{im}\delta_{jn} - \delta_{in}\delta_{jm}) \\ &= A_i B_j C_m D_n \delta_{im}\delta_{jn} - A_i B_j C_m D_n \delta_{in}\delta_{jm}, \end{aligned}$$

where we have used the ε–δ identity (2.28). Since $C_m\delta_{im} = C_i$ (or $A_i\delta_{im} = A_m$, etc.), we have

$$\begin{aligned}(\mathbf{A} \times \mathbf{B}) \cdot (\mathbf{C} \times \mathbf{D}) &= A_i B_j C_i D_j - A_i B_j C_j D_i \\ &= A_i C_i B_j D_j - A_i D_i B_j C_j \\ &= (\mathbf{A} \cdot \mathbf{C})(\mathbf{B} \cdot \mathbf{D}) - (\mathbf{A} \cdot \mathbf{D})(\mathbf{B} \cdot \mathbf{C}). \end{aligned} \qquad \text{(a)}$$

Although the above vector identity is established in an orthonormal coordinate system, it holds in a general coordinate system. That is, the vector identity (a) is invariant.

Example 2.4 Let $\hat{\mathbf{e}}_i$ ($i = 1, 2, 3$) be a set of orthonormal base vectors, and define new right-handed coordinate base vectors by ($\hat{\bar{\mathbf{e}}}_1 \cdot \hat{\bar{\mathbf{e}}}_2 = 0$)

$$\hat{\bar{\mathbf{e}}}_1 = \frac{2\hat{\mathbf{e}}_1 + 2\hat{\mathbf{e}}_2 + \hat{\mathbf{e}}_3}{3}, \qquad \hat{\bar{\mathbf{e}}}_2 = \frac{\hat{\mathbf{e}}_1 - \hat{\mathbf{e}}_2}{\sqrt{2}},$$

and

$$\hat{\bar{\mathbf{e}}}_3 = \hat{\bar{\mathbf{e}}}_1 \times \hat{\bar{\mathbf{e}}}_2 = \frac{\hat{\mathbf{e}}_1 + \hat{\mathbf{e}}_2 - 4\hat{\mathbf{e}}_3}{3\sqrt{2}}.$$

An arbitrary vector $\mathbf{A}$ can be represented in either coordinate system

$$\mathbf{A} = A_i \hat{\mathbf{e}}_i = \bar{A}_i \hat{\bar{\mathbf{e}}}_i.$$

The components of the vector in the two different coordinate systems are related by

$$\{\bar{A}\} = [\beta]\{A\}, \qquad \beta_{ij} = \hat{\bar{\mathbf{e}}}_i \cdot \hat{\mathbf{e}}_j. \qquad (2.29)$$

The coefficients β_{ij} are called the *direction cosines*. Note that the first subscript of β_{ij} comes from the barred coordinate system and the second subscript from the unbarred system.

For the case at hand we have

$$\begin{aligned} &\beta_{11} = \hat{\bar{\mathbf{e}}}_1 \cdot \hat{\mathbf{e}}_1 = \frac{2}{3}, &&\beta_{12} = \hat{\bar{\mathbf{e}}}_1 \cdot \hat{\mathbf{e}}_2 = \frac{2}{3}, &&\beta_{13} = \hat{\bar{\mathbf{e}}}_1 \cdot \hat{\mathbf{e}}_3 = \frac{1}{3}, \\ &\beta_{21} = \hat{\bar{\mathbf{e}}}_2 \cdot \hat{\mathbf{e}}_1 = \frac{1}{\sqrt{2}}, &&\beta_{22} = \hat{\bar{\mathbf{e}}}_2 \cdot \hat{\mathbf{e}}_2 = -\frac{1}{\sqrt{2}}, &&\beta_{23} = \hat{\bar{\mathbf{e}}}_2 \cdot \hat{\mathbf{e}}_3 = 0, \\ &\beta_{31} = \hat{\bar{\mathbf{e}}}_3 \cdot \hat{\mathbf{e}}_1 = \frac{1}{3\sqrt{2}}, &&\beta_{32} = \hat{\bar{\mathbf{e}}}_3 \cdot \hat{\mathbf{e}}_2 = \frac{1}{3\sqrt{2}}, &&\beta_{33} = \hat{\bar{\mathbf{e}}}_3 \cdot \hat{\mathbf{e}}_3 = -\frac{4}{3\sqrt{2}}, \end{aligned}$$

or

$$[\beta] = \frac{1}{3\sqrt{2}} \begin{bmatrix} 2\sqrt{2} & 2\sqrt{2} & \sqrt{2} \\ 3 & -3 & 0 \\ 1 & 1 & -4 \end{bmatrix}.$$

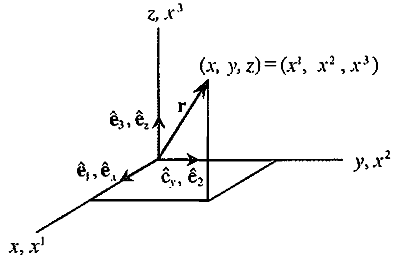


Figure 2.9 Rectangular Cartesian coordinates.

When the basis vectors are constant, that is, with fixed lengths (with the same units) and directions, the basis is called *Cartesian*. The general Cartesian system is oblique. When the basis vectors are unit and orthogonal (orthonormal), the basis system is called *rectangular Cartesian*, or simply *Cartesian*. In much of our study, we shall deal with Cartesian bases.

Let us denote an orthonormal Cartesian basis by

$$\{\hat{\mathbf{e}}_x, \hat{\mathbf{e}}_y, \hat{\mathbf{e}}_z\} \qquad \text{or} \qquad \{\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2, \hat{\mathbf{e}}_3\}.$$

The Cartesian coordinates are denoted by (x, y, z) or (x^1, x^2, x^3). The familiar rectangular Cartesian coordinate system is shown in Fig. 2.9. We shall always use right-handed coordinate systems.

A position vector to an arbitrary point (x, y, z) or (x^1, x^2, x^3), measured from the origin, is given by

$$\begin{aligned} \mathbf{r} &= x\hat{\mathbf{e}}_x + y\hat{\mathbf{e}}_y + z\hat{\mathbf{e}}_z \\ &= x^1\hat{\mathbf{e}}_1 + x^2\hat{\mathbf{e}}_2 + x^3\hat{\mathbf{e}}_3, \end{aligned} \tag{2.30}$$

or, in summation notation, by

$$\mathbf{r} = x^j\hat{\mathbf{e}}_j. \tag{2.31}$$

The distance between two infinitesimally removed points is given by

$$\begin{aligned} d\mathbf{r} \cdot d\mathbf{r} = (ds)^2 &= dx^j dx^j \\ &= (dx)^2 + (dy)^2 + (dz)^2. \end{aligned} \tag{2.32}$$

2.2.5 Vector Calculus

The basic notions of vector and scalar calculus, especially with regard to physical applications, are closely related to the rate of change of a scalar field with distance. Let us denote a scalar field by $\phi = \phi(\mathbf{r})$. In general coordinates we can write $\phi = \phi(q^1, q^2, q^3)$. The coordinate system (q^1, q^2, q^3) is referred to as the *unitary system*.

We now define the unitary basis $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ as follows:

$$\mathbf{e}_1 \equiv \frac{\partial \mathbf{r}}{\partial q^1}, \qquad \mathbf{e}_2 \equiv \frac{\partial \mathbf{r}}{\partial q^2}, \qquad \mathbf{e}_3 \equiv \frac{\partial \mathbf{r}}{\partial q^3}. \tag{2.33}$$

Hence, an arbitrary vector $\mathbf{A}$ is expressed as

$$\mathbf{A} = A^1\mathbf{e}_1 + A^2\mathbf{e}_2 + A^3\mathbf{e}_3, \tag{2.34}$$

and a differential distance is denoted by

$$d\mathbf{r} = dq^1\mathbf{e}_1 + dq^2\mathbf{e}_2 + dq^3\mathbf{e}_3 = dq^i\mathbf{e}_i. \tag{2.35}$$

Observe that the A's and dq's have superscripts whereas the unitary basis $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ has subscripts. The dq^i are referred to as the *contravariant components* of the differential vector $d\mathbf{r}$ and A^i are the contravariant components of vector $\mathbf{A}$. The unitary basis can be described in terms of the rectangular Cartesian basis $(\hat{\mathbf{e}}_x, \hat{\mathbf{e}}_y, \hat{\mathbf{e}}_z) \doteq (\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2, \hat{\mathbf{e}}_3)$ as follows:

$$\begin{aligned}
\mathbf{e}_1 &= \frac{\partial \mathbf{r}}{\partial q^1} = \frac{\partial x}{\partial q^1}\hat{\mathbf{e}}_x + \frac{\partial y}{\partial q^1}\hat{\mathbf{e}}_y + \frac{\partial z}{\partial q^1}\hat{\mathbf{e}}_z, \\
\mathbf{e}_2 &= \frac{\partial \mathbf{r}}{\partial q^2} = \frac{\partial x}{\partial q^2}\hat{\mathbf{e}}_x + \frac{\partial y}{\partial q^2}\hat{\mathbf{e}}_y + \frac{\partial z}{\partial q^2}\hat{\mathbf{e}}_z, \\
\mathbf{e}_3 &= \frac{\partial \mathbf{r}}{\partial q^3} = \frac{\partial x}{\partial q^3}\hat{\mathbf{e}}_x + \frac{\partial y}{\partial q^3}\hat{\mathbf{e}}_y + \frac{\partial z}{\partial q^3}\hat{\mathbf{e}}_z.
\end{aligned} \tag{2.36a}$$

In the summation convention we have

$$\mathbf{e}_i \equiv \frac{\partial \mathbf{r}}{\partial q^i} = \frac{\partial x^j}{\partial q^i}\hat{\mathbf{e}}_j, \qquad i = 1, 2, 3. \tag{2.36b}$$

Associated with any arbitrary basis is another basis that can be derived from it. We can construct this basis in the following way: Taking the scalar product of the vector $\mathbf{A}$ in Eq. (2.34) with the cross product $\mathbf{e}_1 \times \mathbf{e}_2$, we obtain

$$\mathbf{A} \cdot (\mathbf{e}_1 \times \mathbf{e}_2) = A^3\mathbf{e}_3 \cdot (\mathbf{e}_1 \times \mathbf{e}_2)$$

since $\mathbf{e}_1 \times \mathbf{e}_2$ is perpendicular to both $\mathbf{e}_1$ and $\mathbf{e}_2$. Solving for A^3 gives

$$A^3 = \mathbf{A} \cdot \frac{\mathbf{e}_1 \times \mathbf{e}_2}{\mathbf{e}_3 \cdot (\mathbf{e}_1 \times \mathbf{e}_2)} = \mathbf{A} \cdot \frac{\mathbf{e}_1 \times \mathbf{e}_2}{[\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3]}. \tag{2.37a}$$

In similar fashion we can obtain the following expressions for

$$A^1 = \mathbf{A} \cdot \frac{\mathbf{e}_2 \times \mathbf{e}_3}{[\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3]}, \qquad A^2 = \mathbf{A} \cdot \frac{\mathbf{e}_3 \times \mathbf{e}_1}{[\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3]}. \tag{2.37b}$$

We thus observe that we can obtain the components A^1, A^2, and A^3 by taking the scalar product of the vector $\mathbf{A}$ with special vectors, which we denote as follows:

$$\mathbf{e}^1 = \frac{\mathbf{e}_2 \times \mathbf{e}_3}{[\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3]}, \qquad \mathbf{e}^2 = \frac{\mathbf{e}_3 \times \mathbf{e}_1}{[\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3]}, \qquad \mathbf{e}^3 = \frac{\mathbf{e}_1 \times \mathbf{e}_2}{[\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3]}. \tag{2.38}$$

The set of vectors $(\mathbf{e}^1, \mathbf{e}^2, \mathbf{e}^3)$ is called the *dual* or *reciprocal* basis. Notice from the basic definitions that we have the following relations:

$$\mathbf{e}^i \cdot \mathbf{e}_j = \delta^i_j \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}. \tag{2.39}$$

It is possible, since the dual basis is linearly independent (the reader should verify this), to express a vector $\mathbf{A}$ in terms of the dual basis:

$$\mathbf{A} = A_1\mathbf{e}^1 + A_2\mathbf{e}^2 + A_3\mathbf{e}^3. \tag{2.40}$$

Notice now that the components associated with the dual basis have subscripts, and A_i are the *covariant components* of $\mathbf{A}$.

By an analogous process to that above we can show that the original basis can be expressed in terms of the dual basis in the following way:

$$\mathbf{e}_1 = \frac{\mathbf{e}^2 \times \mathbf{e}^3}{[\mathbf{e}^1\mathbf{e}^2\mathbf{e}^3]}, \qquad \mathbf{e}_2 = \frac{\mathbf{e}^3 \times \mathbf{e}^1}{[\mathbf{e}^1\mathbf{e}^2\mathbf{e}^3]}, \qquad \mathbf{e}_3 = \frac{\mathbf{e}^1 \times \mathbf{e}^2}{[\mathbf{e}^1\mathbf{e}^2\mathbf{e}^3]}. \tag{2.41}$$

Of course in the evaluation of the cross products we shall always use the right-hand rule. It follows from the above expressions that

$$\begin{aligned} &A^1 = \mathbf{A} \cdot \mathbf{e}^1, \quad &&A^2 = \mathbf{A} \cdot \mathbf{e}^2, \quad &&A^3 = \mathbf{A} \cdot \mathbf{e}^3, \quad &&\text{or} \quad &&A^i = \mathbf{A} \cdot \mathbf{e}^i, \\ &A_1 = \mathbf{A} \cdot \mathbf{e}_1, \quad &&A_2 = \mathbf{A} \cdot \mathbf{e}_2, \quad &&A_3 = \mathbf{A} \cdot \mathbf{e}_3, \quad &&\text{or} \quad &&A_i = \mathbf{A} \cdot \mathbf{e}_i. \end{aligned} \tag{2.42}$$

Returning to the scalar field ϕ, the differential change is given by

$$d\phi = \frac{\partial \phi}{\partial q^1}dq^1 + \frac{\partial \phi}{\partial q^2}dq^2 + \frac{\partial \phi}{\partial q^3}dq^3. \tag{2.43}$$

The differentials dq^1, dq^2, dq^3 are components of $d\mathbf{r}$ [see Eq. (2.35)]. We would now like to write $d\phi$ in such a way that we elucidate *the direction* as well as the magnitude of $d\mathbf{r}$. Since $\mathbf{e}^1 \cdot \mathbf{e}_1 = 1$, $\mathbf{e}^2 \cdot \mathbf{e}_2 = 1$, and $\mathbf{e}^3 \cdot \mathbf{e}_3 = 1$, we can write

$$\begin{aligned} d\phi &= \mathbf{e}^1 \frac{\partial \phi}{\partial q^1} \cdot \mathbf{e}_1 dq^1 + \mathbf{e}^2 \frac{\partial \phi}{\partial q^2} \cdot \mathbf{e}_2 dq^2 + \mathbf{e}^3 \frac{\partial \phi}{\partial q^3} \cdot \mathbf{e}_3 dq^3 \\ &= (dq^1\mathbf{e}_1 + dq^2\mathbf{e}_2 + dq^3\mathbf{e}_3) \cdot \left(\mathbf{e}^1 \frac{\partial \phi}{\partial q^1} + \mathbf{e}^2 \frac{\partial \phi}{\partial q^2} + \mathbf{e}^3 \frac{\partial \phi}{\partial q^3}\right) \\ &= d\mathbf{r} \cdot \left(\mathbf{e}^1 \frac{\partial \phi}{\partial q^1} + \mathbf{e}^2 \frac{\partial \phi}{\partial q^2} + \mathbf{e}^3 \frac{\partial \phi}{\partial q^3}\right). \end{aligned} \tag{2.44}$$

Let us now denote the magnitude of $d\mathbf{r}$ by $ds \equiv |d\mathbf{r}|$. Then $\hat{\mathbf{e}} = d\mathbf{r}/ds$ is a unit vector in the direction of $d\mathbf{r}$, and we have

$$\left(\frac{d\phi}{ds}\right)_{\hat{\mathbf{e}}} = \hat{\mathbf{e}} \cdot \left(\mathbf{e}^1 \frac{\partial \phi}{\partial q^1} + \mathbf{e}^2 \frac{\partial \phi}{\partial q^2} + \mathbf{e}^3 \frac{\partial \phi}{\partial q^3}\right). \tag{2.45}$$

The derivative $(d\phi/ds)_{\hat{\mathbf{e}}}$ is called the *directional derivative* of ϕ. We see that it is the *rate of change* of ϕ with respect to distance and that it depends on the direction $\hat{\mathbf{e}}$ in which the distance is taken.

The vector that is scalar multiplied by $\hat{\mathbf{e}}$ can be obtained immediately whenever the scalar field is given. Because the magnitude of this vector is equal to the maximum value of the directional derivative, it is called the *gradient vector* and is denoted by grad ϕ:

$$\text{grad}\,\phi \equiv \mathbf{e}^1 \frac{\partial \phi}{\partial q^1} + \mathbf{e}^2 \frac{\partial \phi}{\partial q^2} + \mathbf{e}^3 \frac{\partial \phi}{\partial q^3}. \tag{2.46}$$

From this representation it can be seen that

$$\frac{\partial \phi}{\partial q^1}, \qquad \frac{\partial \phi}{\partial q^2}, \qquad \frac{\partial \phi}{\partial q^3}$$

are the *covariant components* of the gradient vector.

When the scalar function $\phi(\mathbf{r})$ is set equal to a constant, $\phi(\mathbf{r}) = \text{constant}$, a family of surfaces is generated. A different surface is designated by different values of the constant, and each surface is called a *level surface* (see Fig. 2.10). If the direction in which the directional derivative is taken lies within a level surface, then $d\phi/ds$ is zero, since ϕ is a constant on a level surface. In this case the unit vector $\hat{\mathbf{e}}$ is tangent to a level surface. It follows, therefore, that if $d\phi/ds$ is zero, then grad ϕ must be perpendicular to $\hat{\mathbf{e}}$ and thus *perpendicular to a level surface*. Thus if any surface is given by $\phi(\mathbf{r}) = \text{constant}$, the unit normal to the surface is determined by

$$\hat{\mathbf{n}} = \pm \frac{\text{grad}\,\phi}{|\,\text{grad}\,\phi|}. \tag{2.47}$$

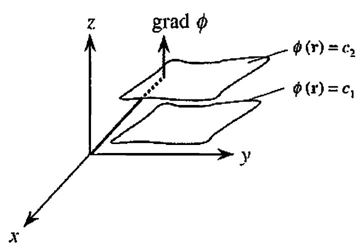


Figure 2.10 Level surfaces.

The plus or minus sign appears because the direction of $\hat{\mathbf{n}}$ may point in either direction away from the surface. If the surface is closed, the usual convention is to take $\hat{\mathbf{n}}$ pointing outward.

It is convenient to write the gradient vector as

$$\text{grad}\,\phi \equiv \left(\mathbf{e}^1\frac{\partial}{\partial q^1} + \mathbf{e}^2\frac{\partial}{\partial q^2} + \mathbf{e}^3\frac{\partial}{\partial q^3}\right)\phi \tag{2.48}$$

and interpret grad ϕ as some operator operating on ϕ, that is, grad $\phi \equiv \nabla\phi$. This operator is denoted by

$$\nabla \equiv \mathbf{e}^1\frac{\partial}{\partial q^1} + \mathbf{e}^2\frac{\partial}{\partial q^2} + \mathbf{e}^3\frac{\partial}{\partial q^3} \tag{2.49}$$

and is called the del operator. The del operator is a *vector differential* operator, and the "components" $\partial/\partial q^1$, $\partial/\partial q^2$, and ∂/q^3 appear as covariant components.

It is important to note that whereas the del operator has some of the properties of a vector, it does not have them all, because it is an operator. For instance $\nabla \cdot \mathbf{A}$ is a scalar (called the divergence of $\mathbf{A}$) whereas $\mathbf{A} \cdot \nabla$ is a scalar *differential operator*. Thus the del operator does not commute in this sense.

In Cartesian systems we have the simple form

$$\nabla \equiv \hat{\mathbf{e}}_x\frac{\partial}{\partial x} + \hat{\mathbf{e}}_y\frac{\partial}{\partial y} + \hat{\mathbf{e}}_z\frac{\partial}{\partial z}. \tag{2.50a}$$

or, in the summation convention, we have

$$\nabla \equiv \hat{\mathbf{e}}_i\frac{\partial}{\partial x_i}. \tag{2.50b}$$

The dot product of del operator with a vector is called the *divergence of a vector* and denoted by

$$\nabla \cdot \mathbf{A} \equiv \text{div}\,\mathbf{A}. \tag{2.51}$$

If we take the divergence of the gradient vector, we have

$$\text{div}(\text{grad}\,\phi) \equiv \nabla \cdot \nabla\phi = (\nabla \cdot \nabla)\phi = \nabla^2\phi. \tag{2.52}$$

The notation $\nabla^2 = \nabla \cdot \nabla$ is called the *Laplacian operator*. In Cartesian systems this reduces to the simple form

$$\nabla^2\phi = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2} = \frac{\partial^2\phi}{\partial x_i \partial x_i}. \tag{2.53}$$

The Laplacian of a scalar appears frequently in the partial differential equations governing physical phenomena.

The curl of a vector is defined as the del operator operating on a vector by means of the cross product

$$\operatorname{curl} \mathbf{A} = \nabla \times \mathbf{A} \tag{2.53a}$$

$$= \varepsilon_{ijk} \mathbf{e}_i \frac{\partial A_k}{\partial x_j} \tag{2.53b}$$

Example 2.5 Using the index summation notation we prove the following identity in a Cartesian system

$$\nabla \times (\nabla \times \mathbf{v}) = \nabla(\nabla \cdot \mathbf{v}) - \nabla^2 \mathbf{v}$$

where $\mathbf{v}$ is a vector function of the coordinates x_i. Observe that

$$\nabla \times (\nabla \times \mathbf{v}) = \mathbf{e}_i \frac{\partial}{\partial x_i} \times \left(\mathbf{e}_j \frac{\partial}{\partial x_j} \times v_k \mathbf{e}_k \right)$$

$$= \mathbf{e}_i \frac{\partial}{\partial x_i} \times \left(\varepsilon_{jkl} \frac{\partial v_k}{\partial x_j} \mathbf{e}_l \right)$$

$$= \varepsilon_{ilm} \varepsilon_{jkl} \frac{\partial^2 v_k}{\partial x_i \, \partial x_j} \mathbf{e}_m$$

Using the ε–δ identity we obtain

$$\nabla \times (\nabla \times \mathbf{v}) = (\delta_{mj}\delta_{ik} - \delta_{mk}\delta_{ij}) \frac{\partial^2 v_k}{\partial x_i \partial x_j} \mathbf{e}_m$$

$$= \frac{\partial^2 v_i}{\partial x_i \, \partial x_j} \mathbf{e}_j - \frac{\partial^2 v_k}{\partial x_i \partial x_i} \hat{\mathbf{e}}_k = \mathbf{e}_j \frac{\partial}{\partial x_j} \left(\frac{\partial v_i}{\partial x_i} \right) - \frac{\partial^2}{\partial x_i \, \partial x_i} (v_k \mathbf{e}_k)$$

$$= \nabla (\nabla \cdot \mathbf{v}) - \nabla^2 \mathbf{v}$$

This result is sometimes used as the definition of the Laplacian of a vector, that is,

$$\nabla^2 \mathbf{v} = \operatorname{grad}(\operatorname{div} \mathbf{v}) - \operatorname{curl} \operatorname{curl} \mathbf{v}$$

A summary of vector operations in both general vector notation and in Cartesian component form is given in Table 2.1, and some useful vector operations for cylindrical and spherical coordinate systems are shown in Table 2.2.

2.2.6 Integral Relations

Useful expressions for the gradient of a vector, divergence of a vector, and curl of a vector can be established from integral relations between volume integrals and

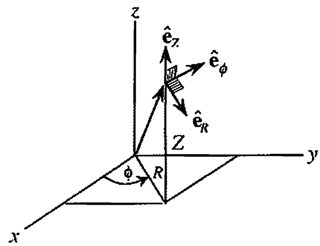
Table 2.1 Vector terms and operations in Cartesian component form. Here **A**, **B** and **C** are vector functions and U is a scalar function of the Cartesian coordinates (x_1, x_2, x_3). $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ are the Cartesian unit vectors

No	Vector Form	Cartesian Component Form
1	$\mathbf{A}$	$A_i \mathbf{e}_i$
2	$\mathbf{A} \cdot \mathbf{B}$	$A_i B_i$
3	$\mathbf{A} \times \mathbf{B}$	$\varepsilon_{ijk} A_i B_j \mathbf{e}_k$
4	$\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C})$	$\varepsilon_{ijk} A_i B_j C_k$
5	$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - \mathbf{C}(\mathbf{A} \cdot \mathbf{B})$	$\varepsilon_{ijk} \varepsilon_{klm} A_j B_l C_m \mathbf{e}_i$
6	∇U	$\frac{\partial U}{\partial x_i} \mathbf{e}_i$
7	$\nabla \mathbf{A}$	$\frac{\partial A_j}{\partial x_i} \mathbf{e}_i \mathbf{e}_j$
8	$\nabla \cdot \mathbf{A}$	$\frac{\partial A_i}{\partial x_i}$
9	$\nabla \times \mathbf{A}$	$\varepsilon_{ijk} \frac{\partial A_j}{\partial x_i} \mathbf{e}_k$
10	$\nabla \cdot (\mathbf{A} \times \mathbf{B}) = \mathbf{B} \cdot (\nabla \times \mathbf{A}) - \mathbf{A} \cdot (\nabla \times \mathbf{B})$	$\varepsilon_{ijk} \frac{\partial}{\partial x_i} (A_j B_k)$
11	$\nabla \cdot (U\mathbf{A}) = U \nabla \cdot \mathbf{A} + \nabla U \cdot \mathbf{A}$	$\frac{\partial}{\partial x_i} (U A_i)$
12	$\nabla \times (U\mathbf{A}) = \nabla U \times \mathbf{A} + U \nabla \times \mathbf{A}$	$\varepsilon_{ijk} \frac{\partial}{\partial x_j} (U A_k) \mathbf{e}_i$
13	$\nabla(U\mathbf{A}) = \nabla U \mathbf{A} + U \nabla \mathbf{A}$	$\mathbf{e}_j \frac{\partial}{\partial x_j} (U A_k \mathbf{e}_k)$
14	$\nabla \times (\mathbf{A} \times \mathbf{B}) = \mathbf{A}(\nabla \cdot \mathbf{B}) - \mathbf{B}(\nabla \cdot \mathbf{A}) +$ $\mathbf{B} \cdot \nabla \mathbf{A} - \mathbf{A} \cdot \nabla \mathbf{B}$	$\varepsilon_{ijk} \varepsilon_{klm} \frac{\partial}{\partial x_j} (A_l B_m) \mathbf{e}_i$
15	$(\nabla \times \mathbf{A}) \times \mathbf{B} = \mathbf{B} \cdot [\nabla \mathbf{A} - (\nabla \mathbf{A})^{T}]$	$\varepsilon_{ijk} \varepsilon_{klm} B_l \frac{\partial A_j}{\partial x_i} \mathbf{e}_m$
16	$\nabla \cdot (\nabla U) = \nabla^2 U$	$\frac{\partial^2 U}{\partial x_i \partial x_i}$
17	$\nabla \cdot (\nabla \mathbf{A}) = \nabla^2 \mathbf{A}$	$\frac{\partial^2 A_j}{\partial x_i \partial x_i} \mathbf{e}_j$
18	$\nabla \times \nabla \times \mathbf{A} = \nabla(\nabla \cdot \mathbf{A}) - (\nabla \cdot \nabla)\mathbf{A}$	$\varepsilon_{mil} \varepsilon_{jkl} \frac{\partial^2 A_k}{\partial x_i \partial x_j} \mathbf{e}_m$
19	$(\mathbf{A} \cdot \nabla)\mathbf{B}$	$A_j \frac{\partial B_i}{\partial x_j} \mathbf{e}_i$
20	$\mathbf{A}(\nabla \cdot \mathbf{B})$	$A_i \mathbf{e}_i \frac{\partial B_j}{\partial x_j}$

surface integrals. Let V denote a region in space surrounded by the surface S. Let dS be a differential element of surface and $\mathbf{n}$ the unit outward normal and dV be a differential volume element. The following relations are taken from advanced calculus.

Table 2.2 The del and Laplace operators in cylindrical and spherical coordinate systems

Cylindrical coordinate system (R, ϕ, z)



$$x = R\cos\phi$$
$$y = R\sin\phi$$
$$z = z$$

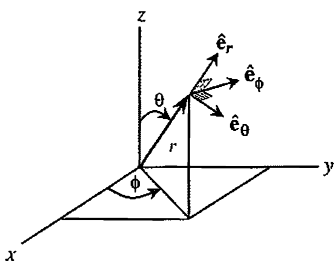
$$\hat{\mathbf{e}}_R = \cos\phi\,\hat{\mathbf{e}}_x + \sin\phi\,\hat{\mathbf{e}}_y$$
$$\hat{\mathbf{e}}_\phi = -\sin\phi\,\hat{\mathbf{e}}_x + \cos\phi\,\hat{\mathbf{e}}_y$$
$$\hat{\mathbf{e}}_z = \hat{\mathbf{e}}_z$$

$$\frac{\partial \hat{\mathbf{e}}_R}{\partial \phi} = -\sin\phi\,\hat{\mathbf{e}}_x + \cos\phi\,\hat{\mathbf{e}}_y = \hat{\mathbf{e}}_\phi$$
$$\frac{\partial \hat{\mathbf{e}}_\phi}{\partial \phi} = -\cos\phi\,\hat{\mathbf{e}}_x - \sin\phi\,\hat{\mathbf{e}}_y = -\hat{\mathbf{e}}_R$$

All other derivatives of the base vectors are zero.

$$\nabla = \hat{\mathbf{e}}_R\frac{\partial}{\partial R} + \frac{1}{R}\hat{\mathbf{e}}_\phi\frac{\partial}{\partial \phi} + \hat{\mathbf{e}}_z\frac{\partial}{\partial z}$$
$$\nabla^2 F = \frac{1}{R}\left[\frac{\partial}{\partial R}\left(R\frac{\partial F}{\partial R}\right) + \frac{1}{R}\frac{\partial^2 F}{\partial \phi^2} + R\frac{\partial^2 F}{\partial z^2}\right]$$

Spherical coordinate system (r, θ, ϕ)



$$x = r\sin\theta\cos\phi$$
$$y = r\sin\theta\sin\phi$$
$$z = r\cos\theta$$

$$\hat{\mathbf{e}}_r = \sin\theta\cos\phi\,\hat{\mathbf{e}}_x + \sin\theta\sin\phi\,\hat{\mathbf{e}}_y + \cos\theta\,\hat{\mathbf{e}}_z$$
$$\hat{\mathbf{e}}_\theta = \cos\theta\cos\phi\,\hat{\mathbf{e}}_x + \cos\theta\sin\phi\,\hat{\mathbf{e}}_y - \sin\theta\,\hat{\mathbf{e}}_z$$
$$\hat{\mathbf{e}}_\phi = -\sin\phi\,\hat{\mathbf{e}}_x + \cos\phi\,\hat{\mathbf{e}}_y$$

$$\frac{\partial \hat{\mathbf{e}}_r}{\partial \theta} = \hat{\mathbf{e}}_\theta$$
$$\frac{\partial \hat{\mathbf{e}}_r}{\partial \phi} = \sin\theta\,\hat{\mathbf{e}}_\phi$$
$$\frac{\partial \hat{\mathbf{e}}_\theta}{\partial \theta} = -\hat{\mathbf{e}}_r$$
$$\frac{\partial \hat{\mathbf{e}}_\theta}{\partial \phi} = \cos\theta\,\hat{\mathbf{e}}_\phi$$
$$\frac{\partial \hat{\mathbf{e}}_\phi}{\partial \phi} = -\sin\theta\,\hat{\mathbf{e}}_r - \cos\theta\,\hat{\mathbf{e}}_\theta$$

All other derivatives of the base vectors are zero.

$$\nabla = \hat{\mathbf{e}}_r\frac{\partial}{\partial r} + \frac{1}{r}\hat{\mathbf{e}}_\theta\frac{\partial}{\partial \theta} + \frac{1}{r\sin\theta}\hat{\mathbf{e}}_\phi\frac{\partial}{\partial \phi}$$
$$\nabla^2 F = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial F}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial F}{\partial \theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 F}{\partial \phi^2}$$

Gradient Theorem

$$\int_V \operatorname{grad}\phi\, dV = \oint_S \hat{\mathbf{n}}\phi\, dS. \tag{2.54a}$$

Divergence Theorem

$$\int_V \operatorname{div} \mathbf{A}\, dV = \oint_S \hat{\mathbf{n}} \cdot \mathbf{A}\, dS. \tag{2.54b}$$

Curl Theorem

$$\int_V \operatorname{curl} \mathbf{A}\, dV = \oint_S \hat{\mathbf{n}} \times \mathbf{A}\, dS. \tag{2.54c}$$

Now let $\mathbf{A} = \operatorname{grad} \phi$ in Eq. (2.54b). Then the divergence theorem gives

$$\int_V \operatorname{div}(\operatorname{grad} \phi)\, dV \equiv \int_V \nabla^2 \phi\, dV = \oint_S \hat{\mathbf{n}} \cdot \operatorname{grad} \phi\, dS. \tag{2.55}$$

The quantity $\hat{\mathbf{n}} \cdot \operatorname{grad} \phi$ is called the *normal derivative* of ϕ on the surface S, and is denoted by

$$\frac{\partial \phi}{\partial n} \equiv \hat{\mathbf{n}} \cdot \operatorname{grad} \phi = \hat{\mathbf{n}} \cdot \nabla \phi. \tag{2.56}$$

In a Cartesian system this becomes

$$\frac{\partial \phi}{\partial n} = \frac{\partial \phi}{\partial x} n_x + \frac{\partial \phi}{\partial y} n_y + \frac{\partial \phi}{\partial z} n_z, \tag{2.57}$$

where n_x, n_y, and n_z are the direction cosines of the unit normal

$$\hat{\mathbf{n}} = n_x \hat{\mathbf{e}}_x + n_y \hat{\mathbf{e}}_y + n_z \hat{\mathbf{e}}_z. \tag{2.58}$$

Example 2.6 This example illustrates the relation between the integral relations (2.54a–c) and the so-called integration by parts. Consider a rectangular region $R = \{(x, y){:}\ 0 < x < a, 0 < y < b\}$ with boundary C, which is the union of line segments C_1, C_2, C_3, and C_4 (see Fig. 2.11).

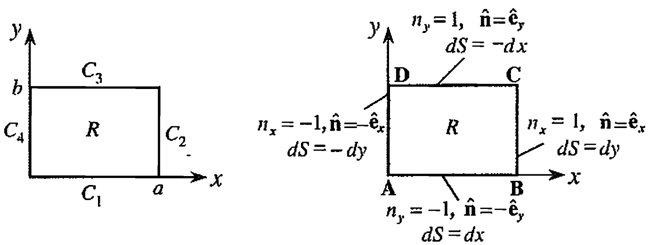


Figure 2.11 Integration over rectangular regions.

Suppose that we wish to evaluate the integral $\int_R \nabla^2\phi\, dxdy$. From Eq. (2.55) we have

$$\int_R \nabla^2\phi\, dxdy = \int_R \nabla\cdot(\nabla\phi)\, dxdy = \oint_c \frac{\partial\phi}{\partial n}\, dS.$$

The line integral can be simplified for the region under consideration as follows (note that in two dimensions, the volume integral becomes an area integral):

$$\begin{aligned}
\oint_C \frac{\partial\phi}{\partial n} dS &= \int_{C_1} \frac{\partial\phi}{\partial n} dS + \int_{C_2} \frac{\partial\phi}{\partial n} dS + \int_{C_3} \frac{\partial\phi}{\partial n} dS + \int_{C_4} \frac{\partial\phi}{\partial n} dS \\
&= \int_0^a \left(-\frac{\partial\phi}{\partial y}\right)\bigg|_{y=0} dx + \int_0^b \left(\frac{\partial\phi}{\partial x}\right)\bigg|_{x=a} dy \\
&\quad + \int_a^0 \left(\frac{\partial\phi}{\partial y}\right)\bigg|_{y=b} (-dx) + \int_b^0 \left(-\frac{\partial\phi}{\partial x}\right)\bigg|_{x=0} (-dy) \\
&= \int_0^a \left[\left(\frac{\partial\phi}{\partial y}\right)_{y=b} - \left(\frac{\partial\phi}{\partial y}\right)_{y=0}\right] dx \\
&\quad + \int_0^b \left[\left(\frac{\partial\phi}{\partial x}\right)_{x=a} - \left(\frac{\partial\phi}{\partial x}\right)_{x=0}\right] dy.
\end{aligned}$$

The same result can be obtained by means of integration by parts:

$$\begin{aligned}
\int_R \nabla^2 \phi\, dx\, dy &= \int_0^b \int_0^a \left(\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2}\right) dxdy \\
&= \int_0^b \int_0^a \frac{\partial}{\partial x}\left(\frac{\partial\phi}{\partial x}\right) dxdy + \int_0^a \int_0^b \frac{\partial}{\partial y}\left(\frac{\partial\phi}{\partial y}\right) dydx \\
&= \int_0^b \left(\frac{\partial\phi}{\partial x}\right)\bigg|_{x=0}^{x=a} dy + \int_0^a \left(\frac{\partial\phi}{\partial y}\right)\bigg|_{y=0}^{y=b} dx \\
&= \int_0^b \left[\left(\frac{\partial\phi}{\partial x}\right)_{x=a} - \left(\frac{\partial\phi}{\partial x}\right)_{x=0}\right] dy \\
&\quad + \int_0^a \left[\left(\frac{\partial\phi}{\partial y}\right)_{y=b} - \left(\frac{\partial\phi}{\partial y}\right)_{y=0}\right] dx.
\end{aligned}$$

Thus integration by parts is a special case of the gradient or the divergence theorem.

2.3 TENSORS

2.3.1 Second-Order Tensors

To introduce the concept of a second-order tensor, also called a *dyadic*, we consider the equilibrium of an element of a continuum acted upon by forces. The surface force acting on a small element of area in a continuous medium depends not only on the magnitude of the area but also upon the orientation of the area. It is customary to denote the direction of a plane area by means of a unit vector drawn normal to that plane (see Fig. 2.12a). To fix the direction of the normal, we assign a *sense of travel* along the contour of the boundary of the plane area in question. The direction of the normal is taken by convention as that in which a right-handed screw advances as it is rotated according to the sense of travel along the boundary curve or contour (see Fig. 2.12b). Let the unit normal vector be given by $\hat{\mathbf{n}}$. Then the area can be denoted by $\mathbf{S} = S\hat{\mathbf{n}}$.

If we denote by $\Delta\mathbf{F}(\hat{\mathbf{n}})$ the force on a small area $\hat{\mathbf{n}}\Delta S$ located at the position $\mathbf{r}$ (see Fig. 2.13), the *stress vector* can be defined as follows:

$$\mathbf{t}(\hat{\mathbf{n}}) = \lim_{\Delta S \to 0} \frac{\Delta\mathbf{F}(\hat{\mathbf{n}})}{\Delta S}. \tag{2.59}$$

We see that the stress vector is a point function of the unit normal $\hat{\mathbf{n}}$ which denotes the orientation of the surface ΔS. The component of $\mathbf{t}$ that is in the direction of $\hat{\mathbf{n}}$ is called the *normal stress*. The component of $\mathbf{t}$ that is normal to $\hat{\mathbf{n}}$ is called a *shear stress*. Because of Newton's third law for action and reaction, we see that $\mathbf{t}(-\hat{\mathbf{n}}) = -\mathbf{t}(\hat{\mathbf{n}})$.

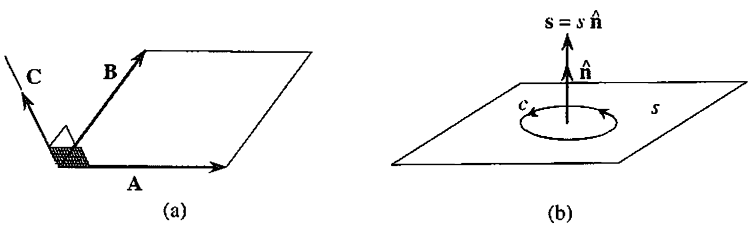


Figure 2.12 (a) Plane area as a vector. (b) Unit normal vector and sense of travel.

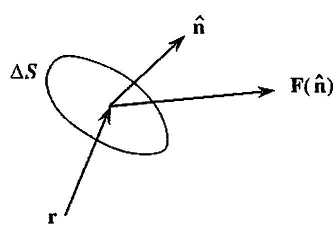


Figure 2.13 Force on an area element.

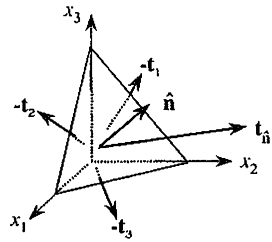


Figure 2.14 Tetrahedral element in Cartesian coordinates.

At a fixed point $\mathbf{r}$ for each given unit vector $\hat{\mathbf{n}}$ there is a stress vector $\mathbf{t}(\hat{\mathbf{n}})$ acting on the plane normal to $\hat{\mathbf{n}}$. Note that $\mathbf{t}(\hat{\mathbf{n}})$ is, in general, not in the direction of $\hat{\mathbf{n}}$. It is fruitful to establish a relationship between $\mathbf{t}$ and $\hat{\mathbf{n}}$. To do this, we now set up an infinitesimal tetrahedron in Cartesian coordinates as shown in Fig. 2.14.

If $-\mathbf{t}_1$, $-\mathbf{t}_2$, $-\mathbf{t}_3$, and $\mathbf{t}$ denote the stress vectors in the outward directions on the faces of the infinitesimal tetrahedron whose areas are ΔS_1, ΔS_2, ΔS_3, and ΔS, respectively, we have by Newton's second law for the mass inside the tetrahedron,

$$\mathbf{t}\Delta S - \mathbf{t}_1\Delta S_1 - \mathbf{t}_2\Delta S_2 - \mathbf{t}_3\Delta S_3 + \rho\Delta V\mathbf{f} = \rho\Delta V\mathbf{a}, \tag{2.60}$$

where ΔV is the volume of the tetrahedron, ρ the density, $\mathbf{f}$ the body force per unit mass, and $\mathbf{a}$ the acceleration. Since the total vector area of a closed surface is zero [see the gradient theorem; set $\phi = 1$ in (2.54a)], we have

$$\Delta S\hat{\mathbf{n}} - \Delta S_1\hat{\mathbf{e}}_1 - \Delta S_2\hat{\mathbf{e}}_2 - \Delta S_3\hat{\mathbf{e}}_3 = \mathbf{0}. \tag{2.61}$$

It follows that

$$\Delta S_1 = (\hat{\mathbf{n}}\cdot\hat{\mathbf{e}}_1)\Delta S, \qquad \Delta S_2 = (\hat{\mathbf{n}}\cdot\hat{\mathbf{e}}_2)\Delta S, \qquad \Delta S_3 = (\hat{\mathbf{n}}\cdot\hat{\mathbf{e}}_3)\Delta S. \tag{2.62}$$

The volume of the element ΔV can be expressed as

$$\Delta V = \frac{\Delta h}{3}\Delta S, \tag{2.63}$$

where Δh is the perpendicular distance from the origin to the slant face. The result in (2.63) can also be obtained from the divergence theorem (2.54b) by setting $\mathbf{A} = \mathbf{r}$.

Substitution of Eqs. (2.62) and (2.63) in (2.60) and dividing throughout by ΔS reduces it to

$$\mathbf{t} = (\hat{\mathbf{n}}\cdot\hat{\mathbf{e}}_1)\mathbf{t}_1 + (\hat{\mathbf{n}}\cdot\hat{\mathbf{e}}_2)\mathbf{t}_2 + (\hat{\mathbf{n}}\cdot\hat{\mathbf{e}}_3)\mathbf{t}_3 + \rho\frac{\Delta h}{3}(\mathbf{a} - \mathbf{f}). \tag{2.64}$$

In the limit when the tetrahedron shrinks to a point, $\Delta h \to 0$, we are left with

$$\begin{aligned}\mathbf{t} &= (\hat{\mathbf{n}}\cdot\hat{\mathbf{e}}_1)\mathbf{t}_1 + (\hat{\mathbf{n}}\cdot\hat{\mathbf{e}}_2)\mathbf{t}_2 + (\hat{\mathbf{n}}\cdot\hat{\mathbf{e}}_3)\mathbf{t}_3 \\ &= (\hat{\mathbf{n}}\cdot\hat{\mathbf{e}}_i)\mathbf{t}_i.\end{aligned} \tag{2.65}$$

It is now convenient to display the above equation as

$$\mathbf{t} = \hat{\mathbf{n}} \cdot (\hat{\mathbf{e}}_1 \mathbf{t}_1 + \hat{\mathbf{e}}_2 \mathbf{t}_2 + \hat{\mathbf{e}}_3 \mathbf{t}_3). \tag{2.66}$$

The terms in the parenthesis are to be treated as a dyadic, called *stress dyadic* or *stress tensor* $\overleftrightarrow{\sigma}$:

$$\overleftrightarrow{\sigma} \equiv \hat{\mathbf{e}}_1 \mathbf{t}_1 + \hat{\mathbf{e}}_2 \mathbf{t}_2 + \hat{\mathbf{e}}_3 \mathbf{t}_3. \tag{2.67}$$

The stress tensor is a property of the medium that is independent of the $\hat{\mathbf{n}}$. Thus, we have

$$\mathbf{t}(\hat{\mathbf{n}}) = \hat{\mathbf{n}} \cdot \overleftrightarrow{\sigma} \tag{2.68}$$

and the dependence of $\mathbf{t}$ on $\hat{\mathbf{n}}$ has been explicitly displayed.

It is useful to resolve the stress vectors $\mathbf{t}_1, \mathbf{t}_2$, and $\mathbf{t}_3$ into their orthogonal components. We have

$$\begin{aligned} \mathbf{t}_i &= \sigma_{i1}\hat{\mathbf{e}}_1 + \sigma_{i2}\hat{\mathbf{e}}_2 + \sigma_{i3}\hat{\mathbf{e}}_3 \\ &= \sigma_{ij}\hat{\mathbf{e}}_j \end{aligned} \tag{2.69}$$

for $i = 1, 2, 3$. Hence, the stress dyadic can be expressed in summation notation as

$$\begin{aligned} \overleftrightarrow{\sigma} &= \hat{\mathbf{e}}_i \mathbf{t}_i \\ &= \sigma_{ij}\hat{\mathbf{e}}_i\hat{\mathbf{e}}_j. \end{aligned} \tag{2.70}$$

The component σ_{ij} represents the stress (force per unit area) on an area perpendicular to the ith coordinate and in the jth coordinate direction (see Fig. 2.15). The stress vector $\mathbf{t}$ represents the vectorial stress on an area perpendicular to the direction $\hat{\mathbf{n}}$. Equation (2.68) is known as the *Cauchy stress formula*, and $\overleftrightarrow{\sigma}$ is termed the *Cauchy stress tensor*.

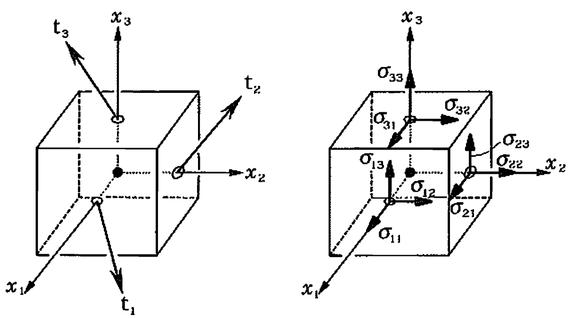


Figure 2.15 Definition of stress components in Cartesian rectangular coordinates.

2.3.2 General Properties of a Dyadic

Because of its utilization in physical applications, a dyad is defined as two vectors standing side by side and acting as a unit. A linear combination of dyads is called a dyadic. Let $\mathbf{A}_1, \mathbf{A}_2, \ldots, \mathbf{A}_n$ and $\mathbf{B}_1, \mathbf{B}_2, \ldots, \mathbf{B}_n$ be arbitrary vectors. Then we can represent a dyadic as

$$\overleftrightarrow{\mathbf{\Phi}} = \mathbf{A}_1\mathbf{B}_1 + \mathbf{A}_2\mathbf{B}_2 + \cdots + \mathbf{A}_n\mathbf{B}_n. \tag{2.71}$$

Here, we limit our discussion to Cartesian tensors. For a Cartesian tensor the basis vectors are constants and thus do not take roles as variables in differentiation and integration.

One of the properties of a dyadic is defined by the dot product with a vector, say $\mathbf{V}$:

$$\begin{aligned} \overleftrightarrow{\mathbf{\Phi}} \cdot \mathbf{V} &= \mathbf{A}_1(\mathbf{B}_1 \cdot \mathbf{V}) + \mathbf{A}_2(\mathbf{B}_2 \cdot \mathbf{V}) + \cdots + \mathbf{A}_n(\mathbf{B}_n \cdot \mathbf{V}), \\ \mathbf{V} \cdot \overleftrightarrow{\mathbf{\Phi}} &= (\mathbf{V} \cdot \mathbf{A}_1)\mathbf{B}_1 + (\mathbf{V} \cdot \mathbf{A}_2)\mathbf{B}_2 + \cdots + (\mathbf{V} \cdot \mathbf{A}_n)\mathbf{B}_n. \end{aligned} \tag{2.72}$$

The dot operation with a vector produces another vector. In the first case the dyadic acts as a *prefactor* and in the second case as a *postfactor*. The two operations in general produce different vectors.

The conjugate, or transpose, of a dyadic is defined as the result obtained by the interchange of the two vectors in each of the dyads:

$$\overleftrightarrow{\mathbf{\Phi}}^{\mathrm{T}} = \mathbf{B}_1\mathbf{A}_1 + \mathbf{B}_2\mathbf{A}_2 + \cdots + \mathbf{B}_n\mathbf{A}_n. \tag{2.73}$$

It is clear that we have

$$\begin{aligned} \mathbf{V} \cdot \overleftrightarrow{\mathbf{\Phi}} &= \overleftrightarrow{\mathbf{\Phi}}^{\mathrm{T}} \cdot \mathbf{V}, \\ \overleftrightarrow{\mathbf{\Phi}} \cdot \mathbf{V} &= \mathbf{V} \cdot \overleftrightarrow{\mathbf{\Phi}}^{\mathrm{T}}. \end{aligned} \tag{2.74}$$

2.3.3 Nonion Form of a Dyadic

Let each of the vectors in the dyadic be represented in a given basis system. In Cartesian system, we have

$$\begin{aligned} \mathbf{A}_i &= A_{ij}\mathbf{e}_j, \\ \mathbf{B}_i &= B_{ik}\mathbf{e}_k. \end{aligned} \tag{2.75}$$

The summations on j and k are implied by the repeated indices.

We can display all of the components of a dyadic $\overleftrightarrow{\Phi}$ by letting the k index run to the right and the j index run downward:

$$\begin{aligned}\overleftrightarrow{\Phi} &= \phi_{11}\hat{\mathbf{e}}_1\hat{\mathbf{e}}_1 + \phi_{12}\hat{\mathbf{e}}_1\hat{\mathbf{e}}_2 + \phi_{13}\hat{\mathbf{e}}_1\hat{\mathbf{e}}_3 \\ &\quad + \phi_{21}\hat{\mathbf{e}}_2\hat{\mathbf{e}}_1 + \phi_{22}\hat{\mathbf{e}}_2\hat{\mathbf{e}}_2 + \phi_{23}\hat{\mathbf{e}}_2\hat{\mathbf{e}}_3 \\ &\quad + \phi_{31}\hat{\mathbf{e}}_3\hat{\mathbf{e}}_1 + \phi_{32}\hat{\mathbf{e}}_3\hat{\mathbf{e}}_2 + \phi_{33}\hat{\mathbf{e}}_3\hat{\mathbf{e}}_3. \end{aligned} \tag{2.76}$$

This form is called the *nonion* form. Equation (2.63) illustrates that a dyadic in three-dimensional space, or what we shall call a second-order tensor, has nine independent components in general, each component associated with a certain dyad pair. The components are thus said to be ordered. When the ordering is understood, such as suggested by the nonion form (2.76), the explicit writing of the dyads can be suppressed and the dyadic written as an array:

$$[\Phi] = \begin{bmatrix} \phi_{11} & \phi_{12} & \phi_{13} \\ \phi_{21} & \phi_{22} & \phi_{23} \\ \phi_{31} & \phi_{32} & \phi_{33} \end{bmatrix} \quad \text{and} \quad \overleftrightarrow{\Phi} = \begin{Bmatrix} \hat{\mathbf{e}}_1 \\ \hat{\mathbf{e}}_2 \\ \hat{\mathbf{e}}_3 \end{Bmatrix}^{\mathrm{T}} [\Phi] \begin{Bmatrix} \hat{\mathbf{e}}_1 \\ \hat{\mathbf{e}}_2 \\ \hat{\mathbf{e}}_3 \end{Bmatrix}. \tag{2.77}$$

This representation is simpler than Eq. (2.76), but it is taken to mean the same.

In the general scheme that is thus developed, vectors are called *first-order tensors* and dyadics are called *second-order tensors*. Scalars are called *zeroth-order tensors*. The generalization to *third-order tensors* thus leads, or is derived from, *triadics*, or three vectors standing side by side. It follows that higher-order tensors are developed from *polyadics*.

Example 2.7 With reference to a rectangular Cartesian system (x_1, x_2, x_3), the components of the stress dyadic at a certain point of a continuous medium are given by

$$[\sigma] = \begin{bmatrix} 200 & 400 & 300 \\ 400 & 0 & 0 \\ 300 & 0 & -100 \end{bmatrix} \text{psi}.$$

We wish to determine the stress vector $\mathbf{t}$ at the point and normal to the plane, $x_1 + 2x_2 + 2x_3 - 6 = 0$, and then compute the normal and tangential components of the stress vector at the point.

First we should find the unit normal to the plane on which we are required to find the stress vector. The unit normal is given by [see Eq. (2.47)]

$$\hat{\mathbf{n}} = \frac{\nabla P}{|\nabla P|}, \qquad P(x_1, x_2, x_3) = x_1 + 2x_2 + 2x_3 - 6,$$

$$\hat{\mathbf{n}} = \frac{1}{3}(\hat{\mathbf{e}}_1 + 2\hat{\mathbf{e}}_2 + 2\hat{\mathbf{e}}_3).$$

The components of the stress vector are displayed in an array

$$\begin{Bmatrix} t_1 \\ t_2 \\ t_3 \end{Bmatrix} = \begin{bmatrix} 200 & 400 & 300 \\ 400 & 0 & 0 \\ 300 & 0 & -100 \end{bmatrix} \frac{1}{3} \begin{Bmatrix} 1 \\ 2 \\ 2 \end{Bmatrix} = \frac{1}{3} \begin{Bmatrix} 1600 \\ 400 \\ 100 \end{Bmatrix} \text{ psi}$$

or

$$\mathbf{t}(\hat{\mathbf{n}}) = \frac{1}{3}(1600\hat{\mathbf{e}}_1 + 400\hat{\mathbf{e}}_2 + 100\hat{\mathbf{e}}_3) \text{ psi.}$$

The normal component t_n of the stress vector $\mathbf{t}$ on the plane is given by

$$t_n = \mathbf{t}(\hat{\mathbf{n}}) \cdot \hat{\mathbf{n}} = \frac{2600}{9} \text{ psi,}$$

and the tangential component is given by (the Pythagorean theorem)

$$t_s = \sqrt{|\mathbf{t}|^2 - t_n^2} = \frac{10^2}{9}\sqrt{(256 + 16 + 1)9 - 26 \times 26} \text{ psi}$$
$$= 100\frac{\sqrt{1781}}{9} = 468.9 \text{ psi.}$$

A second-order Cartesian tensor $\overleftrightarrow{\mathbf{\Phi}}$ may be represented in barred and unbarred coordinate systems as

$$\overleftrightarrow{\mathbf{\Phi}} = \phi_{ij}\hat{\mathbf{e}}_i\hat{\mathbf{e}}_j$$
$$= \bar{\phi}_{kl}\hat{\bar{\mathbf{e}}}_k\hat{\bar{\mathbf{e}}}_l.$$

The unit base vectors in the barred and unbarred systems are related by

$$\hat{\mathbf{e}}_i = \frac{\partial \bar{x}_j}{\partial x_i}\hat{\bar{\mathbf{e}}}_j \equiv \beta_{ji}\hat{\bar{\mathbf{e}}}_j, \qquad \beta_{ij} = \hat{\bar{\mathbf{e}}}_i \cdot \hat{\mathbf{e}}_j, \tag{2.78}$$

where β_{ij} denote the direction cosines between barred and unbarred systems [see Eq. (2.29)]. Thus the components of a second-order tensor transform according to

$$\bar{\phi}_{kl} = \phi_{ij}\beta_{ki}\beta_{lj} \qquad \text{or} \qquad [\bar{\phi}] = [\beta][\phi][\beta]^{\mathrm{T}}. \tag{2.79}$$

In right-handed orthogonal systems the determinant of the transformation matrix is unity, and we have

$$[\beta]^{-1} = [\beta]^{\mathrm{T}}. \tag{2.80}$$

The unit tensor is defined as

$$\overleftrightarrow{\mathbf{I}} = \hat{\mathbf{e}}_i\hat{\mathbf{e}}_i. \tag{2.81}$$

With the help of the Kronecker delta symbol, this can be written alternatively as

$$\overleftrightarrow{\mathbf{I}} = \delta_{ij}\hat{\mathbf{e}}_i\hat{\mathbf{e}}_j. \tag{2.82}$$

Clearly the unit tensor is symmetric.

The sum of the diagonal terms of a Cartesian tensor is called the *trace of the tensor*:

$$\text{trace } \overleftrightarrow{\boldsymbol{\Phi}} = \phi_{ii}. \tag{2.83}$$

The trace of a tensor is *invariant*, called the first invariant, and it is denoted by I_1; that is, it is invariant with coordinate transformations ($\phi_{ii} = \bar{\phi}_{ii}$). The first, second, and third invariants of a Cartesian tensor are given by

$$I_1 = \phi_{ii}, \qquad I_2 = \frac{1}{2}(\phi_{ij}\phi_{ij} - \phi_{ii}\phi_{jj}), \qquad I_3 = \det[\phi] = |\phi|. \tag{2.84}$$

The double-dot product between two dyadics is very useful in many problems. The double-dot product between a dyad (**AB**) and another (**CD**) is defined as the scalar:

$$(\mathbf{AB}) : (\mathbf{CD}) \equiv (\mathbf{B} \cdot \mathbf{C})(\mathbf{A} \cdot \mathbf{D}). \tag{2.85}$$

The double-dot product, by this definition, is commutative. The double-dot product between two dyadics is given by

$$\begin{aligned} \overleftrightarrow{\boldsymbol{\Phi}} : \overleftrightarrow{\boldsymbol{\psi}} &= (\phi_{ij}\hat{\mathbf{e}}_i\hat{\mathbf{e}}_j) : (\psi_{mn}\hat{\mathbf{e}}_m\hat{\mathbf{e}}_n) \\ &= \phi_{ij}\psi_{mn}(\hat{\mathbf{e}}_i \cdot \hat{\mathbf{e}}_n)(\hat{\mathbf{e}}_j \cdot \hat{\mathbf{e}}_m) \\ &= \phi_{ij}\psi_{mn}\delta_{in}\delta_{jm} \\ &= \phi_{ij}\psi_{ji}. \end{aligned} \tag{2.86}$$

Note that the double-dot product of a Cartesian tensor $\overleftrightarrow{\boldsymbol{\Phi}}$ with the unit tensor $\overleftrightarrow{\mathbf{I}}$ produces its trace $I_1 = \phi_{ii}$.

We note that the gradient of a vector is a second-order tensor:

$$\begin{aligned} \nabla\mathbf{A} &= \hat{\mathbf{e}}_i \frac{\partial}{\partial x_i}(A_j\hat{\mathbf{e}}_j) \\ &= \frac{\partial A_j}{\partial x_i}\hat{\mathbf{e}}_i\hat{\mathbf{e}}_j. \end{aligned} \tag{2.87}$$

It can be expressed as the sum of

$$\nabla\mathbf{A} = \frac{1}{2}\left(\frac{\partial A_j}{\partial x_i} + \frac{\partial A_i}{\partial x_j}\right)\hat{\mathbf{e}}_i\hat{\mathbf{e}}_j + \frac{1}{2}\left(\frac{\partial A_j}{\partial x_i} - \frac{\partial A_i}{\partial x_j}\right)\hat{\mathbf{e}}_i\hat{\mathbf{e}}_j. \tag{2.88}$$

Analogously to the divergence of a vector, the divergence of a (second-order) Cartesian tensor is defined as

$$\begin{aligned}
\operatorname{div} \overleftrightarrow{\boldsymbol{\Phi}} &= \nabla \cdot \overleftrightarrow{\boldsymbol{\Phi}} \\
&= \hat{\mathbf{e}}_i \frac{\partial}{\partial x_i} \cdot (\phi_{mn} \hat{\mathbf{e}}_m \hat{\mathbf{e}}_n) \\
&= \frac{\partial \phi_{mn}}{\partial x_i} (\hat{\mathbf{e}}_i \cdot \hat{\mathbf{e}}_m) \hat{\mathbf{e}}_n \\
&= \frac{\partial \phi_{in}}{\partial x_i} \hat{\mathbf{e}}_n .
\end{aligned}$$

Thus the divergence of a second-order tensor is a vector.

The integral theorems of vectors presented in Section 2.2.6 are also valid for tensors (second-order and higher):

$$\int_V \operatorname{grad} \mathbf{A} \, dV = \oint_S \hat{\mathbf{n}} \mathbf{A} \, dS, \tag{2.89a}$$

$$\int_V \operatorname{div} \overleftrightarrow{\boldsymbol{\Phi}} \, dV = \oint_S \hat{\mathbf{n}} \cdot \overleftrightarrow{\boldsymbol{\Phi}} \, dS, \tag{2.89b}$$

$$\int_V \operatorname{curl} \overleftrightarrow{\boldsymbol{\Phi}} \, dV = \oint_S \hat{\mathbf{n}} \times \overleftrightarrow{\boldsymbol{\Phi}} \, dS, \tag{2.89c}$$

$$\int_{CS} \hat{\mathbf{n}} \cdot \operatorname{curl} \overleftrightarrow{\boldsymbol{\Phi}} \, dS = \oint_C d\mathbf{S} \cdot \overleftrightarrow{\boldsymbol{\Phi}}, \tag{2.89d}$$

where CS denotes ‘closed surface’. It is important that the order of the operations be observed in the above expressions.

2.3.4 Eigenvectors Associated with Dyadics

It is conceptually useful to regard a dyadic as an operator that changes a vector into another vector (by means of the dot product). In this regard it is of interest to inquire whether there are certain vectors that have only their lengths, and not their orientation, changed when operated upon by a given dyadic or tensor. If such vectors exist, they must satisfy the equation

$$\overleftrightarrow{\boldsymbol{\Phi}} \cdot \mathbf{A} = \lambda \mathbf{A}. \tag{2.90}$$

The vectors $\mathbf{A}$ are called *characteristic vectors, or eigenvectors*, associated with $\overleftrightarrow{\boldsymbol{\Phi}}$. The parameter λ is called an *eigenvalue*, and it characterizes the change in length (and possibly sense) of the eigenvector $\mathbf{A}$ after it has been operated upon by $\overleftrightarrow{\boldsymbol{\Phi}}$.

Since $\mathbf{A}$ can be expressed as $\mathbf{A} = \overleftrightarrow{\mathbf{I}} \cdot \mathbf{A}$, Eq. (2.90) can also be written as

$$\left(\overleftrightarrow{\boldsymbol{\Phi}} - \lambda \overleftrightarrow{\mathbf{I}}\right) \cdot \mathbf{A} = \mathbf{0}. \tag{2.91}$$

When written in matrix form for Cartesian components, this equation becomes

$$\begin{bmatrix} \phi_{11} - \lambda & \phi_{12} & \phi_{13} \\ \phi_{21} & \phi_{22} - \lambda & \phi_{23} \\ \phi_{31} & \phi_{32} & \phi_{33} - \lambda \end{bmatrix} \begin{Bmatrix} A_1 \\ A_2 \\ A_3 \end{Bmatrix} = \mathbf{0}. \tag{2.92}$$

Because this is a homogeneous set of equations for A_1, A_2, A_3, a nontrivial solution will not exist unless the determinant of the matrix $[\overleftrightarrow{\boldsymbol{\Phi}} - \lambda \overleftrightarrow{\mathbf{I}}]$ vanishes. The vanishing of this determinant yields a cubic equation for λ, called the *characteristic equation*, the solution of which yields three values of λ, that is, three eigenvalues λ_1, λ_2, and λ_3. The character of these eigenvalues depends on the character of the dyadic $\overleftrightarrow{\boldsymbol{\Phi}}$. At least one of the eigenvalues must be real. The other two may be real and distinct, real and repeated, or complex conjugates.

In the preponderance of practical problems, the dyadic $\overleftrightarrow{\boldsymbol{\Phi}}$ is symmetric, that is, $\overleftrightarrow{\boldsymbol{\Phi}} = \overleftrightarrow{\boldsymbol{\Phi}}^{\mathrm{T}}$. Of course, $\overleftrightarrow{\boldsymbol{\Phi}}$ is always real in our considerations. For example, the moment-of-inertia dyadic is symmetric, and the stress tensor $\overleftrightarrow{\sigma}$ is usually but not always symmetric. We limit our discussion to symmetric dyadics.

The vanishing of the determinant assures that three eigenvectors are not unique to within a multiplicative constant, however, and an infinite number of solutions exist having at least three different orientations. Since only orientation is important, it is thus useful to represent the three eigenvectors by three unit eigenvectors $\hat{\mathbf{e}}_1^*$, $\hat{\mathbf{e}}_2^*$, $\hat{\mathbf{e}}_3^*$, denoting three different orientations, each associated with a particular eigenvalue.

Suppose now that λ_1 and λ_2 are two distinct eigenvalues and $\mathbf{A}_1$ and $\mathbf{A}_2$ are their corresponding eigenvectors:

$$\overleftrightarrow{\boldsymbol{\Phi}} \cdot \mathbf{A}_1 = \lambda_1 \mathbf{A}_1, \tag{2.93a}$$

$$\overleftrightarrow{\boldsymbol{\Phi}} \cdot \mathbf{A}_2 = \lambda_2 \mathbf{A}_2. \tag{2.93b}$$

Scalar product of the first equation by $\mathbf{A}_2$ and the second by $\mathbf{A}_1$, and then subtraction, yields

$$\mathbf{A}_2 \cdot \overleftrightarrow{\boldsymbol{\Phi}} \cdot \mathbf{A}_1 - \mathbf{A}_1 \cdot \overleftrightarrow{\boldsymbol{\Phi}} \cdot \mathbf{A}_2 = (\lambda_1 - \lambda_2)\mathbf{A}_1 \cdot \mathbf{A}_2. \tag{2.94}$$

Since $\overleftrightarrow{\boldsymbol{\Phi}}$ is symmetric, one can establish that the left-hand side of this equation vanishes. Thus

$$0 = (\lambda_1 - \lambda_2)\mathbf{A}_1 \cdot \mathbf{A}_2. \tag{2.95}$$

Now suppose that λ_1 and λ_2 are complex conjugates such that $\lambda_1 - \lambda_2 = 2i\lambda_{1i}$, where $i = \sqrt{-1}$ and λ_{1i} is the imaginary part of λ_1. Then $\mathbf{A}_1 \cdot \mathbf{A}_2$ is always positive since $\mathbf{A}_1$ and $\mathbf{A}_2$ are complex conjugate vectors associated with λ_1 and λ_2. It then follows from Eq. (2.95) that $\lambda_{1i} = 0$ and hence that the *three eigenvalues associated with a symmetric dyadic are all real.*

Now assume that λ_1 and λ_2 are real and distinct such that $\lambda_1 - \lambda_2$ is not zero. It then follows from Eq. (2.95) that $\mathbf{A}_1 \cdot \mathbf{A}_2 = 0$. Thus the *eigenvectors associated with distinct eigenvalues of a symmetric dyadic are orthogonal.* If the three eigenvalues are all distinct, then the three eigenvectors are mutually orthogonal.

If λ_1 and λ_2 are distinct, but λ_3 is repeated, say $\lambda_3 = \lambda_2$, then $\mathbf{A}_3$ must also be perpendicular to $\mathbf{A}_1$ as deducted by an argument similar to that for $\mathbf{A}_2$ stemming from Eq. (2.95). Neither $\mathbf{A}_2$ nor $\mathbf{A}_3$ is preferred, and they are both arbitrary, except insofar as they are both perpendicular to $\mathbf{A}_1$. It is useful, however, to select $\mathbf{A}_3$ such that it is perpendicular to both $\mathbf{A}_1$ and $\mathbf{A}_2$. We do this by choosing $\mathbf{A}_3 = \mathbf{A}_1 \times \mathbf{A}_2$ and thus establishing a mutually orthogonal set of eigenvectors. This sort of behavior arises when there is an axis of symmetry present in a problem.

In a Cartesian system the characteristic equation associated with a dyadic can be expressed in the form

$$\lambda^3 - I_1\lambda^2 - I_2\lambda - I_3 = 0, \tag{2.96}$$

where I_1, I_2, and I_3 are the invariants associated with the matrix of $\overleftrightarrow{\Phi}$. The invariants can also be expressed in terms of the eigenvalues,

$$I_1 = \lambda_1 + \lambda_2 + \lambda_3, \qquad I_2 = -(\lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_3\lambda_1), \qquad I_3 = \lambda_1\lambda_2\lambda_3. \tag{2.97}$$

Finding the roots of the cubic Eq. (2.96) is not always easy. However, when the matrix under consideration is of the form

$$\begin{bmatrix} \phi_{11} & 0 & 0 \\ 0 & \phi_{22} & \phi_{23} \\ 0 & \phi_{32} & \phi_{33} \end{bmatrix},$$

one of the roots is $\lambda_1 = \phi_{11}$, and the remaining two roots can be found from the quadratic equation

$$(\phi_{22} - \lambda)(\phi_{33} - \lambda) - \phi_{23}\phi_{32} = 0.$$

That is,

$$\lambda_{2,3} = \frac{\phi_{22} + \phi_{33}}{2} \pm \frac{1}{2}\sqrt{(\phi_{22} + \phi_{33})^2 - 4(\phi_{22}\phi_{33} - \phi_{23}\phi_{32})}. \tag{2.98}$$

In cases where one of the roots is not obvious, an alternative procedure given below proves to be useful.

In the alternative method we seek the eigenvalues of the so-called *deviatoric tensor* associated with $\overleftrightarrow{\Phi}$:

$$\phi'_{ij} \equiv \phi_{ij} - \tfrac{1}{3}\phi_{kk}\delta_{ij}. \tag{2.99}$$

Note that

$$\phi'_{ii} = \phi_{ii} - \phi_{kk} = 0. \tag{2.100}$$

That is, the first invariant I'_1 of the deviatoric tensor is zero. As a result the characteristic equation associated with the deviatoric tensor is of the form

$$(\lambda')^3 - I'_2\lambda' - I'_3 = 0, \tag{2.101}$$

where λ' is the eigenvalue, and I'_2 and I'_3 are the second and third invariants of the deviatoric tensor. The eigenvalues associated with $[\phi]$ itself can be computed from

$$\lambda = \lambda' + \tfrac{1}{3}\phi_{kk}, \qquad I'_2 = \tfrac{1}{2}\phi'_{ij}\phi'_{ij}, \qquad I'_3 = |\phi'|. \tag{2.102}$$

The cubic equation in (2.101) is of a special form that allows a direct computation of its roots. Equation (2.101) can be solved explicitly by introducing the transformation

$$\lambda' = 2\left(\frac{1}{3}I'_2\right)^{1/2}\cos\alpha, \tag{2.103}$$

which transforms (2.101) into

$$2\left(\frac{1}{3}I'_2\right)^{3/2}[4\cos^3\alpha - 3\cos\alpha] = I'_3. \tag{2.104}$$

The expression in square brackets is equal to $\cos 3\alpha$. Hence

$$\cos 3\alpha = \frac{I'_3}{2}\left(\frac{3}{I'_2}\right)^{3/2}. \tag{2.105}$$

If α_1 is the angle satisfying $0 \leq 3\alpha_1 \leq \pi$ whose cosine is given by Eq. (2.105), then $3\alpha_1$, $3\alpha_1 + 2\pi$, and $3\alpha_1 - 2\pi$ all have the same cosine, and furnish three independent roots of Eq. (2.103),

$$\lambda'_i = 2\left(\frac{1}{3}I'_2\right)^{1/2}\cos\alpha_i, \qquad i = 1, 2, 3, \tag{2.106a}$$

where

$$\alpha_1 = \frac{1}{3}\left\{\cos^{-1}\left[\frac{I'_3}{2}\left(\frac{3}{I'_2}\right)^{3/2}\right]\right\}, \qquad \alpha_2 = \alpha_1 + \frac{2\pi}{3}, \qquad \alpha_3 = \alpha_1 - \frac{2\pi}{3}. \tag{2.106b}$$

Finally we can compute λ_i from Eq. (2.102).

Example 2.8 We wish to determine the eigenvalues and eigenvectors of the matrix

$$[\phi] = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 4 & 1 \\ 0 & 1 & 2 \end{bmatrix}.$$

The characteristic equation is obtained by setting det $(\phi_{ij} - \lambda\, \delta_{ij})$ to zero:

$$\begin{vmatrix} 2-\lambda & 1 & 0 \\ 1 & 4-\lambda & 1 \\ 0 & 1 & 2-\lambda \end{vmatrix} = (2-\lambda)[(4-\lambda)(2-\lambda) - 1] - 1 \cdot (2-\lambda) = 0,$$

or

$$(2-\lambda)[(4-\lambda)(2-\lambda) - 2] = 0.$$

Hence

$$\lambda_1 = 3 + \sqrt{3} = 4.7321, \qquad \lambda_2 = 3 - \sqrt{3} = 1.2679, \qquad \lambda_3 = 2.$$

Alternatively,

$$[\phi'] = \begin{bmatrix} 2-\frac{8}{3} & 1 & 0 \\ 1 & 4-\frac{8}{3} & 1 \\ 0 & 1 & 2-\frac{8}{3} \end{bmatrix}$$

$$I_2' = \frac{1}{2}(\phi'_{ij}\phi'_{ij} - \phi'_{ii}\phi'_{jj}) = \frac{1}{2}\phi'_{ij}\phi'_{ij}$$

$$= \frac{1}{2}\left[\left(-\frac{2}{3}\right)^2 + \left(-\frac{2}{3}\right)^2 + \left(\frac{4}{3}\right)^2 + 2 + 2\right] = \frac{10}{3}$$

$$I_3' = \det(\phi'_{ij}) = \frac{52}{27}.$$

From Eq. (2.106b),

$$\alpha_1 = \frac{1}{3}\left\{\cos^{-1}\left[\frac{52}{54}\left(\frac{9}{10}\right)^{3/2}\right]\right\} = 11.565°,$$

$$\alpha_2 = 131.565°, \qquad \alpha_3 = -108.435°,$$

and from Eq. (2.103),

$$\lambda_1' = 2.065384, \qquad \lambda_2' = -1.3987, \qquad \lambda_3' = -0.66667.$$

Finally, using Eq. (2.102), we obtain the eigenvalues

$$\lambda_1 = 4.7321, \qquad \lambda_2 = 1.2679, \qquad \lambda_3 = 2.00.$$

The eigenvector corresponding to $\lambda_3 = 2$, for example, is calculated as follows. From $(\phi_{ij} - \lambda_3\delta_{ij})A_j = 0$ we have

$$\begin{bmatrix} 2-2 & 1 & 0 \\ 1 & 4-2 & 1 \\ 0 & 1 & 2-2 \end{bmatrix} \begin{Bmatrix} A_1 \\ A_2 \\ A_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}.$$

This gives

$$A_2 = 0, \qquad A_1 = -A_3.$$

Using $A_1^2 + A_2^2 + A_3^2 = 1$, we obtain

$$\hat{\mathbf{A}}_3 = \pm\frac{1}{\sqrt{2}}(1, 0, -1), \qquad \text{for } \lambda_3 = 2.$$

Similarly, the eigenvectors corresponding to $\lambda_{1,2} = 3 \pm \sqrt{3}$ are given by

$$\hat{\mathbf{A}}_1 = \pm\frac{(3-\sqrt{3})}{12}\left(1, \left(1+\sqrt{3}\right), 1\right), \qquad \text{for } \lambda_1 = 3+\sqrt{3},$$

$$\hat{\mathbf{A}}_2 = \pm\frac{(3+\sqrt{3})}{12}\left(1, \left(1-\sqrt{3}\right), 1\right), \qquad \text{for } \lambda_2 = 3-\sqrt{3}.$$

EXERCISES

2.1 Find the equation of a line (or a set of lines) passing through the terminal point of a vector **A** and in the direction of vector **B**.

2.2 Find the equation of a plane connecting the terminal points of vectors **A**, **B**, and **C**. Assume that all three vectors are referred to a common origin.

2.3 Prove the following vector identity without the use of a coordinate system (see Example 2.2):

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = (\mathbf{A} \cdot \mathbf{C})\mathbf{B} - (\mathbf{A} \cdot \mathbf{B})\mathbf{C}.$$

2.4 If $\hat{\mathbf{e}}$ is any unit vector and **A** an arbitrary vector, show that

$$\mathbf{A} = (\mathbf{A} \cdot \hat{\mathbf{e}})\hat{\mathbf{e}} + \hat{\mathbf{e}} \times (\mathbf{A} \times \hat{\mathbf{e}}).$$

This identity shows that a vector can be resolved into a component parallel to and one perpendicular to an arbitrary direction $\hat{\mathbf{e}}$.

2.5 Verify the following identities:

(a) $\delta_{ii} = 3.$ (b) $\delta_{ij}\delta_{ij} = \delta_{ii}.$

(c) $\delta_{ij}\delta_{jk} = \delta_{ik}.$ (d) $\varepsilon_{mjk}\varepsilon_{njk} = 2\delta_{mn}.$

(e) $\varepsilon_{ijk}\varepsilon_{ijk} = 6.$ (f) $A_iA_j\varepsilon_{ijk} = 0.$

2.6 Using the index notation, prove the identity

$$(\mathbf{A} \times \mathbf{B}) \cdot (\mathbf{B} \times \mathbf{C}) \times (\mathbf{C} \times \mathbf{A}) = (\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}))^2.$$

2.7 Prove the following vector identity in an orthonormal system using index-summation notation:

$$(\mathbf{A} \times \mathbf{B}) \times (\mathbf{C} \times \mathbf{D}) = [\mathbf{A} \cdot (\mathbf{C} \times \mathbf{D})]\mathbf{B} - [\mathbf{B} \cdot (\mathbf{C} \times \mathbf{D}]\mathbf{A}.$$

2.8 Determine whether the following set of vectors is linearly independent:

$$\mathbf{A} = \hat{\mathbf{e}}_1 + \hat{\mathbf{e}}_2, \qquad \mathbf{B} = \hat{\mathbf{e}}_2 + \hat{\mathbf{e}}_4, \qquad \mathbf{C} = \hat{\mathbf{e}}_3 + \hat{\mathbf{e}}_4, \qquad \mathbf{D} = \hat{\mathbf{e}}_1 + \hat{\mathbf{e}}_2 + \hat{\mathbf{e}}_3 + \hat{\mathbf{e}}_4.$$

Here $\hat{\mathbf{e}}_i$ are orthonormal unit base vectors in a four-dimensional space.

2.9 Determine whether the following set of vectors is linearly independent:

$$\mathbf{A} = 2\hat{\mathbf{e}}_1 - \hat{\mathbf{e}}_2 + \hat{\mathbf{e}}_3, \qquad \mathbf{B} = \hat{\mathbf{e}}_2 - \hat{\mathbf{e}}_3, \qquad \mathbf{C} = -\hat{\mathbf{e}}_1 + \hat{\mathbf{e}}_2.$$

Here $\hat{\mathbf{e}}_i$ are orthonormal unit base vectors in $\Re^3$.

2.10 Determine which of the following sets of vectors span $\Re^3$:

(a) $\mathbf{A} = \hat{\mathbf{e}}_1 + 3\hat{\mathbf{e}}_2 - \hat{\mathbf{e}}_3, \qquad \mathbf{B} = -4\hat{\mathbf{e}}_1 + 3\hat{\mathbf{e}}_2 - 5\hat{\mathbf{e}}_3, \qquad \mathbf{C} = 2\hat{\mathbf{e}}_1 + \hat{\mathbf{e}}_2 + \hat{\mathbf{e}}_3.$

(b) $\mathbf{A} = \hat{\mathbf{e}}_1 + \hat{\mathbf{e}}_2, \qquad \mathbf{B} = \hat{\mathbf{e}}_1 + \hat{\mathbf{e}}_2 - 2\hat{\mathbf{e}}_3, \qquad \mathbf{C} = \hat{\mathbf{e}}_1 - \hat{\mathbf{e}}_3.$

Here $\hat{\mathbf{e}}_i$ are orthonormal unit base vectors in $\Re^3$.

2.11 Consider two rectangular Cartesian coordinate systems that are rotated with respect to each other and have a common origin. Let one system be denoted as a barred system, so that a position vector can be written in each of the systems as

$$\begin{aligned}\mathbf{r} &= x_i \hat{\mathbf{e}}_i, \\ &= \bar{x}_j \hat{\bar{\mathbf{e}}}_j,\end{aligned}$$

where $\{\hat{\mathbf{e}}_j\}$ and $\{\hat{\bar{\mathbf{e}}}_j\}$ are the respective orthonormal Cartesian bases in the unbarred and barred systems. By requiring that the position vector $\mathbf{r}$ be invariant under a rotation of the coordinate systems, deduce that the transformation between the coordinates is given by

$$\begin{aligned}\bar{x}_1 &= a_{11}x_1 + a_{12}x_2 + a_{13}x_3 \\ \bar{x}_2 &= a_{21}x_1 + a_{22}x_2 + a_{23}x_3 \quad , \\ \bar{x}_3 &= a_{31}x_1 + a_{32}x_2 + a_{33}x_3\end{aligned}$$

or, more compactly,

$$\bar{x}_i = a_{ij}x_j, \qquad i, j = 1, 2, 3,$$

where the terms a_{ij} can be identified as the direction cosines

$$a_{ij} \equiv \hat{\bar{\mathbf{e}}}_i \cdot \hat{\mathbf{e}}_j = \cos(\hat{\bar{\mathbf{e}}}_i, \hat{\mathbf{e}}_j).$$

Deduce further that the basis vectors obey the same transformation

$$\hat{\bar{\mathbf{e}}}_i = a_{ij}\hat{\mathbf{e}}_j,$$

and that the following orthogonality conditions hold:

$$a_{ij}a_{kj} = \delta_{ik}.$$

2.12 Determine the transformation matrix relating the orthonormal basis vectors $(\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2, \hat{\mathbf{e}}_3)$ and $(\hat{\mathbf{e}}'_1, \hat{\mathbf{e}}'_2, \hat{\mathbf{e}}'_3)$, when $\hat{\mathbf{e}}'_i$ are given by

(a) $\hat{\mathbf{e}}'_1$ is along the vector $\hat{\mathbf{e}}_1 - \hat{\mathbf{e}}_2 + \hat{\mathbf{e}}_3$ and $\hat{\mathbf{e}}'_2$ is perpendicular to the plane $2x_1 + 3x_2 + x_3 - 5 = 0$.

(b) $\hat{\mathbf{e}}'_1$ is along the line segment connecting point $(1, -1, 3)$ to $(2, -2, 4)$ and $\hat{\mathbf{e}}'_3 = (-\hat{\mathbf{e}}_1 + \hat{\mathbf{e}}_2 + 2\hat{\mathbf{e}}_3)/\sqrt{6}$.

(c) $\hat{\mathbf{e}}'_3 = \hat{\mathbf{e}}_3$, and the angle between x'_1-axis and x_1-axis is 30°.

2.13 The angles between the barred and unbarred coordinate lines are given by

	$\hat{\mathbf{e}}_1$	$\hat{\mathbf{e}}_2$	$\hat{\mathbf{e}}_3$
$\hat{\bar{\mathbf{e}}}_1$	60°	30°	90°
$\hat{\bar{\mathbf{e}}}_2$	150°	60°	90°
$\hat{\bar{\mathbf{e}}}_3$	90°	90°	0°

Determine the direction cosines of the transformation.

2.14 The angles between the barred and unbarred coordinate lines are given by

	x_1	x_2	x_3
$\bar{x}_1$	45°	90°	45°
$\bar{x}_2$	60°	45°	120°
$\bar{x}_3$	120°	45°	60°

Determine the transformation matrix.

2.15 In a rectangular Cartesian coordinate system, find the length and direction cosines of a vector **A** that extends from the point $(1, -1, 3)$ to the midpoint of the line segment from the origin to the point $(6, -6, 4)$.

2.16 The vectors **A** and **B** are defined as follows:

$$\mathbf{A} = 3\hat{\mathbf{i}} - 4\hat{\mathbf{k}},$$

$$\mathbf{B} = 2\hat{\mathbf{i}} - 2\hat{\mathbf{j}} + \hat{\mathbf{k}},$$

where $(\hat{\mathbf{i}}, \hat{\mathbf{j}}, \hat{\mathbf{k}})$ is an orthonormal basis.

(a) Find the orthogonal projection of **A** in the direction of **B**.

(b) Find the angle between the positive directions of the vectors.

2.17 Prove the following identities (see Eq. (2.13) for the definition of [**ABC**]):

(a) $\frac{d}{dt}[\mathbf{ABC}] = \left[\frac{d\mathbf{A}}{dt}\mathbf{BC}\right] + \left[\mathbf{A}\frac{d\mathbf{B}}{dt}\mathbf{C}\right] + \left[\mathbf{AB}\frac{d\mathbf{C}}{dt}\right]$.

(b) $\frac{d}{dt}\left[\mathbf{A}\frac{d\mathbf{A}}{dt}\frac{d^2\mathbf{A}}{dt^2}\right] = \left[\mathbf{A}\frac{d\mathbf{A}}{dt}\frac{d^3\mathbf{A}}{dt^3}\right]$.

2.18 Let $\mathbf{r}$ denote a position vector $\mathbf{r} = x_i\hat{\mathbf{e}}_i$ ($r^2 = x_ix_i$) and **A** an arbitrary constant vector. Show that:

(a) $\text{grad}(r) = \frac{\mathbf{r}}{r}$.

(b) $\text{grad}(r^n) = nr^{n-2}\mathbf{r}$.

(c) $\nabla^2(r^n) = n(n+1)r^{n-2}$.

(d) $\text{grad}(\mathbf{r}\cdot\mathbf{A}) = \mathbf{A}$.

(c) $\text{div}(\mathbf{r}\times\mathbf{A}) = 0$.

(f) $\text{curl}(\mathbf{r}\times\mathbf{A}) = -2\mathbf{A}$.

(g) $\text{div}(r\mathbf{A}) = \frac{1}{r}(\mathbf{r}\cdot\mathbf{A})$.

(h) $\text{curl}(r\mathbf{A}) = \frac{1}{r}(\mathbf{r}\times\mathbf{A})$.

2.19 Let **A** and **B** be continuous vector functions of the position vector **r** with continuous first derivatives, and let F and G be continuous scalar functions of **r** with continuous first and second derivatives. Show that:

(a) $\text{curl}(\text{grad } F) = 0$.

(b) $\text{div}(\text{curl }\mathbf{A}) = 0$.

(c) $\text{div}(\text{grad } F\times \text{grad } G) = 0$.

(d) $\text{grad}(FG) = F \text{ grad } G + G \text{ grad } F$.

(e) $\text{div}(F\mathbf{A}) = \mathbf{A}\cdot \text{grad } F + F\text{div}\mathbf{A}$.

(f) $\text{curl}(F\mathbf{A}) = F \text{ curl }\mathbf{A} - \mathbf{A}\times \text{grad } F$.

(g) $\text{grad}(\mathbf{A}\cdot\mathbf{B}) = \mathbf{A}\cdot\text{grad }\mathbf{B} + \mathbf{B}\cdot\text{grad }\mathbf{A} + \mathbf{A}\,\text{curl }\mathbf{B} + \mathbf{B}\,\text{curl }\mathbf{A}$.

(h) $\text{div}(\mathbf{A}\times\mathbf{B}) = \text{curl }\mathbf{A}\cdot\mathbf{B} - \text{curl }\mathbf{B}\cdot\mathbf{A}$.

(i) $\text{curl}(\mathbf{A}\times\mathbf{B}) = \mathbf{B}\cdot\nabla\mathbf{A} - \mathbf{A}\cdot\nabla\mathbf{B} + \mathbf{A}\ \text{div }\mathbf{B} - \mathbf{B}\,\text{div }\mathbf{A}$.

(j) $(\nabla\times\mathbf{A})\times\mathbf{A} = \mathbf{A}\cdot\nabla\mathbf{A} - \nabla\mathbf{A}\cdot\mathbf{A}$.

(k) $\nabla^2(FG) = F\,\nabla^2 G + 2\nabla F\cdot\nabla G + G\,\nabla^2 F$.

2.20 Show that the vector area of a closed surface is zero, that is,

$$\oint_S \hat{\mathbf{n}}\,dS = \mathbf{0}.$$

2.21 Show that the volume enclosed by a surface S is

$$\text{volume} = \frac{1}{6}\oint_S \text{grad}(r^2)\cdot\hat{\mathbf{n}}\,dS,$$

or

$$\text{volume} = \frac{1}{3}\oint_S \mathbf{r}\cdot\hat{\mathbf{n}}\,dS.$$

2.22 Let $\phi(\mathbf{r})$ be a scalar field. Show that

$$\int_V \nabla^2\phi\,dV = \oint_S \frac{\partial\phi}{\partial n}\,dS,$$

where $\partial\phi/\partial n \equiv \hat{\mathbf{n}}\cdot\text{grad}\,\phi$ is the derivative of ϕ in the outward direction normal to the surface.

2.23 In the divergence theorem (2.54b), set $\mathbf{A} = \phi\,\text{grad}\,\psi$ and $\mathbf{A} = \psi\,\text{grad}\,\phi$ successively and obtain the integral forms

$$\text{(a)}\quad \int_\Omega \left[\phi\nabla^2\psi + \nabla\phi\cdot\nabla\psi\right]d\Omega = \oint_\Gamma \phi\frac{\partial\psi}{\partial n}\,d\Gamma,$$

$$\text{(b)}\quad \int_\Omega \left[\phi\nabla^2\psi - \psi\nabla^2\phi\right]d\Omega = \oint_\Gamma \left[\phi\frac{\partial\psi}{\partial n} - \psi\frac{\partial\phi}{\partial n}\right]d\Gamma,$$

$$\text{(c)}\quad \int_\Omega \left[\phi\nabla^4\psi - \nabla^2\phi\nabla^2\psi\right]d\Omega = \oint_\Gamma \left[\phi\frac{\partial}{\partial n}(\nabla^2\psi) - \nabla^2\psi\frac{\partial\phi}{\partial n}\right]d\Gamma,$$

where Ω denotes a (2D or 3D) region with boundary Γ. The first two identities are sometimes called Green's first and second theorems.

2.24 Determine the rotation transformation matrix such that the new base vector $\hat{\bar{\mathbf{e}}}_1$ is along $\hat{\mathbf{e}}_1 - \hat{\mathbf{e}}_2 + \hat{\mathbf{e}}_3$, and $\hat{\bar{\mathbf{e}}}_2$ is along the normal to the plane $2x_1 + 3x_2 + x_3 = 5$. If $\overleftrightarrow{\mathbf{T}}$ is the dyadic whose components in the unbarred system are given by $T_{11} = 1$, $T_{12} = 0$, $T_{13} = -1$, $T_{22} = 3$, $T_{23} = -2$, and $T_{33} = 0$, find the components in the barred coordinates.

2.25 Show that the characteristic equation for a second-order tensor σ_{ij} can be expressed as

$$\lambda^3 - I_1\lambda^2 - I_2\lambda - I_3 = 0,$$

where

$$I_1 = \sigma_{kk}$$

$$I_2 = -\frac{1}{2}(\sigma_{ii}\sigma_{jj} - \sigma_{ij}\sigma_{ji})$$

$$I_3 = \frac{1}{6}(2\sigma_{ij}\sigma_{jk}\sigma_{ki} - 3\sigma_{ij}\sigma_{ji}\sigma_{kk} + \sigma_{ii}\sigma_{jj}\sigma_{kk}) = \det(\sigma_{ij})$$

are the three invariants of the tensor.

2.26 Find the eigenvalues and eigenvectors of the following matrices:

$$\text{(a)}\ \begin{bmatrix} 4 & -4 & 0 \\ -4 & 0 & 0 \\ 0 & 0 & 3 \end{bmatrix} \qquad \text{(b)}\ \begin{bmatrix} 2 & -\sqrt{3} & 0 \\ -\sqrt{3} & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

$$\text{(c)}\ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & -1 \\ 0 & -1 & 3 \end{bmatrix} \qquad \text{(d)}\ \begin{bmatrix} 2 & -1 & 1 \\ -1 & 0 & 1 \\ 1 & 1 & 2 \end{bmatrix}$$

$$\text{(e)}\ \begin{bmatrix} 3 & 5 & 8 \\ 5 & 1 & 0 \\ 8 & 0 & 2 \end{bmatrix} \qquad \text{(f)}\ \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$$

2.27 Evaluate the three invariants of the matrices in Exercise 2.26 and check them against the invariants obtained by using the eigenvalues.

2.28 The components of a stress dyadic at a point, referred to the (x_1, x_2, x_3) system, are (in ksi = 1000 psi):

$$\text{(i)}\ \begin{bmatrix} 12 & 9 & 0 \\ 9 & -12 & 0 \\ 0 & 0 & 6 \end{bmatrix} \qquad \text{(ii)}\ \begin{bmatrix} 9 & 0 & 12 \\ 0 & -25 & 0 \\ 12 & 0 & 16 \end{bmatrix}$$

$$\text{(iii)}\ \begin{bmatrix} 1 & -3 & \sqrt{2} \\ -3 & 1 & -\sqrt{2} \\ \sqrt{2} & -\sqrt{2} & 4 \end{bmatrix}$$

Find the following:

(a) The stress vector acting on a plane perpendicular to the vector $2\hat{\mathbf{e}}_1 - 2\hat{\mathbf{e}}_2 + \hat{\mathbf{e}}_3$ passing through the point.

(b) The magnitude of the stress vector and the angle between the stress vector and the normal to the plane.

(c) The magnitudes of the normal and tangential components of the stress vector.

2.29 Verify that

$$\overleftrightarrow{\mathbf{I}} \cdot \overleftrightarrow{\Phi} = \overleftrightarrow{\Phi} \cdot \overleftrightarrow{\mathbf{I}} = \overleftrightarrow{\Phi}.$$

2.30 If $\mathbf{A}$ is an arbitrary vector and $\overleftrightarrow{\Phi}$ is an arbitrary dyadic, verify that

(a) $(\overleftrightarrow{\mathbf{I}} \times \mathbf{A}) \cdot \overleftrightarrow{\Phi} = \mathbf{A} \times \overleftrightarrow{\Phi}$.

(b) $(\mathbf{A} \times \overleftrightarrow{\mathbf{I}}) \cdot \overleftrightarrow{\Phi} = \mathbf{A} \times \overleftrightarrow{\Phi}$.

(c) $(\overleftrightarrow{\Phi} \times \mathbf{A})^{\mathrm{T}} = -\mathbf{A} \times \overleftrightarrow{\Phi}^{\mathrm{T}}$.

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3

REVIEW OF EQUATIONS OF SOLID MECHANICS

3.1 INTRODUCTION

3.1.1 Classification of Equations

The objective of this chapter is to record the governing equations of a solid continuum in a form suitable for use later in this book. The word *continuum* needs to be explained first. Microscopically, a medium occupied by a matter, solid or fluid, is made of discrete particles of protons, neutrons, and electrons. Macroscopically, the medium is assumed to contain no gaps or voids between material points of the medium so that it can be divided indefinitely into smaller and smaller parts without encountering a void. This concept allows us to shrink an arbitrarily small region to a point so that all spatial derivatives of various quantities associated with the medium can be defined. For example, if mass density $\rho(\mathbf{x}, t)$ of the medium is defined to be the mass per unit volume, we assume that the limit

$$\rho(\mathbf{x}, t) = \lim_{\Delta V \to 0} \frac{\Delta m}{\Delta V} \tag{3.1}$$

exists (i.e., ρ is finite). Here Δm denotes the mass of the matter occupying an infinitesimal volume ΔV, $\mathbf{x}$ is the position vector, and t denotes time. Of course, the continuum assumption may be violated when one considers matter at nanometer (10^{-9} m) or atomic scale, and the continuum equations to be derived in this chapter may not be applicable to problems at such scales.

The governing equations of a continuum are derived using the following conservation principles (or laws of physics):

1. Principle of conservation of mass.
2. Principle of conservation of linear momentum.

3. Principle of conservation of angular momentum.
4. Principle of conservation of energy.

The above principles do not explicitly account for geometric changes or mechanical response of the continuum. Without these, the equations derived from the conservation principles are insufficient to determine the total response of the continuum. Therefore, we must consider the following additional two sets of equations:

5. Kinematics (strain–displacement equations).
6. Constitutive equations (e.g., stress–strain relations).

Kinematics is a study of the geometry of motion and deformation without consideration of the forces causing the motion. The constitutive equations describe the mechanical behavior of the continuum and relate the dependent variables introduced in the kinetic description (i.e., conservation of momenta and energy) to those in the kinematic description. Since the resulting equations involve spatial (i.e., position) coordinates and time, appropriate boundary and initial conditions are needed to determine the solution. For a detailed derivation of the equations, the reader may consult the books on continuum mechanics [1–3], elasticity [4–6], and fluid mechanics [7,8].

3.1.2 Descriptions of Motion

There are two alternative descriptions used to study the motion of a continuum. In the first, one considers the motion of all matter passing through a *fixed spatial location*, as shown in Fig. 3.1a. Here one is interested in various properties (e.g., velocity, pressure, temperature, density, and so on) of the matter that instantly occupies the fixed spatial location. This description is called the *spatial description* or the *Eulerian description*. In the second, one focuses attention on a *fixed set of material particles*, irrespective of their spatial locations (see Fig. 3.1b). The relative displacements of these particles and the stress caused by external forces and temperature are of interest in this case. This description is known as the *material description* or the *Lagrangian description*. The Eulerian description is most commonly used to study fluid flows and coupled heat transfer and fluid flow, while the Lagrangian description is generally used to study heat transfer, stress, and deformation of solid bodies.

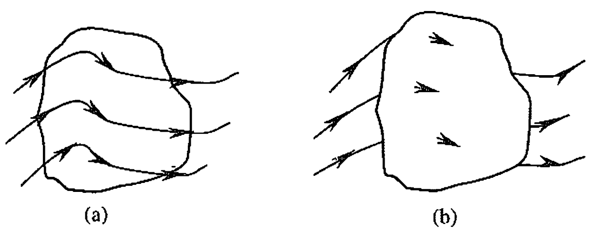

Figure 3.1 (a) Fixed region in space and mass moving through it. (b) Motion of a fixed collection of material particles.

In order to understand the difference between the material and spatial descriptions, consider a continuum and identify a region of the continuum for our study. Let $\mathbf{X}$ denote the position of an arbitrarily fixed point in the region (in both descriptions) at time $t = 0$, and let us label the material particle that occupies position $\mathbf{X}$ with X (a name given to the particle). For time $t > 0$, the point $\mathbf{X}$ in a spatial description remains the same, but denoted by $\mathbf{x}$. Although the current position $\mathbf{x}$ and initial position $\mathbf{X}$ are the same in the spatial description, the particles occupying the position at these two times are not the same (i.e., particle X no longer occupies the position $\mathbf{x}$ at time $t > 0$), unless there is no motion. On the other hand, in a material description, where attention is focused on a material particle, the particle X occupying the position $\mathbf{X}$ at time $t = 0$ moves to a new position $\mathbf{x}$ at time $t > 0$.

To gain further understanding, consider representing a scalar quantity such as the mass density ρ in the two descriptions. In a material description, the functional variation of ρ is described with respect to the coordinate $\mathbf{X}$ occupied by the material particle X in the region as

$$\rho = \rho(\mathbf{X}, t). \tag{3.2}$$

In spatial description, ρ is described with respect to the position in space, $\mathbf{x}$, currently occupied by a material particle in the continuum at time t as

$$\rho = \rho(\mathbf{x}, t). \tag{3.3}$$

In Eq. (3.2) a change in time t implies that the same material particle X has a different density. The particle's current position $\mathbf{x}$ has been expressed in terms of its position $\mathbf{X}$ at time $t = 0$. Thus, in material description, attention is focused on the material particle. In Eq. (3.3), however, a change in time t implies that a different density is observed at the same spatial location $\mathbf{x}$, now probably occupied by a different material particle. Therefore, in spatial description, attention is focused on spatial location.

3.2 CONSERVATION OF LINEAR AND ANGULAR MOMENTA

3.2.1 Equations of Motion

The principle of conservation of linear momentum, or Newton's second law of motion, applied to a set of particles (or rigid body), can be stated as *the time rate of change of (linear) momentum of a collection of particles equals the net force exerted on the collection.* Written in vector form, the principle implies

$$\frac{d}{dt}(m\mathbf{v}) = \mathbf{F}, \tag{3.4}$$

where m is the total mass, $\mathbf{v}$ the velocity, and $\mathbf{F}$ the resultant force on the collection particles. For constant mass, Eq. (3.4) becomes

$$\mathbf{F} = m\frac{d\mathbf{v}}{dt} = m\mathbf{a}, \tag{3.5}$$

which is the familiar form of Newton's second law (i.e., force = mass × acceleration). To derive the equation of motion applied to a control volume (i.e., either a fixed region in space through which material flows or a fixed material body that is in motion), we must identify the forces acting on it.

Forces acting on a control volume can be classified as *internal* and *external*. The internal forces resist the tendency of one part of the region/body to be separated from another part. The internal force per unit area is termed stress, as defined in Section 2.3. The external forces are those transmitted by the body. The external forces can be further classified as *body (or volume) forces* and *surface forces*.

Body forces act on the distribution of mass inside the body. Examples of body forces are provided by the gravitational and electromagnetic forces. Body forces are usually measured per unit mass or unit volume of the body. Let $\mathbf{f}$ denote the body force per unit mass. Consider an elemental volume dV inside V. The body force of the elemental volume is equal to $\rho dV\mathbf{f}$. Hence, the total body force of the control volume is

$$\int_V \rho\mathbf{f}\, dV. \tag{3.6}$$

Surface forces are contact forces acting on the boundary surface of the body. Examples of surface forces are provided by applied forces on the surface of the body. Surface forces are reckoned per unit area. Let $\mathbf{t}$ denote the surface force per unit area (or surface stress vector). The surface force on an elemental surface dS of the control volume is $\mathbf{t}\, dS$. The total surface force acting on the closed surface of the control volume is

$$\oint_S \mathbf{t}\, dS. \tag{3.7}$$

Since the stress vector $\mathbf{t}$ on the surface is related to the (internal) stress tensor $\overleftrightarrow{\sigma}$ by Cauchy's formula [see Eq. (2.68)],

$$\mathbf{t} = \hat{\mathbf{n}} \cdot \overleftrightarrow{\sigma}, \tag{3.8}$$

where $\hat{\mathbf{n}}$ denotes the unit normal to the surface, we can express the surface force as

$$\oint_S \hat{\mathbf{n}} \cdot \overleftrightarrow{\sigma}\, dS.$$

Using the divergence theorem (2.89b), we can write

$$\oint_S \hat{\mathbf{n}} \cdot \overleftrightarrow{\sigma}\, dS = \int_V \nabla \cdot \overleftrightarrow{\sigma}\, dV. \tag{3.9}$$

The principle of conservation of linear momentum as applied to a solid continuum yields the result

$$\nabla \cdot \overleftrightarrow{\sigma} + \rho\mathbf{f} = \rho \frac{\partial^2 \mathbf{u}}{\partial t^2}, \tag{3.10}$$

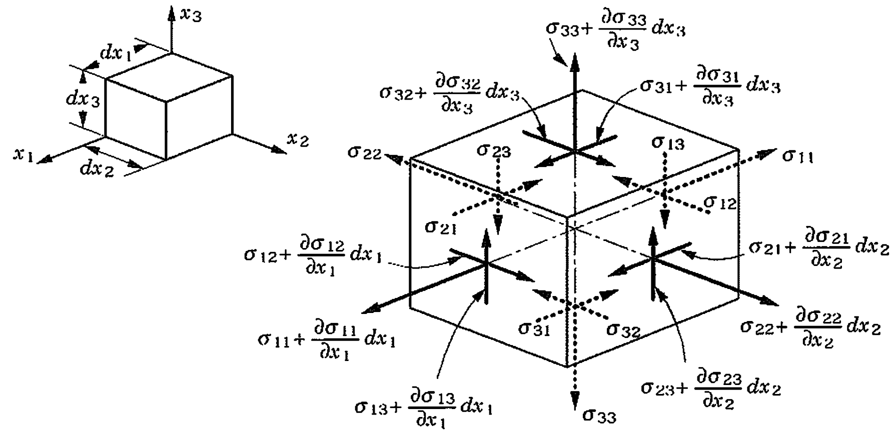

Figure 3.2 Stresses on a parallelepiped element.

where **u** is the displacement vector. In the Cartesian rectangular system, we have

$$\frac{\partial\sigma_{ji}}{\partial x_j}+\rho f_i=\rho\frac{\partial^2 u_i}{\partial t^2}\qquad (i=1,2,3). \tag{3.11}$$

Equations of motion (3.11) can be derived by directly applying Newton's second law of motion to a volume element. Consider the stresses and body forces on an infinitesimal parallelepiped element of a material body. Figure 3.2 shows the stresses acting on the various faces of the infinitesimal parallelepiped with dimensions dx_1, dx_2, and dx_3 along coordinate lines (x_1, x_2, x_3). The sum of all forces in the x_1-direction is given by

$$\begin{aligned}&\left(\sigma_{11}+\frac{\partial\sigma_{11}}{\partial x_1}dx_1\right)dx_2dx_3-\sigma_{11}\,dx_2dx_3+\left(\sigma_{21}+\frac{\partial\sigma_{21}}{\partial x_2}dx_2\right)dx_1dx_3\\&\quad-\sigma_{21}\,dx_1dx_3+\left(\sigma_{31}+\frac{\partial\sigma_{31}}{\partial x_3}dx_3\right)dx_1dx_2-\sigma_{31}\,dx_1dx_2+\rho f_1\,dx_1dx_2dx_3\\&\quad=\left(\frac{\partial\sigma_{11}}{\partial x_1}+\frac{\partial\sigma_{21}}{\partial x_2}+\frac{\partial\sigma_{31}}{\partial x_3}+\rho f_1\right)dx_1dx_2dx_3.\end{aligned}$$

By Newton's second law of motion, the sum of the forces is equal to the product of mass and acceleration in the x_1-direction,

$$(\rho\,dx_1dx_2dx_3)\frac{\partial^2 u_1}{\partial t^2},$$

where ρ is the density. Thus, upon dividing throughout by $dx_1dx_2dx_3$, we obtain

$$\frac{\partial\sigma_{11}}{\partial x_1}+\frac{\partial\sigma_{21}}{\partial x_2}+\frac{\partial\sigma_{31}}{\partial x_3}+\rho f_1=\rho\frac{\partial^2 u_1}{\partial t^2}.$$

Similarly, the application of Newton's second law in the x_2- and x_3-directions gives respectively

$$\frac{\partial \sigma_{12}}{\partial x_1} + \frac{\partial \sigma_{22}}{\partial x_2} + \frac{\partial \sigma_{32}}{\partial x_3} + \rho f_2 = \rho \frac{\partial^2 u_2}{\partial t^2},$$

$$\frac{\partial \sigma_{13}}{\partial x_1} + \frac{\partial \sigma_{23}}{\partial x_2} + \frac{\partial \sigma_{33}}{\partial x_3} + \rho f_3 = \rho \frac{\partial^2 u_3}{\partial t^2},$$

or, in index notation,

$$\frac{\partial \sigma_{ji}}{\partial x_j} + \rho f_i = \rho \frac{\partial^2 u_i}{\partial t^2} \qquad (i = 1, 2, 3), \tag{3.12}$$

which is the same as that in Eq. (3.11). For static equilibrium, we set the time derivative terms to zero and obtain the equilibrium equations

$$\frac{\partial \sigma_{ji}}{\partial x_j} + \rho f_i = 0 \qquad (i = 1, 2, 3). \tag{3.13}$$

3.2.2 Symmetry of Stress Tensor

The principle of conservation of angular momentum states that *the time rate of change of the total moment of momentum for a continuum is equal to the vector sum of the moments of external forces acting on the continuum.* In the absence of body couples (i.e., volume-dependent couples), the principle leads to the symmetry of the stress tensor. That is, the matrix of the stress components is symmetric:

$$\sigma_{23} = \sigma_{32}, \qquad \sigma_{31} = \sigma_{13}, \qquad \sigma_{12} = \sigma_{21}.$$

Thus, there are only six stress components that are independent.

The symmetry of the stress tensor can also be established using Newton's second law for moments. Consider the moment of all forces acting on the parallelepiped about the x_3-axis (see Fig. 3.2). Using the right-handed screw rule for positive moment, we obtain

$$\left[\left(\sigma_{12} + \frac{\partial \sigma_{12}}{\partial x_1} dx_1\right) dx_2 dx_3\right] \frac{dx_1}{2} + (\sigma_{12}\, dx_2 dx_3) \frac{dx_1}{2}$$
$$- \left[\left(\sigma_{21} + \frac{\partial \sigma_{21}}{\partial x_2} dx_2\right) dx_1 dx_3\right] \frac{dx_2}{2} - (\sigma_{21}\, dx_1 dx_3) \frac{dx_2}{2} = 0.$$

Dividing throughout by $\frac{1}{2}\, dx_1 dx_2 dx_3$ and taking the limit $dx_1 \to 0$ and $dx_2 \to 0$, we obtain

$$\sigma_{12} - \sigma_{21} = 0.$$

Similar considerations of moments about the x_1-axis and x_2-axis give, respectively, the relations

$$\sigma_{23} - \sigma_{32} = 0, \qquad \sigma_{13} - \sigma_{31} = 0.$$

3.3 KINEMATICS OF DEFORMATION

3.3.1 Strain Tensor

Kinematics is a study of the geometry of motion and deformation without consideration of the forces causing them. Under the action of applied loads, a material body gets displaced and deformed or strained. The word *deformation* refers to changes in the geometry of the body. A body is said to undergo *rigid body motion* if the distance between any two arbitrary points and the angle between any two infinitesimal line segments in the body remains unchanged. The rigid-body motion does not alter the shape of the body. Thus, a measure of the deformation is provided by the change in the distance between points and angle between line segments in the body. Here we develop a measure of straining of the body and introduce strain tensors using the material description.

Consider a fixed mass occupying a region with volume V and closed surface S. For simplicity, we also denote the region with V. A typical material particle in V is labeled as X, and its location with respect to a rectangular Cartesian coordinate system is denoted as $\mathbf{X}$. Under the action of applied forces (and subjected to geometric constraints), the body undergoes deformation and the particle X moves to a new location $\mathbf{x}$. Since all material particles of the body move, possibly by different amounts, the body occupies a new region, as shown in Fig. 3.3.

To determine if the motion involved is a deformation (or straining), we must compute the distance between two neighboring points after deformation. Let P: (X_1, X_2, X_3) and Q: $(X_1 + dX_1, X_2 + dX_2, X_3 + dX_3)$ denote two arbitrary material points in the body at time $t = 0$ (our reference configuration). Under the action of externally applied forces, the body deforms, and the points P and Q move to new places: $\overline{P}$: (x_1, x_2, x_3) and $\overline{Q}$: $(x_1 + dx_1, x_2 + dx_2, x_3 + dx_3)$, respectively. The motion of a particle X occupying position $\mathbf{X}$ in the undeformed body (i.e., the reference configuration) to point $\mathbf{x}$ in the deformed body can be expressed by the displacement vector

$$\mathbf{u} = \mathbf{x} - \mathbf{X} \quad \text{or} \quad u_i = x_i - X_i. \tag{3.14}$$

In the material description, the current position $\mathbf{x}$ of a material point is expressed in terms of its position at time $t = 0$:

$$\mathbf{x} = \mathbf{x}(\mathbf{X}, t), \qquad x_i = x_i(X_1, X_2, X_3, t) \qquad (i = 1, 2, 3). \tag{3.15}$$

By assumption, a continuous medium cannot have gaps or overlaps. Therefore, a one-to-one correspondence exists between points in the undeformed body and points in the deformed body. Consequently, a unique inverse to (3.15) exists.

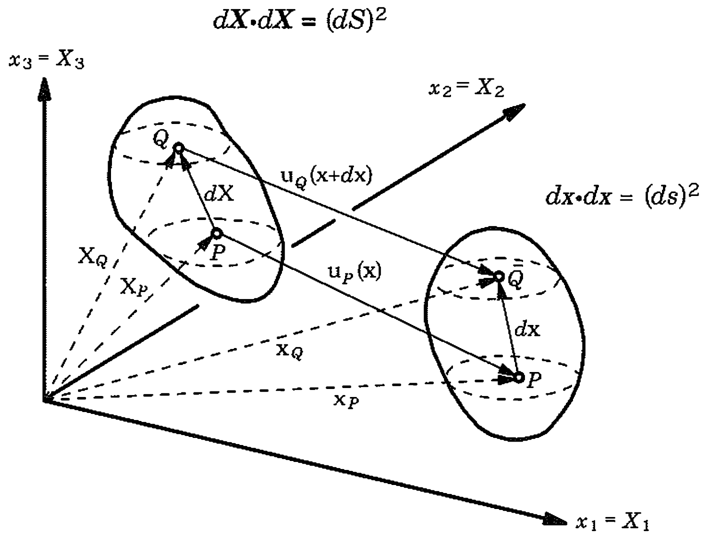

Figure 3.3 Deformation of a material body.

Next, we wish to determine the change in the distance between points $\overline{P}$ and $\overline{Q}$ in the deformed body and compare it with the distance between points P and Q in the undeformed body. The distances between points P and Q and points $\overline{P}$ and $\overline{Q}$ are given, respectively, by

$$(dS)^2 = d\mathbf{X} \cdot d\mathbf{X} = dX_i dX_i, \qquad (ds)^2 = d\mathbf{x} \cdot d\mathbf{x} = dx_i dx_i, \tag{3.16}$$

where summation on repeated subscripts is implied. If $dS \neq ds$, then we know that the body has strained. Since the length of a vector is obtained by computing the square of its length and then taking the square root of it, it is convenient to consider the square of the distance between points. We have

$$\begin{aligned}(ds)^2 - (dS)^2 &= d\mathbf{x} \cdot d\mathbf{x} - d\mathbf{X} \cdot d\mathbf{X} = dx_m dx_m - dX_i dX_i \\ &= \left(\frac{\partial x_m}{\partial X_i} dX_i\right)\left(\frac{\partial x_m}{\partial X_j} dX_j\right) - dX_i dX_i \\ &= \frac{\partial x_m}{\partial X_i}\frac{\partial x_m}{\partial X_j} dX_i dX_j - \delta_{ij} dX_i dX_j \\ &\equiv 2\varepsilon_{ij} dX_i dX_j, \end{aligned} \tag{3.17}$$

where ε_{ij} are the components of the *Green strain tensor* at point P,

$$\overset{\leftrightarrow}{\varepsilon} = \varepsilon_{ij} \hat{\mathbf{E}}_i \hat{\mathbf{E}}_j, \tag{3.18a}$$

with components

$$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial x_m}{\partial X_i}\frac{\partial x_m}{\partial X_j} - \delta_{ij}\right). \tag{3.18b}$$

Here $\hat{\mathbf{E}}_i$ denote the unit basis vectors in the rectangular Cartesian system (X_1, X_2, X_3). The fact that ε_{ij} are components of a second-order tensor will be established shortly. Note that the Green strain tensor components are symmetric by definition. Also, the change in the length of a line element is zero if and only if $\varepsilon_{ij} = 0$.

The strain components can be expressed in terms of the displacement components at point P using Eq. (3.14):

$$x_m = u_m + X_m, \qquad \frac{\partial x_m}{\partial X_i} = \frac{\partial u_m}{\partial X_i} + \delta_{mi}. \tag{3.19}$$

We have

$$\begin{aligned} \varepsilon_{ij} &= \frac{1}{2}\left[\left(\frac{\partial u_m}{\partial X_i} + \delta_{mi}\right)\left(\frac{\partial u_m}{\partial X_j} + \delta_{mj}\right) - \delta_{ij}\right] \\ &= \frac{1}{2}\left(\frac{\partial u_i}{\partial X_j} + \frac{\partial u_j}{\partial X_i} + \frac{\partial u_m}{\partial X_i}\frac{\partial u_m}{\partial X_j}\right). \end{aligned} \tag{3.20}$$

In expanded notation, typical strain components are given by

$$\begin{aligned} \varepsilon_{11} &= \frac{\partial u_1}{\partial X_1} + \frac{1}{2}\left[\left(\frac{\partial u_1}{\partial X_1}\right)^2 + \left(\frac{\partial u_2}{\partial X_1}\right)^2 + \left(\frac{\partial u_3}{\partial X_1}\right)^2\right], \\ \varepsilon_{12} &= \frac{1}{2}\left(\frac{\partial u_1}{\partial X_2} + \frac{\partial u_2}{\partial X_1} + \frac{\partial u_1}{\partial X_1}\frac{\partial u_1}{\partial X_2} + \frac{\partial u_2}{\partial X_1}\frac{\partial u_2}{\partial X_2} + \frac{\partial u_3}{\partial X_1}\frac{\partial u_3}{\partial X_2}\right). \end{aligned} \tag{3.21}$$

When the derivatives of the displacement components are small compared to unity, their products can be neglected:

$$\frac{\partial u_i}{\partial X_j} \ll 1, \qquad \left(\frac{\partial u_i}{\partial X_j}\right)^2 \approx 0. \tag{3.22}$$

In this case, the strain components ε_{ij} become the *infinitesimal strain* components e_{ij}:

$$\varepsilon_{ij} \approx e_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial X_j} + \frac{\partial u_j}{\partial X_i}\right) = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right). \tag{3.23}$$

Example 3.1 Consider a rectangular block of dimensions $a \times b \times h$, where h is very small compared to a and b. Suppose that the block is deformed into the diamond shape shown in Fig. 3.4. By inspection, the geometry of the deformed body can be described as follows: let (X_1, X_2, X_3) denote the coordinates of a material point in the undeformed configuration. In the deformed configuration, the material point has

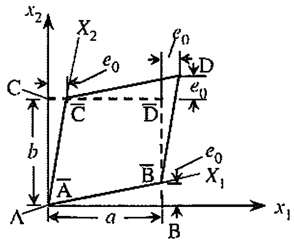

Figure 3.4 Undeformed and deformed rectangular block.

the coordinates

$$x_1 = X_1 + \frac{e_0}{b} X_2,$$
$$x_2 = X_2 + \frac{e_0}{a} X_1,$$
$$x_3 \approx X_3 \quad (\text{because } h \ll a \text{ or } b).$$

Thus, the displacement components of the material point are

$$u_1 = x_1 - X_1 = \frac{e_0}{b} X_2,$$
$$u_2 = x_2 - X_2 = \frac{e_0}{a} X_1,$$
$$u_3 = x_3 - X_3 = 0.$$

The strains are given by

$$\varepsilon_{11} = \frac{1}{2}\left(\frac{e_0}{a}\right)^2, \qquad \varepsilon_{12} = \frac{e_0}{2b} + \frac{e_0}{2a}, \qquad \varepsilon_{22} = \frac{1}{2}\left(\frac{e_0}{b}\right)^2,$$
$$\varepsilon_{13} = \varepsilon_{23} = \varepsilon_{33} = 0.$$

The same results can be obtained using the elementary mechanics of materials approach, where the strains are defined to be the ratio of the difference between the final length and original length to the original length. For example, a line element AB in the undeformed body moves to position $\bar{A}\bar{B}$. Then the strain in the line AB is given by

$$\begin{aligned}\varepsilon_{11} = \varepsilon_{AB} &= \frac{\bar{A}\bar{B} - AB}{AB} \\ &= \frac{1}{a}\sqrt{a^2 + e_0^2} - 1 = \sqrt{1 + \left(\frac{e_0}{a}\right)^2} - 1 \\ &= \left[1 + \frac{1}{2}\left(\frac{e_0}{a}\right)^2 + \cdots\right] - 1 \approx \frac{1}{2}\left(\frac{e_0}{a}\right)^2.\end{aligned}$$

When e_0/b and e_0/a are small compared to unity, their squares can be neglected, and the infinitesimal strain components are given by

$$e_{11} \approx 0, \qquad e_{22} \approx 0, \qquad e_{12} = \frac{e_0}{2ab}(a+b).$$

Now consider a line element oriented at an angle from the X_1-axis. For example, the diagonal line AD deforms and becomes $\bar{A}\bar{D}$ in the deformed body. The normal strain in AD is given by

$$\begin{aligned} \varepsilon'_{xx} &= \frac{\sqrt{(a+e_0)^2+(b+e_0)^2}-\sqrt{a^2+b^2}}{\sqrt{a^2+b^2}} \\ &= \sqrt{1+2\left[\frac{e_0^2+e_0(a+b)}{a^2+b^2}\right]}-1 \approx \frac{e_0^2+e_0(a+b)}{a^2+b^2}. \end{aligned}$$

Since $\overleftrightarrow{\varepsilon}$ is a second-order tensor, we can use the transformation relation in Eq. (2.79) and obtain the same result. We have

$$\begin{aligned} \varepsilon'_{11} &= \beta_{1i}\beta_{1j}\varepsilon_{ij} \\ &= \beta_{11}\beta_{11}\varepsilon_{11} + \beta_{12}\beta_{12}\varepsilon_{22} + \beta_{11}\beta_{12}\varepsilon_{12} + \beta_{12}\beta_{11}\varepsilon_{21}, \end{aligned}$$

where

$$\beta_{ij} = \hat{\bar{\mathbf{e}}}_i \cdot \hat{\mathbf{e}}_j.$$

In particular, we have $\beta_{11} = \cos\theta$, $\beta_{12} = \sin\theta$, $\beta_{22} = \cos\theta$, $\beta_{21} = -\sin\theta$ and $\theta = \tan^{-1}(b/a)$. Hence,

$$\varepsilon'_{11} = \cos^2\theta\,\varepsilon_{11} + \sin^2\theta\,\varepsilon_{22} + 2\sin\theta\cos\theta\,\varepsilon_{12} \equiv \varepsilon_\theta.$$

Substituting for $\cos\theta$, $\sin\theta$, and ε_{ij} into ε'_{ij}, we obtain

$$\begin{aligned} \varepsilon'_{11} &= \frac{a^2}{a^2+b^2}\frac{1}{2}\left(\frac{e_0}{a}\right)^2 + \frac{b^2}{a^2+b^2}\frac{1}{2}\left(\frac{e_0}{b}\right)^2 + 2\frac{ab}{a^2+b^2}\frac{e_0(a+b)}{2ab} \\ &= \frac{e_0^2+e_0(a+b)}{a^2+b^2}. \end{aligned}$$

3.3.2 Strain Compatibility Equations

When a sufficiently differentiable displacement field is given, the computation of the strains is straightforward: one can make use of the equations in (3.20) or (3.23) to compute the strains. However, when the strain components are given, the determination of the displacements is not always possible because there are six strain components related to three displacement components. Stated in other words, there are six differential equations involving three unknowns. Thus, the six equations should be compatible with one another in the sense that any three equations should give the same displacement field.

To understand the situation further, let us consider the two-dimensional case. The strain–displacement relations of the linear theory are given by

$$\begin{aligned} \frac{\partial u_1}{\partial x_1} &= e_{11}, \\ \frac{\partial u_2}{\partial x_2} &= e_{22}, \\ \frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1} &= 2e_{12}. \end{aligned} \tag{3.24}$$

If e_{ij} are given as functions of x_1 and x_2, they cannot be arbitrary: e_{11}, e_{22}, and e_{12} should have a relationship such that the three equations are compatible. This relation can be derived as follows: differentiate the third equation with respect to x_1 and x_2 to obtain

$$\frac{\partial^3 u_1}{\partial x_1 \partial x_2^2} + \frac{\partial^3 u_2}{\partial x_1^2 \partial x_2} = 2\frac{\partial^2 e_{12}}{\partial x_1 \partial x_2}. \tag{3.25}$$

The third derivatives of u_1 and u_2 needed in Eq. (3.25) can be computed from the first two equations in Eq. (3.24):

$$\frac{\partial u_1}{\partial x_1 \partial x_2^2} = \frac{\partial^2 e_{11}}{\partial x_2^2}, \qquad \frac{\partial^2 u_2}{\partial x_1^2 \partial x_2} = \frac{\partial^2 e_{22}}{\partial x_1^2}.$$

Substituting these relations into Eq. (3.25), we obtain a relationship between the derivatives of the strain components:

$$\frac{\partial^2 e_{11}}{\partial x_2^2} + \frac{\partial^2 e_{22}}{\partial x_1^2} = 2\frac{\partial^2 e_{12}}{\partial x_1 \partial x_2}. \tag{3.26}$$

Equation (3.26) is called the compatibility equation for a two-dimensional strain field.

The discussion given above can be generalized to three-dimensional elasticity. First we form the following derivatives of the strains defined in (3.23):

$$e_{ij,kl} = \tfrac{1}{2}(u_{i,jkl} + u_{j,ikl}), \qquad e_{kl,ij} = \tfrac{1}{2}(u_{k,lij} + u_{l,kij}),$$
$$e_{lj,ki} = \tfrac{1}{2}(u_{l,jki} + u_{j,lki}), \qquad e_{ki,lj} = \tfrac{1}{2}(u_{k,ilj} + u_{i,klj}).$$

Next, we add the first two equations and the last two equations separately:

$$e_{ij,kl} + e_{kl,ij} = \tfrac{1}{2}(u_{i,jkl} + u_{j,ikl} + u_{k,lij} + u_{l,kij}),$$
$$e_{lj,ki} + e_{ki,lj} = \tfrac{1}{2}(u_{l,jki} + u_{j,lki} + u_{k,ilj} + u_{i,klj}).$$

We immediately note that the right-hand sides of the two expressions are the same (note that the order of the subscripts following the comma is interchangeable). Therefore, in Cartesian component form we have

$$e_{ij,kl} + e_{kl,ij} = e_{lj,ki} + e_{ki,lj}, \tag{3.27}$$

or in vector form (not shown here; see Exercise 3.33),

$$\nabla \times (\nabla \times \overset{\leftrightarrow}{\mathbf{e}})^{\mathrm{T}} = \overset{\leftrightarrow}{\mathbf{0}} \text{ (zero dyad).} \tag{3.28}$$

Equation (3.27) forms the necessary and sufficient conditions for the existence of a single-valued displacement field (when the strains are given). Although Eq. (3.27) yields 81 equations for the three-dimensional case, only six of them are different from each other, and those remaining are either trivial or linear combinations of the six equations. These six equations are given by

$$\frac{\partial^2 e_{11}}{\partial x_2^2} + \frac{\partial^2 e_{22}}{\partial x_1^2} = 2\frac{\partial^2 e_{12}}{\partial x_1 \partial x_2},$$
$$\frac{\partial^2 e_{22}}{\partial x_3^2} + \frac{\partial^2 e_{33}}{\partial x_2^2} = 2\frac{\partial^2 e_{23}}{\partial x_2 \partial x_3},$$
$$\frac{\partial^2 e_{11}}{\partial x_3^2} + \frac{\partial^2 e_{33}}{\partial x_1^2} = 2\frac{\partial^2 e_{13}}{\partial x_1 \partial x_3},$$

$$\begin{aligned}
\frac{\partial}{\partial x_1}\left(-\frac{\partial e_{23}}{\partial x_1} + \frac{\partial e_{13}}{\partial x_2} + \frac{\partial e_{12}}{\partial x_3}\right) &= \frac{\partial^2 e_{11}}{\partial x_2 \partial x_3},\\
\frac{\partial}{\partial x_2}\left(-\frac{\partial e_{13}}{\partial x_2} + \frac{\partial e_{12}}{\partial x_3} + \frac{\partial e_{23}}{\partial x_1}\right) &= \frac{\partial^2 e_{22}}{\partial x_1 \partial x_3},\\
\frac{\partial}{\partial x_3}\left(-\frac{\partial e_{12}}{\partial x_3} + \frac{\partial e_{23}}{\partial x_1} + \frac{\partial e_{13}}{\partial x_2}\right) &= \frac{\partial^2 e_{33}}{\partial x_1 \partial x_2}.
\end{aligned} \tag{3.29}$$

For the two-dimensional case, the six equations reduce to a single equation

$$\frac{\partial^2 e_{11}}{\partial x_2^2} + \frac{\partial^2 e_{22}}{\partial x_1^2} - 2\frac{\partial^2 e_{12}}{\partial x_1 \partial x_2} = 0. \tag{3.30}$$

It should be noted that the strain compatibility equations are satisfied automatically when the strains are computed from a displacement field. Thus, one needs to verify the compatibility conditions only when the strains are computed from stresses that are in equilibrium.

Example 3.2 Given the strain tensor $\overleftrightarrow{\mathbf{E}} = E_{rr}\hat{\mathbf{e}}_r\hat{\mathbf{e}}_r + E_{\theta\theta}\hat{\mathbf{e}}_\theta\hat{\mathbf{e}}_\theta$ in an axisymmetric body (i.e., E_{rr} and $E_{\theta\theta}$ are functions of r and z only), we wish to determine the compatibility conditions on E_{rr} and $E_{\theta\theta}$.

Using the vector form of compatibility conditions, Eq. (3.28), we obtain

$$\overleftrightarrow{\mathbf{F}} \equiv \nabla \times \overleftrightarrow{\mathbf{E}} = \left(\frac{\partial E_{\theta\theta}}{\partial r} + \frac{E_{\theta\theta} - E_{rr}}{r}\right)\hat{\mathbf{e}}_z\hat{\mathbf{e}}_\theta + \frac{\partial E_{rr}}{\partial z}\hat{\mathbf{e}}_\theta\hat{\mathbf{e}}_r - \frac{\partial E_{\theta\theta}}{\partial z}\hat{\mathbf{e}}_r\hat{\mathbf{e}}_\theta,$$

$$\begin{aligned}\nabla\times(\overleftrightarrow{\mathbf{F}})^{\mathrm{T}} &= \frac{\partial}{\partial r}\left(\frac{\partial E_{\theta\theta}}{\partial r} + \frac{E_{\theta\theta} - E_{rr}}{r}\right)(\hat{\mathbf{e}}_r \times \hat{\mathbf{e}}_\theta)\,\hat{\mathbf{e}}_z \\ &\quad - \frac{\partial^2 E_{\theta\theta}}{\partial r\,\partial z}(\hat{\mathbf{e}}_r \times \hat{\mathbf{e}}_\theta)\,\hat{\mathbf{e}}_r + \frac{1}{r}\left(\frac{\partial E_{\theta\theta}}{\partial r} + \frac{E_{\theta\theta} - E_{rr}}{r}\right)\left(\hat{\mathbf{e}}_\theta \times \frac{\partial \hat{\mathbf{e}}_\theta}{\partial \theta}\right)\hat{\mathbf{e}}_z \\ &\quad + \frac{1}{r}\frac{\partial E_{rr}}{\partial z}\left(\hat{\mathbf{e}}_\theta \times \frac{\partial \hat{\mathbf{e}}_r}{\partial \theta}\right)\hat{\mathbf{e}}_\theta + \frac{1}{r}\frac{\partial E_{rr}}{\partial z}(\hat{\mathbf{e}}_\theta \times \hat{\mathbf{e}}_r)\frac{\partial \hat{\mathbf{e}}_\theta}{\partial \theta} \\ &\quad - \frac{1}{r}\frac{\partial E_{\theta\theta}}{\partial z}\left(\hat{\mathbf{e}}_\theta \times \frac{\partial \hat{\mathbf{e}}_\theta}{\partial \theta}\right)\hat{\mathbf{e}}_r + \frac{\partial}{\partial z}\left(\frac{\partial E_{\theta\theta}}{\partial r} + \frac{E_{\theta\theta} - E_{rr}}{r}\right)(\hat{\mathbf{e}}_z \times \hat{\mathbf{e}}_\theta)\,\hat{\mathbf{e}}_z \\ &\quad + \frac{\partial^2 E_{rr}}{\partial z^2}(\hat{\mathbf{e}}_z \times \hat{\mathbf{e}}_r)\,\hat{\mathbf{e}}_\theta - \frac{\partial^2 E_{\theta\theta}}{\partial z^2}(\hat{\mathbf{e}}_z \times \hat{\mathbf{e}}_\theta)\,\hat{\mathbf{e}}_r = \overleftrightarrow{\mathbf{0}}.\end{aligned}$$

Noting that

$$\frac{\partial \hat{\mathbf{e}}_r}{\partial \theta} = \hat{\mathbf{e}}_\theta, \quad \frac{\partial \hat{\mathbf{e}}_\theta}{\partial \theta} = -\hat{\mathbf{e}}_r, \quad \hat{\mathbf{e}}_r \times \hat{\mathbf{e}}_\theta = \hat{\mathbf{e}}_z, \quad \hat{\mathbf{e}}_\theta \times \hat{\mathbf{e}}_z = \hat{\mathbf{e}}_r, \quad \hat{\mathbf{e}}_z \times \hat{\mathbf{e}}_r = \hat{\mathbf{e}}_\theta,$$

and that a tensor is zero only when all its components are zero, we obtain

$$\hat{\mathbf{e}}_z\hat{\mathbf{e}}_z: \quad \frac{\partial}{\partial r}\left(\frac{\partial E_{\theta\theta}}{\partial r} + \frac{E_{\theta\theta} - E_{rr}}{r}\right) + \frac{1}{r}\left(\frac{\partial E_{\theta\theta}}{\partial r} + \frac{E_{\theta\theta} - E_{rr}}{r}\right) = 0.$$

$$\hat{\mathbf{e}}_z\hat{\mathbf{e}}_r: \quad -\frac{\partial^2 E_{\theta\theta}}{\partial r\,\partial z} + \frac{1}{r}\frac{\partial}{\partial z}(E_{rr} - E_{\theta\theta}) = 0.$$

$$\hat{\mathbf{e}}_r\hat{\mathbf{e}}_z: \quad -\frac{\partial}{\partial z}\left(\frac{\partial E_{\theta\theta}}{\partial r} + \frac{E_{\theta\theta} - E_{rr}}{r}\right) = 0.$$

$$\hat{\mathbf{e}}_\theta\hat{\mathbf{e}}_\theta: \quad \frac{\partial^2 E_{rr}}{\partial z^2} = 0, \qquad \hat{\mathbf{e}}_r\hat{\mathbf{e}}_r: \quad -\frac{\partial^2 E_{\theta\theta}}{\partial z^2} = 0.$$

Example 3.3 We wish to check if the following two-dimensional strain field is compatible:

$$e_{11} = c_1 x_1\left(x_1^2 + x_2^2\right), \qquad e_{22} = \frac{1}{3}c_2 x_1^3, \qquad e_{12} = c_3 x_1^2 x_2,$$

where c_1, c_2, and c_3 are constants. Using Eq. (3.30), we obtain

$$\frac{\partial^2 e_{11}}{\partial x_2^2} + \frac{\partial^2 e_{22}}{\partial x_1^2} - 2\frac{\partial^2 e_{12}}{\partial x_1 \partial x_2} = 2c_1 x_1 + 2c_2 x_1 - 4c_3 x_1.$$

Thus the strain field is not compatible, unless $c_1 + c_2 - 2c_3 = 0$.

3.4 CONSTITUTIVE EQUATIONS

3.4.1 Introduction

The kinematic relations and the mechanical and thermodynamic principles are applicable to any continuum irrespective of its physical constitution. Here we consider equations characterizing the individual material and its reaction to applied loads. These equations are called the *constitutive equations*. The formulation of the constitutive equations for a given material is guided by certain rules (i.e., constitutive axioms). We will not discuss them here but will review the linear constitutive relations for solids undergoing small deformations.

A material body is said to be *homogeneous* if the material properties are the same throughout the body (i.e., independent of position). In a *heterogeneous* body, the material properties are a function of position. An *anisotropic* body is one that has different values of a material property in different directions at a point, i.e., material properties are direction-dependent. An *isotropic* body is one for which every material property in all directions at a point is the same. An isotropic or anisotropic material can be nonhomogeneous or homogeneous.

A material body is said to be *ideally elastic* when, under isothermal conditions, the body recovers its original form completely upon removal of the forces causing deformation, and there is a one-to-one relationship between the state of stress and the state of strain. The constitutive equations described here do not include creep at constant stress and stress relaxation at constant strain. Thus, the material coefficients that specify the constitutive relationship between the stress and strain components are assumed to be constant during the deformation. This does not automatically imply that we neglect temperature effects on deformation. We account for the thermal expansion of the material, which can produce strains or stresses as large as those produced by the applied mechanical forces. The dependence of the constitutive properties on temperature and strains can be accounted for if required. Here, we review the basic constitutive equations of linear elasticity (i.e., generalized Hooke's law) for small displacements.

3.4.2 Generalized Hooke's Law

The generalized Hooke's law relates the six components of stress to the six components of strain with respect to the coordinate axes (x_1, x_2, x_3) aligned with the material planes as

$$\sigma_{ij} = C_{ijkl}\varepsilon_{kl}, \tag{3.31}$$

where σ_{ij} are the stress components, ε_{ij} the strain components, and C_{ijkl} are the elastic coefficients referred to the material coordinates. Alternatively, Eq. (3.31) can be expressed in matrix form as

$$\begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \\ \sigma_5 \\ \sigma_6 \end{Bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} \\ C_{21} & C_{22} & C_{23} & C_{24} & C_{25} & C_{26} \\ C_{31} & C_{32} & C_{33} & C_{34} & C_{35} & C_{36} \\ C_{41} & C_{42} & C_{43} & C_{44} & C_{45} & C_{46} \\ C_{51} & C_{52} & C_{53} & C_{54} & C_{55} & C_{56} \\ C_{61} & C_{62} & C_{63} & C_{64} & C_{65} & C_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \varepsilon_4 \\ \varepsilon_5 \\ \varepsilon_6 \end{Bmatrix}. \tag{3.32}$$

The single subscript notation used in Eq. (3.32) for stresses and strains is called the *contracted notation* or the Voigt–Kelvin notation:

$$\begin{gathered} \sigma_1 = \sigma_{11}, \quad \sigma_2 = \sigma_{22}, \quad \sigma_3 = \sigma_{33}, \quad \sigma_4 = \sigma_{23}, \quad \sigma_5 = \sigma_{13}, \quad \sigma_6 = \sigma_{12}, \\ \varepsilon_1 = \varepsilon_{11}, \quad \varepsilon_2 = \varepsilon_{22}, \quad \varepsilon_3 = \varepsilon_{33}, \quad \varepsilon_4 = 2\varepsilon_{23}, \quad \varepsilon_5 = 2\varepsilon_{13}, \quad \varepsilon_6 = 2\varepsilon_{12}. \end{gathered} \tag{3.33}$$

The two-subscript components C_{ij} are obtained from C_{ijkl} by the following change of subscripts:

$$11 \to 1, \quad 22 \to 2, \quad 33 \to 3, \quad 23 \to 4, \quad 13 \to 5, \quad 12 \to 6.$$

The resulting C_{ij} are also symmetric ($C_{ij} = C_{ji}$) by virtue of the assumption that there exists a potential function $U_0 = U_0(\varepsilon_{ij})$, called the *strain energy density function*, whose derivative with respect to a strain component determines the corresponding stress component:

$$\sigma_{ij} = \frac{\partial U_0}{\partial \varepsilon_{ij}}. \tag{3.34}$$

Such materials are termed *hyperelastic* materials. For hyperelastic materials, there are only 21 independent coefficients of the matrix $[C]$.

When three mutually orthogonal planes of material symmetry exist, the number of elastic coefficients is reduced to 9, and such materials are called *orthotropic*. The stress–strain relations for an orthotropic material take the form

$$\begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \\ \sigma_5 \\ \sigma_6 \end{Bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\ C_{12} & C_{22} & C_{23} & 0 & 0 & 0 \\ C_{13} & C_{23} & C_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & C_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \varepsilon_4 \\ \varepsilon_5 \\ \varepsilon_6 \end{Bmatrix}. \tag{3.35}$$

The inverse relations, strain–stress relations, are given by

$$\begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \varepsilon_4 \\ \varepsilon_5 \\ \varepsilon_6 \end{Bmatrix} = \begin{bmatrix} S_{11} & S_{12} & S_{13} & 0 & 0 & 0 \\ S_{12} & S_{22} & S_{23} & 0 & 0 & 0 \\ S_{13} & S_{23} & S_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & S_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & S_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & S_{66} \end{bmatrix} \begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \\ \sigma_5 \\ \sigma_6 \end{Bmatrix}$$

$$= \begin{bmatrix} \frac{1}{E_1} & -\frac{\nu_{21}}{E_2} & -\frac{\nu_{31}}{E_3} & 0 & 0 & 0 \\ -\frac{\nu_{12}}{E_1} & \frac{1}{E_2} & -\frac{\nu_{32}}{E_3} & 0 & 0 & 0 \\ -\frac{\nu_{13}}{E_1} & -\frac{\nu_{23}}{E_2} & \frac{1}{E_3} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{G_{23}} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{G_{13}} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{G_{12}} \end{bmatrix} \begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \\ \sigma_5 \\ \sigma_6 \end{Bmatrix}, \tag{3.36}$$

where S_{ij} denote the compliance coefficients, $[S] = [C]^{-1}$, E_1, E_2, E_3 are Young's moduli in 1, 2, and 3 material directions, respectively, ν_{ij} is Poisson's ratio, defined as the ratio of transverse strain in the jth direction to the axial strain in the ith direction when stressed in the i-direction, and G_{23}, G_{13}, G_{12} = shear moduli in the 2–3, 1–3, and 1–2 planes, respectively. Since the compliance matrix $[S]$ is the inverse of the stiffness matrix $[C]$ and the inverse of a symmetric matrix is symmetric, it follows that the compliance matrix $[S]$ is also a symmetric matrix. This in turn implies that the following reciprocal relations hold:

$$\frac{\nu_{21}}{E_2} = \frac{\nu_{12}}{E_1}; \qquad \frac{\nu_{31}}{E_3} = \frac{\nu_{13}}{E_1}; \qquad \frac{\nu_{32}}{E_3} = \frac{\nu_{23}}{E_2}, \tag{3.37a}$$

or, in short,

$$\frac{\nu_{ij}}{E_i} = \frac{\nu_{ji}}{E_j} \quad \text{(no sum on } i, j) \tag{3.37b}$$

for $i, j = 1, 2, 3$. Thus, there are only nine independent material coefficients,

$$E_1,\ E_2,\ E_3,\ G_{23},\ G_{13},\ G_{12},\ \nu_{12},\ \nu_{13},\ \nu_{23},$$

for an orthotropic material.

When there exist no preferred directions in the material (i.e., the material has an infinite number of planes of material symmetry), the number of independent elastic coefficients reduces to 2. Such materials are called *isotropic*. For isotropic materials we have $E_1 = E_2 = E_3 = E$, $G_{12} = G_{13} = G_{23} \equiv G$, and $\nu_{12} = \nu_{23} = \nu_{13} \equiv \nu$.

Of the three constants (E, ν, G), only two are independent and the third one is related to the other two by the relation

$$G = \frac{E}{2(1+\nu)}. \tag{3.38}$$

3.4.3 Plane Stress Constitutive Relations

A *plane stress state* is defined to be one in which all transverse stresses are negligible. The strain–stress relations of an orthotropic body in plane stress state can be written as

$$\begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_6 \end{Bmatrix} = \begin{bmatrix} S_{11} & S_{12} & 0 \\ S_{12} & S_{22} & 0 \\ 0 & 0 & S_{66} \end{bmatrix} \begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_6 \end{Bmatrix} = \begin{bmatrix} \frac{1}{E_1} & -\frac{\nu_{21}}{E_2} & 0 \\ -\frac{\nu_{12}}{E_1} & \frac{1}{E_2} & 0 \\ 0 & 0 & \frac{1}{G_{12}} \end{bmatrix} \begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_6 \end{Bmatrix}. \tag{3.39}$$

The strain–stress relations (3.39) are inverted to obtain the stress–strain relations

$$\begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_6 \end{Bmatrix} = \begin{bmatrix} Q_{11} & Q_{12} & 0 \\ Q_{12} & Q_{22} & 0 \\ 0 & 0 & Q_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_6 \end{Bmatrix}, \tag{3.40}$$

where the Q_{ij}, called the *plane stress-reduced stiffnesses*, are given by

$$\begin{aligned} Q_{11} &= \frac{S_{22}}{S_{11}S_{22} - S_{12}^2} = \frac{E_1}{1 - \nu_{12}\nu_{21}}, \\ Q_{12} &= \frac{S_{12}}{S_{11}S_{22} - S_{12}^2} = \frac{\nu_{12}E_2}{1 - \nu_{12}\nu_{21}}, \\ Q_{22} &= \frac{S_{11}}{S_{11}S_{22} - S_{12}^2} = \frac{E_2}{1 - \nu_{12}\nu_{21}}, \\ Q_{66} &= \frac{1}{S_{66}} = G_{12}. \end{aligned} \tag{3.41}$$

Note that the reduced stiffnesses involve four independent material constants, E_1, E_2, ν_{12}, and G_{12}.

The transverse shear stresses are related to the transverse shear strains in an orthotropic material by the relations

$$\begin{Bmatrix} \sigma_4 \\ \sigma_5 \end{Bmatrix} = \begin{bmatrix} Q_{44} & 0 \\ 0 & Q_{55} \end{bmatrix} \begin{Bmatrix} \varepsilon_4 \\ \varepsilon_5 \end{Bmatrix}, \tag{3.42}$$

where $Q_{44} = C_{44} = G_{23}$ and $Q_{55} = C_{55} = G_{13}$.

3.4.4 Thermoelastic Constitutive Relations

When temperature changes occur in the elastic body, we account for the thermal expansion of the material, even though the variation of elastic constants with temperature is neglected. When the strains, geometric changes, and temperature variations are sufficiently small, all governing equations are linear and superposition of mechanical and thermal effects is possible. The linear thermoelastic constitutive equations have the form

$$\sigma_j = C_{ji}[-\alpha_i(T - T_0) + \varepsilon_i], \tag{3.43}$$

$$\varepsilon_j = S_{ji}\sigma_i + \alpha_j(T - T_0), \tag{3.44}$$

where α_i $(i = 1, 2, 3)$ are the linear coefficients of thermal expansion, T denotes temperature, and T_0 is the reference temperature of the undeformed body. In writing Eqs. (3.43) and (3.44), it is assumed that α_i and C_{ij} are independent of strains and temperature. For an isotropic material, we have $\alpha_1 = \alpha_2 = \alpha_3 \equiv \alpha$. The plane stress constitutive relations for a thermoelastic case are given by

$$\begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_6 \end{Bmatrix} = \begin{bmatrix} Q_{11} & Q_{12} & 0 \\ Q_{12} & Q_{22} & 0 \\ 0 & 0 & Q_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_1 - \alpha_1 \Delta T \\ \varepsilon_2 - \alpha_2 \Delta T \\ \varepsilon_6 \end{Bmatrix}, \tag{3.45}$$

where $\Delta T = T - T_0$ is the temperature change from the reference temperature T_0 and α_i $(i = 1, 2)$ are the coefficients of thermal expansion of an orthotropic material in the x_i-coordinate direction. In addition, Eq. (3.42) holds for the thermoelastic case.

Example 3.4 Consider the problem of an isotropic cantilever beam bent by a load F_0 at the free end (see Fig. 3.5). From the elementary beam theory [see Eqs. (4.35) and (4.36)], we calculate the stresses first:

$$\sigma_{11} = \frac{M_3 x_2}{I_3} = -\frac{F_0 x_1 x_2}{I_3}, \qquad \sigma_{12} = \frac{VQ}{I_3 b} = -\frac{F_0}{2I_3}(h^2 - x_2^2). \tag{3.46}$$

Then the strains are given by ($e_{11} = \sigma_{11}/E$ and $e_{12} = \sigma_{12}/2G = \sigma_{12}[(1+\nu)/E]$):

$$e_{11} = -\frac{F_0 x_1 x_2}{EI_3}, \qquad e_{22} = -\nu e_{11} = \nu \frac{F_0 x_1 x_2}{EI_3}, \qquad e_{12} = -\frac{(1+\nu)F_0}{2EI_3}(h^2 - x_2^2), \tag{3.47}$$

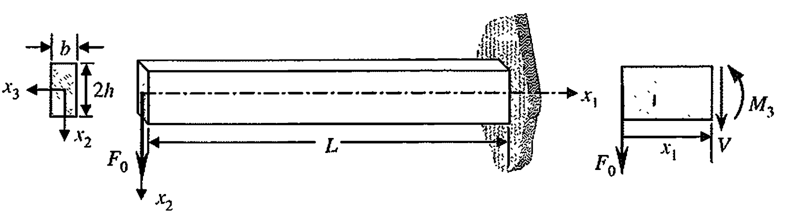

Figure 3.5 Cantilever beam bent by a point load, F_0.

where ν is the Poisson ratio, EI_3 is the bending stiffness (E is Young's modulus and I_3 is the second moment of area about the x_3-axis), and $2h$ is the height of the beam. Next, we determine if these strains are compatible. Substituting e_{ij} into Eq. (3.30), we obtain $0 + 0 = 0$. Thus the strains satisfy the compatibility equations in two dimensions. Although the two-dimensional strains are compatible, the three-dimensional strains are not compatible. For example, using the additional strains, $e_{33} = -\nu e_{11}$, $e_{13} = e_{23} = 0$, one can verify that all of the equations except the last one in Eq. (3.29) are satisfied.

Example 3.5 Consider the cantilever beam problem of Example 3.4. The displacement field in the beam according to the Euler–Bernoulli beam theory is given by (see Example 4.3)

$$u_1(x_1, x_2) = \frac{F_0}{2EI_3}(L^2 - x_1^2)x_2, \tag{3.48a}$$

$$u_2(x_1, x_2) = \frac{F_0}{6EI_3}\left(x_1^3 - 3L^2x_1 + 2L^3\right), \tag{3.48b}$$

and according to the Timoshenko beam theory (see Exercise 4.26), it is

$$u_1(x_1, x_2) = \frac{F_0}{2EI_3}(L^2 - x_1^2)x_2, \tag{3.49a}$$

$$u_2(x_1, x_2) = \frac{F_0}{6EI_3}\left(x_1^3 - 3L^2x_1 + 2L^3\right) + \frac{F_0}{K_sGA}(L - x_1), \tag{3.49b}$$

where K_s is the shear correction factor, and A is the area of cross section. The transverse deflection u_2 of the Timoshenko beam theory has an extra term due to the inclusion of the transverse shear force/strain effect. For a rectangular beam of height $2h$ and width b, we have $A = 2bh$ and $I_3 = 2bh^3/3$.

Since the strains are compatible in two-dimensional theory of elasticity, we wish to compute the displacement field $\mathbf{u} = \hat{\mathbf{e}}_1 u_1 + \hat{\mathbf{e}}_2 u_2$ of the beam using the known strains. Integrating the strains in Eq. (3.47), we obtain

$$e_{11} = \frac{\partial u_1}{\partial x_1} = -\frac{F_0 x_1 x_2}{EI_3} \quad \text{or} \quad u_1 = -\frac{F_0 x_1^2 x_2}{2EI_3} + f_1(x_2), \tag{3.50a}$$

$$e_{22} = \frac{\partial u_2}{\partial x_2} = \frac{\nu F_0 x_1 x_2}{EI_3} \quad \text{or} \quad u_2 = \frac{\nu F_0 x_1 x_2^2}{2EI_3} + f_2(x_1), \tag{3.50b}$$

where (f_1, f_2) are functions of integration. Substituting u_1 and u_2 into the definition of the shear strain $2e_{12}$, we obtain

$$2e_{12} = \frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1} = -\frac{F_0 x_1^2}{2EI_3} + \frac{df_1}{dx_2} + \frac{\nu F_0 x_2^2}{2EI_3} + \frac{df_2}{dx_1}.$$

But this must be equal to that calculated earlier [see Eq. (3.47)]:

$$-\frac{F_0 x_1^2}{2EI_3} + \frac{df_1}{dx_2} + \frac{\nu F_0 x_2^2}{2EI_3} + \frac{df_2}{dx_1} = -\frac{(1+\nu)}{EI_3} F_0 \left(h^2 - x_2^2\right). \tag{3.51}$$

Separating the x_1 and x_2 terms, we obtain

$$-\frac{df_2}{dx_1} + \frac{F_0}{2EI_3} x_1^2 - \frac{(1+\nu) F_0 h^2}{EI_3} = \frac{df_1}{dx_2} - \frac{(2+\nu) F_0}{2EI_3} x_2^2.$$

Since the left side depends only on x_1 and the right side depends only on x_2, and yet the equality must hold, it follows that both sides should be equal to a constant, say c_0:

$$\frac{df_1}{dx_2} - \frac{(2+\nu) F_0}{2EI_3} x_2^2 = c_0, \qquad -\frac{df_2}{dx_1} + \frac{F_0}{2EI_3} x_1^2 - \frac{(1+\nu) F_0 h^2}{EI_3} = c_0.$$

Integrating the expressions for f_1 and f_2, we obtain

$$f_1(x_2) = \frac{(2+\nu) F_0}{6EI_3} x_2^3 + c_0 x_2 + c_1,$$

$$f_2(x_1) = \frac{F_0}{6EI_3} x_1^3 - \frac{(1+\nu) F_0 h^2}{EI_3} x_1 - c_0 x_1 + c_2,$$

where c_1 and c_2 are constants of integration that are to be determined. The displacements (u_1, u_2) are now given by

$$u_1(x_1, x_2) = -\frac{F_0}{2EI_3} x_1^2 x_2 + \frac{(2+\nu) F_0}{6EI_3} x_2^3 + c_0 x_2 + c_1, \tag{3.52a}$$

$$u_2(x_1, x_2) = -\frac{(1+\nu) F_0 h^2}{EI_3} x_1 + \frac{\nu F_0}{2EI_3} x_1 x_2^2 + \frac{F_0}{6EI_3} x_1^3 - c_0 x_1 + c_2. \tag{3.52b}$$

The constants c_0, c_1, and c_2 can be evaluated using the boundary conditions on the displacements. We impose the following boundary conditions:

$$u_1(L, 0) = 0, \qquad u_2(L, 0) = 0, \qquad \left(\frac{\partial u_1}{\partial x_2}\right)\bigg|_{(L,0)} = 0. \tag{3.53}$$

Substituting the expressions for u_1 and u_2 into the boundary conditions (3.53), we obtain

$$u_1(L, 0) = 0 \;\rightarrow\; c_1 = 0$$

$$u_2(L, 0) = 0 \;\rightarrow\; \frac{F_0 L^3}{6EI_3} - c_0 L + c_2 = \frac{(1+\nu) F_0 h^2 L}{EI_3},$$

$$\left(\frac{\partial u_1}{\partial x_2}\right)\bigg|_{(L,0)} = 0 \;\rightarrow\; -\frac{F_0 L^2}{2EI_3} + c_0 = 0 \quad \text{or} \quad c_0 = \frac{F_0 L^2}{2EI_3}.$$

Finally, the displacement field of two-dimensional elasticity theory becomes

$$u_1(x_1, x_2) = \frac{F_0}{2EI_3}(L^2 - x_1^2)x_2 + \frac{(2+\nu)F_0}{6EI_3}x_2^3, \tag{3.54a}$$

$$u_2(x_1, x_2) = \frac{F_0}{6EI_3}(x_1^3 - 3L^2x_1 + 2L^3) + \frac{F_0(1+\nu)h^2}{EI_3}(L - x_1) + \frac{\nu F_0}{2EI_3}x_1x_2^2. \tag{3.54b}$$

The values of the constants c, c_1, and c_2 depend on the boundary conditions used. For example, if one uses $(\partial u_2/\partial x_1) = 0$ in place of $(\partial u_1/\partial x_2) = 0$ at $x_1 = L$ and $x_2 = 0$, the resulting displacements would be

$$u_1(x_1, x_2) = \frac{F_0}{2EI_3}(L^2 - x_1^2)x_2 - \frac{(2+\nu)F_0}{6EI_3}x_2(3h^2 - x_2^2), \tag{3.55a}$$

$$u_2(x_1, x_2) = \frac{F_0}{6EI_3}(x_1^3 - 3L^2x_1 + 2L^3) + \frac{\nu F_0}{2EI_3}x_1x_2^2. \tag{3.55b}$$

A comparison of the elasticity solution (3.54a,b) against the Timoshenko beam solution (3.49a,b) indicates that the elasticity solution has some higher-order terms that are negligible ($x_2 \sim h$). Also one can see that ($GA = 3EI_3/2[(1+\nu)h^2]$) the shear correction factor is $K_s = 2/3$.

EXERCISES

3.1 Let an arbitrary region in a continuous medium be denoted by R and the bounding, closed surface of this region be continuous and denoted by S. Let each point on the bounding surface move with the velocity $\mathbf{v}_s$. It can be shown that the time derivative of the volume integral over some continuous function $Q(\mathbf{r}, t)$ is given by

$$\frac{d}{dt}\int_R Q(\mathbf{r}, t)\, dV \equiv \int_R \frac{\partial Q}{\partial t}\, dV + \oint_S Q\mathbf{v}_s \cdot \hat{\mathbf{n}}\, dS. \tag{a}$$

This expression for the differentiation of a volume integral with variable limits is sometimes known as the three-dimensional *Leibniz rule.*

Let each element of mass in the medium move with the velocity $\mathbf{v}(\mathbf{r}, t)$ and consider a special region R such that the bounding surface S is attached to a fixed set of material elements. Then each point of this surface moves itself with the material velocity, that is, $\mathbf{v}_s = \mathbf{v}$, and the region R thus contains a fixed total amount of mass, since no mass crosses the boundary surface S. To distinguish the time rate of change of an integral over this material region, we replace d/dt by D/Dt and write

$$\frac{D}{Dt}\int_R Q(\mathbf{r},t)\,dV \equiv \int_R \frac{\partial Q}{\partial t}\,dV + \oint_S Q\mathbf{v}\cdot\hat{\mathbf{n}}\,dS, \tag{b}$$

which holds for a material region, that is, a region of fixed total mass. Show that the relation between the time derivative following an arbitrary region and the time derivative following a material region (fixed total mass) is

$$\frac{d}{dt}\int_R Q(\mathbf{r},t)\,dV \equiv \frac{D}{Dt}\int_R Q(\mathbf{r},t)\,dV + \oint_S Q(\mathbf{v}_s-\mathbf{v})\cdot\hat{\mathbf{n}}\,dS. \tag{c}$$

The velocity difference $\mathbf{v}-\mathbf{v}_s$ is the velocity of the material measured relative to the velocity of the surface. The surface integral

$$\oint_S Q(\mathbf{v}-\mathbf{v}_s)\cdot\hat{\mathbf{n}}\,dS \tag{d}$$

thus measures the total *outflow* of the property Q from the region R.

3.2 Let $Q = \rho(\mathbf{r},t)$ denote the mass density of a continuous region. Then conservation of mass for a *material* region requires that

$$\frac{D}{Dt}\int_R \rho\,dV = 0. \tag{a}$$

Show that, for a *fixed region* ($\mathbf{v}_s = 0$), conservation of mass can also be stated as

$$\frac{d}{dt}\int_R \rho\,dV = -\oint_S \rho\mathbf{v}\cdot\hat{\mathbf{n}}\,dS \tag{b}$$

or

$$\int_R \frac{\partial\rho}{\partial t}\,dV = -\oint_S \rho\mathbf{v}\cdot\hat{\mathbf{n}}\,dS. \tag{c}$$

Interpret these equations physically.

3.3 In the field description of a continuous variable $\phi = \phi(\mathbf{r},t)$, let the field position be a function of time such that $\mathbf{r} = \mathbf{r}(t)$. Deduce that the total time derivative of ϕ can be written as

$$\frac{d\phi}{dt} = \frac{\partial\phi}{\partial t} + \frac{d\mathbf{r}}{dt}\cdot\operatorname{grad}\phi. \tag{a}$$

This arbitrary total time derivative corresponds to a change in ϕ following a change in $\mathbf{r}$ with time. If we let $\mathbf{r}$ correspond to the position of a fixed material element, then $d\mathbf{r}/dt = \mathbf{v}$ corresponds to the velocity of the material element. To distinguish the time rates of change following a material element from other arbitrary changes, we write

$$\frac{D\phi}{Dt} \equiv \frac{\partial\phi}{\partial t} + \mathbf{v}\cdot\operatorname{grad}\phi. \tag{b}$$

This is the differential time rate of change of a field variable $\phi(\mathbf{r}, t)$ following a material element. It is referred to as the *material derivative*, the *substantial derivative*, or the *Eulerian derivative*. The corresponding material derivative for integrals was defined in Exercise 3.2. By means of vector identities, show that the continuity equation for mass conservation can be written as

$$\frac{D\rho}{Dt} + \rho \operatorname{div} \mathbf{v} = 0. \tag{c}$$

3.4 The material derivative operator D/Dt corresponds to changes with respect to a fixed mass, that is, $\rho\, dV$ is constant with respect to this operator. Show formally, by means of Leibniz's rule, the divergence theorem, and conservation of mass, that

$$\frac{D}{Dt}\int_R \rho\phi\, dV \equiv \int_R \rho \frac{D\phi}{Dt}\, dV. \tag{a}$$

3.5 Letting a finite volume R shrink to an infinitesimal volume dV, show, by setting $\phi = 1$ in Leibniz's rule and by use of the divergence theorem, that $\operatorname{div} \mathbf{v}$ can be interpreted as

$$\operatorname{div} \mathbf{v} \equiv \lim_{dV \to 0} \frac{1}{dV}\frac{D}{Dt}(dV). \tag{a}$$

The right-hand side can be interpreted as the rate of volumetric strain following a material particle, called the dilatation rate.

3.6 The acceleration of a material element in a continuum is described by

$$\frac{D\mathbf{v}}{Dt} \equiv \frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \operatorname{grad} \mathbf{v}. \tag{a}$$

Show by means of vector identities that the acceleration can also be written as

$$\frac{D\mathbf{v}}{Dt} \equiv \frac{\partial \mathbf{v}}{\partial t} + \operatorname{grad}\left(\frac{v^2}{2}\right) - \mathbf{v} \times \operatorname{curl} \mathbf{v}. \tag{b}$$

This form displays the role of the *vorticity vector*, $\operatorname{curl} \mathbf{v}$.

3.7 Deduce that

$$\mathbf{v} \cdot \frac{D\mathbf{v}}{Dt} = \frac{D}{Dt}\left(\frac{\mathbf{v} \cdot \mathbf{v}}{2}\right). \tag{a}$$

Note that $(\mathbf{v} \cdot \mathbf{v})/2 = v^2/2$ is the kinetic energy per unit mass of a material particle.

3.8 Deduce that

$$\operatorname{curl}\left(\frac{D\mathbf{v}}{Dt}\right) \equiv \frac{D\boldsymbol{\Omega}}{Dt} + \boldsymbol{\Omega} \operatorname{div} \mathbf{v} - \boldsymbol{\Omega} \cdot \nabla \mathbf{v}, \tag{a}$$

where $\boldsymbol{\Omega} \equiv \operatorname{curl} \mathbf{v}$ is the vorticity vector.

3.9 Newton's second law of motion in its elementary form $m\mathbf{a} = \mathbf{F}$ holds strictly for a point particle of fixed mass m. For a material region of continuously distributed mass, Newton's second law reads

$$\frac{D}{Dt} \int_R \rho \mathbf{v} \, dV = \mathbf{F}, \tag{a}$$

where $\mathbf{F}$ is the sum of all the forces acting on the material in R. Deduce by means of the results of Exercise 3.1 that the corresponding law of motion for a region of *variable* mass is

$$\frac{d}{dt} \int_R \rho \mathbf{v} \, dV = \mathbf{F} + \oint_S \rho \mathbf{v}(\mathbf{v}_s - \mathbf{v}) \cdot \hat{\mathbf{n}} \, dS, \tag{b}$$

where $\mathbf{v}_s$ is the velocity of points on the surface S bounding the variable mass region R.

3.10 Newton's second law of motion applied to a continuum states that the rate of change of momentum following a material region of fixed mass is equal to the sum of all the forces on the region. When the forces are divided into surface forces and body forces, Newton's second law reads:

$$\frac{D}{Dt} \int_R \rho \mathbf{v} \, dV = \oint_S \hat{\mathbf{n}} \cdot \overleftrightarrow{\sigma} \, dS + \int_R \rho \mathbf{f} \, dV, \tag{a}$$

where $\overleftrightarrow{\sigma}$ is the surface stress tensor, $\mathbf{f}$ is the body force per unit mass, ρ is the mass density, and $\mathbf{v}$ is the material velocity. Since the material particle mass $\rho \, dV$ is constant with respect to the material time derivative D/Dt, make use of the divergence theorem and obtain the differential form of Newton's second law of motion for a continuum:

$$\rho \frac{D\mathbf{v}}{Dt} = \operatorname{div} \overleftrightarrow{\sigma} + \rho \mathbf{f}. \tag{b}$$

3.11 Multiply the continuity equation

$$\frac{\partial \rho}{\partial t} + \operatorname{div}(\rho \mathbf{v}) = 0 \tag{a}$$

by the velocity $\mathbf{v}$ and add the result to the left-hand side of the momentum equation (Newton's second law) in Exercise 3.10. After use of vector identities, obtain the result

$$\frac{\partial}{\partial t}(\rho \mathbf{v}) + \operatorname{div}(\rho \mathbf{v}\mathbf{v} - \overleftrightarrow{\sigma}) = \rho \mathbf{f}, \tag{b}$$

which is called the conservation form of the momentum equation. The combination $\rho \mathbf{v}\mathbf{v}$ is called the momentum-flux tensor.

3.12 Take the scalar product with $\mathbf{v}$ of the equation for Newton's second law in Exercise 3.10 and obtain the equation of change for the kinetic energy of a material particle in a continuum:

$$\rho \frac{D}{Dt}\left(\frac{v^2}{2}\right) = \mathbf{v} \cdot \operatorname{div} \overset{\leftrightarrow}{\sigma} + \rho \mathbf{v} \cdot \mathbf{f}. \qquad \text{(a)}$$

How do you interpret the rate-of-work terms on the right-hand side?

3.13 Let e denote the thermodynamic internal energy per unit mass of a material. Then the equation of change for total energy of a material region can be written:

$$\frac{D}{Dt}\int_R \rho\left(e + \frac{v^2}{2}\right) dV = \oint_S \hat{\mathbf{n}} \cdot \overset{\leftrightarrow}{\sigma} \cdot \mathbf{v}\, dS + \int_R \rho \mathbf{f} \cdot \mathbf{v}\, dV - \oint_S \mathbf{q} \cdot \hat{\mathbf{n}}\, dS. \qquad \text{(a)}$$

The first two terms on the right-hand side describe the rate of work done on the material region by the surface stresses and the body forces. The third integral describes the net *outflow* of heat from the region, causing a decrease of energy inside the region. The heat-flux vector $\mathbf{q}$ describes the magnitude and direction of the flow of heat energy per unit time and per unit area.

By suitable operations, obtain the differential form of the energy equation:

$$\rho \frac{D}{Dt}\left(e + \frac{v^2}{2}\right) = \operatorname{div}(\overset{\leftrightarrow}{\sigma} \cdot \mathbf{v}) + \rho \mathbf{f} \cdot \mathbf{v} - \operatorname{div} \mathbf{q}. \qquad \text{(a)}$$

Subtract the contribution from kinetic energy and obtain

$$\rho \frac{De}{Dt} = \operatorname{div}(\overset{\leftrightarrow}{\sigma} \cdot \mathbf{v}) - \mathbf{v} \cdot \operatorname{div} \overset{\leftrightarrow}{\sigma} - \operatorname{div} \mathbf{q}. \qquad \text{(b)}$$

This is called the *thermodynamic form* of the energy equation for a continuum.

3.14 The total rate of work done by the surface stresses per unit volume is given by $\operatorname{div}(\overset{\leftrightarrow}{\sigma} \cdot \mathbf{v})$. The rate of work done by the resultant of the surface stresses per unit volume is given by $\mathbf{v} \cdot \operatorname{div} \overset{\leftrightarrow}{\sigma}$. The difference between these two terms yields the rate of work done by the surface stresses in deformation of the material particle, per unit volume. Show that this can be written as

$$\begin{aligned}
\operatorname{div}(\overset{\leftrightarrow}{\sigma} \cdot \mathbf{v}) - \mathbf{v} \cdot \operatorname{div}\overset{\leftrightarrow}{\sigma} &= \overset{\leftrightarrow}{\sigma} : (\nabla \mathbf{v})^{\mathrm{T}} \\
&= \overset{\leftrightarrow}{\sigma} : \nabla \mathbf{v} \quad (\overset{\leftrightarrow}{\sigma} \text{ symmetric}) \\
&= \tfrac{1}{2}\overset{\leftrightarrow}{\sigma} : [\nabla \mathbf{v} + (\nabla \mathbf{v})^{\mathrm{T}}] \quad (\overset{\leftrightarrow}{\sigma} \text{ symmetric}). \qquad \text{(a)}
\end{aligned}$$

3.15 Derive the transformation relations relating the normal and shear stresses, σ_n and σ_s, on a plane whose normal is $\hat{\mathbf{n}} = \cos\theta\hat{\mathbf{e}}_1 + \sin\theta\hat{\mathbf{e}}_2$ to the stress components σ_{11}, σ_{22}, and $\sigma_{12} = \sigma_{21}$ on the $\hat{\mathbf{e}}_1$ and $\hat{\mathbf{e}}_2$ planes (see Fig. E3.15):

$$\begin{aligned} \sigma_n &= \sigma_{11}\cos^2\theta + \sigma_{22}\sin^2\theta + 2\sigma_{12}\sin\theta\cos\theta, \\ \sigma_s &= (\sigma_{22} - \sigma_{11})\sin\theta\cos\theta + \sigma_{12}(\cos^2\theta - \sin^2\theta). \end{aligned} \tag{a}$$

Note that θ is the angle measured from the positive x_1-axis to the normal to the inclined plane. Then show that the principal stresses at a point in a two-dimensional body are given by

$$\begin{aligned} \sigma_{\max} &= \frac{\sigma_{11} + \sigma_{22}}{2} + \sqrt{\left(\frac{\sigma_{11} - \sigma_{22}}{2}\right)^2 + \sigma_{12}^2}, \\ \sigma_{\min} &= \frac{\sigma_{11} + \sigma_{22}}{2} - \sqrt{\left(\frac{\sigma_{11} - \sigma_{22}}{2}\right)^2 + \sigma_{12}^2}, \end{aligned} \tag{b}$$

and that the orientation of the principal planes is given by

$$\theta_p = \frac{1}{2}\tan^{-1}\left[\frac{2\sigma_{12}}{\sigma_{11} - \sigma_{22}}\right].$$

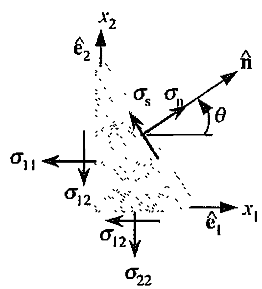

Figure E3.15

3.16 Determine whether the following stress fields are possible in a structural member free of body forces:

(a) $\sigma_{11} = c_1x_1 + c_2x_2 + c_3x_1x_2, \quad \sigma_{12} = -c_3\frac{x_2^2}{2} - c_1x_2,$
$\sigma_{22} = c_4x_1 + c_1x_2.$

(b) $\sigma_{11} = x_1^2 - 2x_1x_2 + cx_3, \quad \sigma_{12} = -x_1x_2 + x_2^2,$
$\sigma_{13} = -x_1x_3, \quad \sigma_{22} = x_2^2, \quad \sigma_{23} = -x_2x_3, \quad \sigma_{33} = (x_1 + x_2)x_3.$

(c) $\sigma_{11} = 3x_1 + 5x_2, \quad \sigma_{12} = 4x_1 - 3x_2, \quad \sigma_{22} = 2x_1 - 4x_2.$

3.17 Given the following state of stress ($\sigma_{ij} = \sigma_{ji}$):

$$\sigma_{11} = -2x_1^2, \qquad \sigma_{12} = -7 + 4x_1x_2 + x_3, \qquad \sigma_{13} = 1 + x_1 - 3x_2,$$
$$\sigma_{22} = 3x_1^2 - 2x_2^2 + 5x_3, \qquad \sigma_{23} = 0, \qquad \sigma_{33} = -5 + x_1 + 3x_2 + 3x_3,$$

determine the body force components for which the stress field describes a state of equilibrium.

3.18 For the cantilever beam bent by a point load at the free end (see Fig. 3.5), the bending moment M_3 about the x_3-axis is given by $M_3 = -F_0x_1$. The bending stress σ_{11} is given by

$$\sigma_{11} = \frac{M_3x_2}{I_3} = -\frac{F_0x_1x_2}{I_3},$$

where I_3 is the moment of inertia of the cross section about the x_3-axis. Starting with this equation, use the two-dimensional equilibrium equations to determine the stresses σ_{22} and σ_{12} as functions of x_1 and x_2.

3.19 Repeat Exercise 3.18 for the case in which the cantilever beam is bent by uniformly distributed load q_0/unit length applied at the top (i.e., at $x_2 = -h$).

3.20 For the state of stress given in Exercise 3.17, determine the stress vector at point (x_1, x_2, x_3) on the plane $x_1 + x_2 + x_3 =$ constant. What are the normal and shearing components of the stress vector at point (1,1,3)?

3.21 Find the principal stresses and their orientation at point (1,2,1) for the state of stress given in Exercise 3.17.

3.22 Find the maximum principal stress and its orientation for the state of stress

$$[\sigma] = \begin{bmatrix} 3 & 5 & 8 \\ 5 & 1 & 0 \\ 8 & 0 & 2 \end{bmatrix} \text{ psi.}$$

3.23 Find the linear strains associated with the displacements

$$u_1 = \left[x_1x_2(2 - x_1) - c_1x_2 + c_2x_2^3\right],$$
$$u_2 = -\left[c_3x_2^2(1 - x_1) + (3 - x_1)\frac{x_1^2}{3} + c_1x_1\right].$$

3.24 The two-dimensional displacement field in a body is given by

$$u_1 = x_1\left[x_1^2x_2 + c_1\left(2c_2^3 + 3c_2^2x_2 - x_2^3\right)\right],$$
$$u_2 = -x_2\left(2c_2^3 + \frac{3}{2}c_2^2x_2 - \frac{1}{4}x_2^3 + \frac{3}{2}c_1x_1^2x_2\right).$$

where c_1 and c_2 are constants. Find the linear and nonlinear strains.

3.25 Determine the displacements and strains in the (x_1, x_2) system for the bodies shown in Figure E3.25.

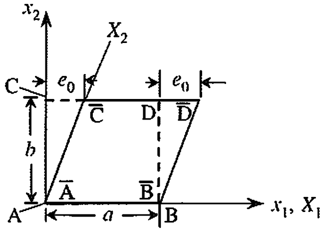

Figure E3.25

3.26 Determine the displacements and strains in the (x_1, x_2) system for the bodies shown in Figure E3.26.

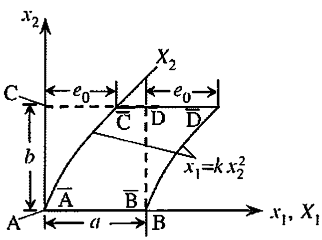

Figure E3.26

3.27 Find the linear strains associated with the displacement field

$$u_1 = u_1^0(x_1, x_2) + x_3\phi_1(x_1, x_2),$$
$$u_2 = u_2^0(x_1, x_2) + x_3\phi_2(x_1, x_2),$$
$$u_3 = u_3^0(x_1, x_2).$$

3.28 Determine whether the following strain fields are possible in a continuous body:

(a) $[e] = \begin{bmatrix} (x_1^2 + x_2^2) & x_1 x_2 \\ x_1 x_2 & x_2^2 \end{bmatrix},$

(b) $[e] = \begin{bmatrix} x_3(x_1^2 + x_2^2) & 2x_1x_2x_3 & x_3 \\ 2x_1x_2x_3 & x_2^2 & x_1 \\ x_3 & x_1 & x_3^2 \end{bmatrix}.$

3.29 Find the normal and shear strains in the diagonal element of the rectangular block in Exercise 3.25.

3.30 The biaxial state of strain at a point is given by $\varepsilon_{11} = 800 \times 10^{-6}$ in./in., $\varepsilon_{22} = 200 \times 10^{-6}$ in./in., $\varepsilon_{12} = 400 \times 10^{-6}$ in./in. Find the principal strains and their directions.

3.31 Find the axial strain in the diagonal element of Exercise 3.25, using (a) the basic definition of normal strain, and (b) the strain transformation equations.

3.32 Using the definition of ∇ and the nonion form of $\overleftrightarrow{\sigma}$, show that the equations of motion in the cylindrical coordinate system are given by

$$\frac{\partial \sigma_{rr}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_{r\theta}}{\partial \theta} + \frac{\partial \sigma_{rz}}{\partial z} + \frac{1}{r}(\sigma_{rr} - \sigma_{\theta\theta}) + \rho_0 f_r = \rho_0 \frac{\partial^2 u_r}{\partial t^2},$$

$$\frac{\partial \sigma_{r\theta}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_{\theta\theta}}{\partial \theta} + \frac{\partial \sigma_{\theta z}}{\partial z} + \frac{2\sigma_{r\theta}}{r} + \rho_0 f_\theta = \rho_0 \frac{\partial^2 u_\theta}{\partial t^2},$$

$$\frac{\partial \sigma_{rz}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_{\theta z}}{\partial \theta} + \frac{\partial \sigma_{zz}}{\partial z} + \frac{\sigma_{rz}}{r} + \rho_0 f_z = \rho_0 \frac{\partial^2 u_z}{\partial t^2}.$$

3.33 Consider the displacement vector in a polar axisymmetric problem,

$$\mathbf{u} = u_r \hat{\mathbf{e}}_r + u_z \hat{\mathbf{e}}_z.$$

The strain tensor in the cylindrical coordinate system can be written in the dyadic form as

$$\begin{aligned}\overleftrightarrow{\varepsilon} &= \varepsilon_{rr}\hat{\mathbf{e}}_{rr} + \varepsilon_{\theta\theta}\hat{\mathbf{e}}_\theta\hat{\mathbf{e}}_\theta + \varepsilon_{zz}\hat{\mathbf{e}}_z\hat{\mathbf{e}}_z + \varepsilon_{r\theta}(\hat{\mathbf{e}}_r\hat{\mathbf{e}}_\theta + \hat{\mathbf{e}}_\theta\hat{\mathbf{e}}_r) \\ &\quad + \varepsilon_{rz}(\hat{\mathbf{e}}_r\hat{\mathbf{e}}_z + \hat{\mathbf{e}}_z\hat{\mathbf{e}}_r) + \varepsilon_{\theta z}(\hat{\mathbf{e}}_\theta\hat{\mathbf{e}}_z + \hat{\mathbf{e}}_z\hat{\mathbf{e}}_\theta).\end{aligned}$$

Show that the only nonzero linear strains are given by

$$\varepsilon_{rr} = \frac{\partial u_r}{\partial r}, \qquad \varepsilon_{\theta\theta} = \frac{u_r}{r}, \qquad \varepsilon_{rz} = \frac{1}{2}\left(\frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r}\right), \qquad \varepsilon_{zz} = \frac{\partial u_z}{\partial z}.$$

3.34 Using the definitions of ∇ and $\mathbf{u}$ in the cylindrical coordinate system, show that the linear strains are given by

$$\varepsilon_{rr} = \frac{\partial u_r}{\partial r}, \quad \varepsilon_{r\theta} = \frac{1}{2}\left(\frac{1}{r}\frac{\partial u_r}{\partial \theta} + \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r}\right), \quad \varepsilon_{rz} = \frac{1}{2}\left(\frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r}\right),$$

$$\varepsilon_{\theta\theta} = \frac{u_r}{r} + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta}, \quad \varepsilon_{z\theta} = \frac{1}{2}\left(\frac{\partial u_\theta}{\partial z} + \frac{1}{r}\frac{\partial u_z}{\partial \theta}\right), \quad \varepsilon_{zz} = \frac{\partial u_z}{\partial z}.$$

3.35 Establish the vector form of the strain compatibility conditions

$$\nabla \times \left(\nabla \times \overleftrightarrow{\mathbf{e}}\right)^{\mathrm{T}} = \overleftrightarrow{\mathbf{0}}.$$

where $\overleftrightarrow{\mathbf{e}}$ is the infinitesimal strain tensor.

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4

WORK, ENERGY, AND VARIATIONAL CALCULUS

4.1 CONCEPTS OF WORK AND ENERGY

Consider a material particle, moving from point A to point B along some path in space under the influence of a force $\mathbf{F}$, which can be time-dependent. The position of the particle is measured from a fixed origin by position vector $\mathbf{r}$. Then the work dW performed by the force $\mathbf{F}$ in moving the particle by an *infinitesimal distance* (or displacement) $d\mathbf{r} = d\mathbf{u}$ along the path over an interval of time dt is defined as (see Fig. 4.1a)

$$dW = \mathbf{F} \cdot d\mathbf{u} = F_1\, du_1 + F_2\, du_2 + F_3\, du_3. \tag{4.1}$$

In other words, work done is the product of the displacement and force in the direction of the displacement. The total work done, W, by the force $\mathbf{F}$ in moving the particle from point A to point B is given by

$$W = \int_A^B \mathbf{F} \cdot d\mathbf{u}. \tag{4.2}$$

By definition, work done is a scalar quantity, and it is positive whenever both displacement and force have the same direction and negative if they are in the opposite directions. Since $\mathbf{u}$ depends on the chosen reference frame, W also depends on the choice of the reference frame. Thus, work is a relative quantity. However, work done does not depend on the path but only the end points, $W = W_B - W_A$. If the reference frame is chosen such that $W_A = 0$, then $W = W_B$.

Analogous to the work done by a force, we can define the work done by a couple (i.e., a pair of equal, opposite forces) $\mathbf{M}$ in moving through an infinitesimal rotation

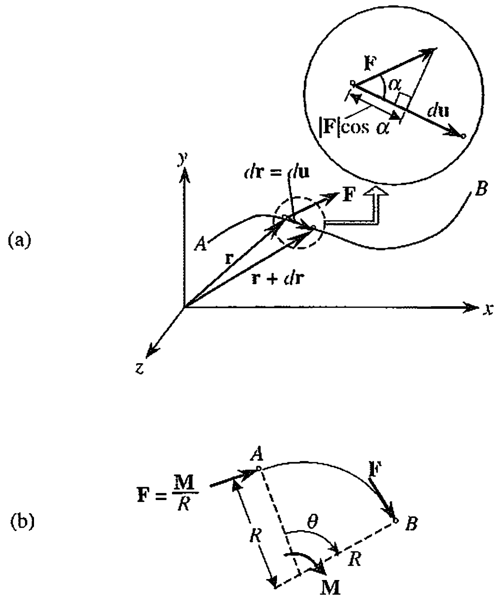

Figure 4.1 (a) Motion of a particle under the action of a force. (b) Motion of a particle under the action of a moment.

$d\theta$ as

$$dW = \mathbf{M} \cdot d\theta = M_1\, d\theta_1 + M_2\, d\theta_2 + M_3\, d\theta_3. \tag{4.3}$$

The total work done in a finite rotation from point A to point B is

$$W = \int_A^B \mathbf{M} \cdot d\theta. \tag{4.4}$$

From Eq. (4.1), it is clear that the rate of work done by force $\mathbf{F}$ in moving the particle through an infinitesimal distance $d\mathbf{r}$ is $\mathbf{F} \cdot d\mathbf{r}/dt$. But $d\mathbf{r}/dt = \mathbf{v}$ is the velocity of the particle. Hence, the rate of work done is equal to $\mathbf{F} \cdot \mathbf{v}$. Now the work done by a force $\mathbf{F}$ in moving through the infinitesimal distance $d\mathbf{r}$ is

$$dW = \mathbf{F} \cdot d\mathbf{r} = \mathbf{F} \cdot \mathbf{v}\, dt.$$

Hence, the total work done during the time interval (t_1, t_2) is

$$W = \int_{t_1}^{t_2} \mathbf{F} \cdot \mathbf{v}\, dt. \tag{4.5}$$

The definition of work can be extended to material bodies, which can be viewed as a collection of material particles. If work is done on all particles of the body, the total work on the body is the sum of all such works. A particle of a deformable body is generally subjected to internal and external forces. Then the total work done on a particle is the sum of the work done by internal and external forces:

$$W = W_I + W_E. \tag{4.6}$$

If a body is subjected to external point forces $\mathbf{F}^1, \mathbf{F}^2, \ldots, \mathbf{F}^n$ that displace the points of action by displacements $\Delta\mathbf{r}_1, \Delta\mathbf{r}_2, \ldots, \Delta\mathbf{r}_n$, respectively, then the work done by the forces on the body during the time interval Δt is the sum of the work done by individual forces in moving through their respective displacements:

$$W_E = -\sum_{i=1}^{n} \mathbf{F}^i \cdot \Delta\mathbf{r}_i = -\sum_{i=1}^{n} \mathbf{F}^i \cdot \mathbf{v}_i \Delta t,$$

where $\mathbf{v}_i$ denotes the mean velocity $\Delta\mathbf{r}_i/\Delta t$. The minus sign indicates that the work is expended on the body as opposed to work stored in the body. If the time increment Δt approaches zero, the sum approaches a limiting value that is represented by an integral, as in Eq. (4.5).

If $\mathbf{f} = \mathbf{f}(\mathbf{x})$ is the distributed force (measured per unit volume) acting on a particle occupying position $\mathbf{x}$ in a body with volume Ω, and $\mathbf{u} = \mathbf{u}(\mathbf{x})$ is the displacement of the particle, then the work done on the particle is $\mathbf{f} \cdot \mathbf{u}$. The work done on all particles occupying the elemental volume $d\Omega$ is $(\mathbf{f} \cdot \mathbf{u})\, d\Omega$. Hence, the total work done on the body is the sum of work done on all particles occupying the body:

$$W_E = -\int_\Omega (\mathbf{f} \cdot \mathbf{u})\, d\Omega.$$

In calculating the external work done, the applied (external) forces (or moments) are assumed to be independent of the displacements (or rotations) they cause in a body. W_E is sometimes called the potential energy due to applied loads and is denoted by V. On the other hand, the internal forces generated inside the body due to the application of the external loads are related to the displacements they move through by the constitutive relations and strain–displacement relations. In general, the work done by internal forces in moving through their respective finite displacements is *not* equal to the work done by (or potential energy due to) external forces in moving through their respective displacements. However, in using energy principles and variational methods for solving problems, we only calculate the work done (internal as well as external) by forces/moments in moving through a set of *imagined* infinitesimal displacements/rotations. In this case, the sum of the internal work done and external work done, with the minus sign in place, is equal to zero.

To understand further the difference between work done by external forces and internal forces, consider a spring–mass system in static equilibrium (see Fig. 4.2a). Suppose that the mass m is placed slowly (to eliminate dynamic effects) at the end of the spring. The spring will elongate by an amount e_0, measured from its undeformed state. In this case, the force is due to gravity and it is equal to $F = mg$. Clearly, F is independent of the extension e_0 in the spring, and F does not change

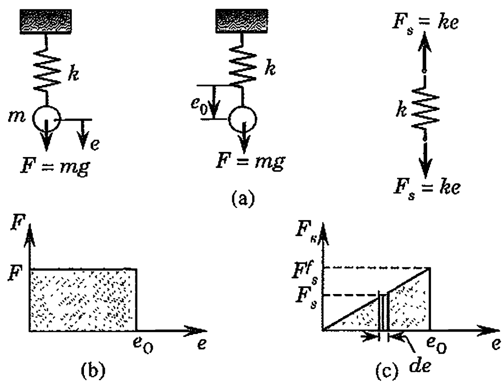

Figure 4.2 A spring-mass system in equilibrium. (a) Elongation due to weight *mg*. (b) External work done. (c) Internal work done.

during the course of the extension e, going from 0 to its final value e_0 (see Fig. 4.2b). The work done by F is

$$W_E = -(Fe_0).$$

Next consider the force in the spring F_s. It goes from 0 to its final value as e goes from 0 to e_0 (see Fig. 4.2c). The work done by F_s in moving through de is $F_s de$. To obtain the total work done, we must integrate, because F_s depends on e, from 0 to e_0:

$$W_I = \int_0^{e_0} F_s(e)\, de.$$

If the spring is assumed to be linearly elastic with spring constant k, we have $F_s = ke$. Hence

$$W_I = \frac{1}{2}ke_0^2.$$

At equilibrium we have $F_s^f = F$, where F_s^f denotes the final force in the spring.

Now suppose that we wish to elongate the spring from e_0 to $e_0 + \Delta e$, Δe being infinitesimally small. Then the additional (or incremental) work done by $F = mg$ and $F_s = F_s(e_0)$ are simply

$$\Delta W_E = -F\Delta e = -mg\Delta e, \qquad \Delta W_I = F_s\Delta e = ke_0\Delta e,$$

which shows that $\Delta W_I = -\Delta W_E$ since $F = F_s$.

Example 4.1 Figure 4.3 shows a rigid link in equilibrium. The initial, no-load position corresponds to $\theta = 0$, and the displacement to the equilibrium position is small (i.e., θ is small). We wish to determine the work done by external forces as well as internal forces when the link is given an infinitesimal rotation $\Delta\theta$ from its equilibrium position. We shall assume that the springs are linearly elastic.

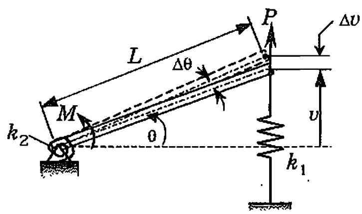

Figure 4.3 A rigid link in equilibrium.

The work done by external forces is given by

$$\begin{aligned}\Delta W_E &= -(P\Delta v + M\Delta\theta)\\ &= -(PL[\sin(\theta+\Delta\theta) - \sin\theta] + M\Delta\theta)\\ &\approx -PL(\theta+\Delta\theta-\theta) - M\Delta\theta = -(PL+M)\Delta\theta.\end{aligned}$$

Once again, the negative sign indicates that the work is done on the body.

The work done by internal force F_s in the extensional spring k_1 and internal moment M_s in the torsional spring k_2 is given by

$$\begin{aligned}\Delta W_I &= F_s\Delta v + M_s\Delta\theta\\ &= k_1 v\Delta v + k_2\theta\Delta\theta\\ &\approx (k_1L^2 + k_2)\theta\Delta\theta.\end{aligned}$$

Note that $\Delta W_I = -\Delta W_E$ (because $P = k_1 L\theta$ and $M = k_2\theta$) or $\Delta W_I + \Delta W_E = 0$. As we shall see later, this is known as the principle of virtual displacements.

Energy is the capacity to do work. It is a measure of the capacity of all forces that can be associated with matter to perform work. Work is performed on a body through a change in energy. For example, consider Newton's second law of motion for a material particle, $\mathbf{F} = m(d\mathbf{v}/dt)$, where m is the mass and $\mathbf{v}$ is the velocity vector. If $\mathbf{r}$ is the position vector from a fixed origin to the particle, the work done by $\mathbf{F}$ in moving the particle through an infinitesimal distance $d\mathbf{r}$ is

$$dW = \mathbf{F}\cdot d\mathbf{r} = m\frac{d\mathbf{v}}{dt}\cdot d\mathbf{r} = m\frac{d\mathbf{v}}{dt}\cdot \mathbf{v}\,dt = \frac{d}{dt}\left(\frac{1}{2}mv^2\right)dt.$$

where v is the magnitude of the velocity vector $\mathbf{v} = d\mathbf{r}/dt$. Since the kinetic energy of the particle is $K = (1/2)mv^2$, it follows that

$$dW = \frac{dK}{dt}dt = dK \quad \text{or} \quad W = \Delta K, \tag{4.7}$$

where ΔK is the increment of kinetic energy resulting from work W. Equation (4.7) states that the work done by a force is equal to the increase in the kinetic energy.

4.2 STRAIN ENERGY AND COMPLEMENTARY STRAIN ENERGY

The first law of thermodynamics gives rise to the following energy equation (see Exercise 3.13) in material description (with no heat generation):

$$\rho \frac{\partial e}{\partial t} = \overleftrightarrow{\sigma} : \nabla \mathbf{v} - \nabla \cdot \mathbf{q}, \tag{4.8}$$

where e is the internal energy per unit mass, $\mathbf{q}$ the heat flux vector, $\overleftrightarrow{\sigma}$ the Cauchy stress tensor, and $\mathbf{v}$ the velocity vector. For elastic bodies under isothermal conditions, the internal energy consists of only stored elastic strain energy, denoted $\rho e = U_0$ and measured per unit volume. In this case (i.e., elastic bodies under isothermal conditions), the energy equation takes the form

$$\frac{\partial U_0}{\partial t} = \overleftrightarrow{\sigma} : \nabla \mathbf{v} \tag{4.9}$$

or (since $\mathbf{v} = d\mathbf{u}/dt$)

$$dU_0 = \overleftrightarrow{\sigma} : \nabla(d\mathbf{u}). \tag{4.10}$$

Note that

$$\begin{aligned} \nabla(d\mathbf{u}) &= d(\nabla \mathbf{u}) = d\left\{\tfrac{1}{2}\left[\nabla \mathbf{u} + (\nabla \mathbf{u})^{\mathrm{T}}\right] + \tfrac{1}{2}\left[\nabla \mathbf{u} - (\nabla \mathbf{u})^{\mathrm{T}}\right]\right\} \\ &= d\overleftrightarrow{\varepsilon} + d\overleftrightarrow{\omega}, \end{aligned}$$

where $\overleftrightarrow{\varepsilon}$ is the linear strain tensor and $\overleftrightarrow{\omega}$ is the rotation tensor. Since the stress tensor is symmetric, the double-dot product $\overleftrightarrow{\sigma} : \overleftrightarrow{\omega}$ is zero, giving

$$\begin{aligned} dU_0 &= \overleftrightarrow{\sigma} : d\overleftrightarrow{\varepsilon} \\ &= \sigma_{ij}\, d\varepsilon_{ij}. \end{aligned} \tag{4.11}$$

Integrating the above equation, we obtain the expression for the strain energy density of an elastic body:

$$U_0 = \int_0^{\varepsilon_{ij}} \sigma_{ij}\, d\varepsilon_{ij}. \tag{4.12}$$

The existence of a scalar function U_0 of strains such that the stresses are derivable from U_0 is of special importance. Such stresses satisfy the energy equation, and consequently they are said to be *conservative*. We have from Eq. (4.11) the result

$$\frac{\partial U_0}{\partial \varepsilon_{ij}} = \sigma_{ij}. \tag{4.13}$$

Often one assumes the existence of $U_0(\varepsilon_{ij})$ such that Eq. (4.13) holds (even for nonlinear elastic bodies with large strains), and Eqs. (4.11) and (4.12) follow from there. We assume that the expression in Eq. (4.12) is valid for all elastic (linear or nonlinear) bodies with linear or nonlinear strain–displacement relations. When thermal effects are to be included, one assumes the existence of a *free energy function* Ψ, called Gibb's free energy:

$$\rho\Psi(\varepsilon_{ij}, T) = U_0(\varepsilon_{ij}, T) - \rho T\eta(\varepsilon_{ij}, T), \tag{4.14}$$

where T denotes temperature and η the *specific entropy*, such that

$$\rho\frac{\partial \Psi}{\partial \varepsilon_{ij}} = \sigma_{ij}, \qquad \frac{\partial \Psi}{\partial T} = -\eta \tag{4.15}$$

hold.

Equation (4.12) can also be arrived by considering the work done by internal forces (i.e., stresses) in moving through displacements. First we consider axial deformation of a bar of area of cross section A. The free-body diagram of an element of length dx_1 of the bar is shown in Fig. 4.4a. Note that the element is in static equilibrium, and we wish to determine the work done by the internal force associated with stress σ_{11}^f, where the superscript f indicates that it is the final value of the quantity. Suppose that the element is deformed slowly so that axial strain varies from 0 to its final value ε_{11}^f. At any instant during the strain variation from ε_{11} to $\varepsilon_{11} + d\varepsilon_{11}$, we assume that σ_{11} (due to ε_{11}) is kept constant so that equilibrium is maintained. Then the work done by $A\sigma_{11}$ in moving through the displacement $d\varepsilon_{11}dx_1$ is

$$A\sigma_{11}\, d\varepsilon_{11}dx_1 = \sigma_{11}\, d\varepsilon_{11}(A\, dx_1) = dU_0(A\, dx_1),$$

where dU_0 denotes the work done per unit volume. Referring to the stress–strain diagram in Fig. 4.4b, dU_0 represents the elemental area *under* the stress–strain curve. The complementary area over the curve is given by

$$dU_0^* = \varepsilon_{11}\, d\sigma_{11}.$$

The total area under the curve is obtained by integrating from zero to the final value of the strain (during which the stress changes according to its relation to the strain):

$$U_0 = \int_0^{\varepsilon_{11}} \sigma_{11}\, d\varepsilon_{11},$$

where the superscript f is omitted as the expression holds for any value of ε. The quantity U_0 is known as the *strain energy density*. The *complementary strain energy density* is given by

$$U_0^* = \int_0^{\sigma_{11}} \varepsilon_{11}\, d\sigma_{11}.$$

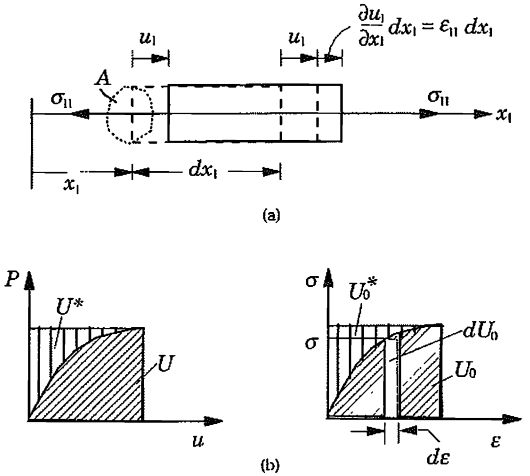

Figure 4.4 Computation of strain energy for the uniaxial case. (a) Work done by force $\sigma_{11} A\, dx_1$. (b) Definitions of various energies.

The internal work done by $A\sigma_{11}$ over the whole element during the entire deformation is

$$dW_I = \int_0^{\varepsilon_{11}} \sigma_{11}\, d\varepsilon_{11} (A\, dx_1) = U_0 (A\, dx_1),$$

and the total work done (or energy stored) in the entire body is obtained by integrating over the length of the bar:

$$W_I = \int_0^L A U_0\, dx_1.$$

This is the internal energy stored in the body. It is called the *strain energy* and denoted by $U = W_I$. Similarly, the *complementary strain energy* is

$$W_I^* = U^* = \int_0^L A U_0^*\, dx_1.$$

Next, we extend the above discussion to the three-dimensional case. Consider the rectangular parallelepiped element (taken from inside an elastic body) of sides dx_1, dx_2, and dx_3 shown in Fig. 4.5a. Suppose that the element is subjected to a system of external forces that vary slowly until they reach their final values, so that the equilibrium is maintained at all times. The forces due to the normal components of stresses are (negative of the same forces act on the opposite faces)

$$\sigma_{11}\, dx_2 dx_3, \qquad \sigma_{22}\, dx_3 dx_1, \qquad \sigma_{33}\, dx_1 dx_2,$$

and the forces due to the shear stresses on the six faces are

$$\sigma_{12}\, dx_2 dx_3,\ \ \sigma_{13}\, dx_3 dx_2,\ \ \sigma_{21}\, dx_1 dx_3,\ \ \sigma_{23}\, dx_3 dx_1,\ \ \sigma_{31}\, dx_1 dx_2,\ \ \sigma_{32}\, dx_2 dx_1.$$

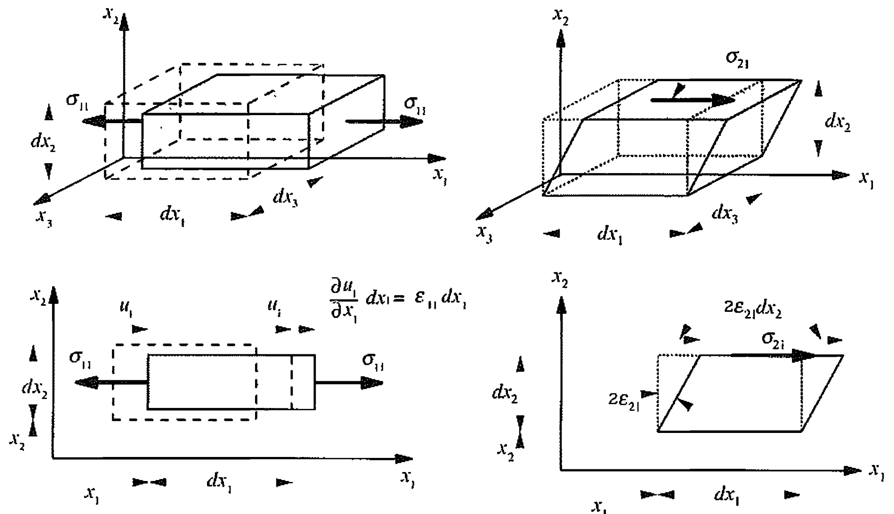

Figure 4.5 Computation of strain energy for the three-dimensional case.

At any stage during the action of these forces, the faces of the parallelepiped will undergo displacements in the normal directions by the amounts $d\varepsilon_{11}dx_1$, $d\varepsilon_{22}dx_2$, and $d\varepsilon_{33}dx_3$, and distort by the amounts $2d\varepsilon_{12}dx_1, 2d\varepsilon_{23}dx_2$, and $2d\varepsilon_{31}dx_3$. As examples, the deformations caused by normal force $\sigma_{11}\, dx_2dx_3$ and shear force $\sigma_{21}\, dx_1dx_3$, each acting alone, are shown in Fig. 4.5b. The work done by individual forces can be summed to obtain the total work done by the simultaneous application of all of the forces, because, for example, an x_1-directed force does no work in the x_2 or x_3 directions. The work done, for instance, during the application of the force $\sigma_{11}\, dx_2dx_3$ is given by (force times displacement)

$$(\sigma_{11}\, dx_2dx_3)(d\varepsilon_{11}dx_1) = \sigma_{11}\, d\varepsilon_{11}dV.$$

Similarly, the work done by the shear force $\sigma_{21}\, dx_1dx_3$ is given by

$$(\sigma_{21}\, dx_1dx_3)(d\varepsilon_{21}dx_2) = \sigma_{21}\, d\varepsilon_{21}dV.$$

The internal work done by all forces in varying slowly from zero to their final values is given by

$$\begin{aligned} dU &= \left(\int_0^{\varepsilon_{11}} \sigma_{11}\, d\varepsilon_{11} + \int_0^{\varepsilon_{12}} \sigma_{12}\, d\varepsilon_{12} + \cdots + \int_0^{\varepsilon_{33}} \sigma_{33}\, d\varepsilon_{33}\right) dV \\ &= \left(\int_0^{\varepsilon_{ij}} \sigma_{ij}\, d\varepsilon_{ij}\right) dV, \end{aligned}$$

where sum on repeated indices is implied but the limit corresponds to a fixed i and j. The expression

$$U_0 = \int_0^{\varepsilon_{ij}} \sigma_{ij}\, d\varepsilon_{ij} \tag{4.16}$$

is the strain energy per unit volume or simply the strain energy density. The complementary strain energy density, U_0^*, can be computed from

$$U_0^* = \int_0^{\sigma_{ij}} \varepsilon_{ij}\, d\sigma_{ij}. \tag{4.17}$$

The total internal work done by internal forces is given by the integral of U_0 over the volume of the body, and it is denoted by U:

$$U = \int_V U_0\, dV, \tag{4.18}$$

and it represents the mechanical energy stored in the body. It is called the *strain energy* of the body. The complementary strain energy U^* of an elastic body is defined by

$$U^* = \int_V U_0^*\, dV. \tag{4.19}$$

Analogous to Eq. (4.13), the strains may be derived from the complementary strain energy density function

$$\varepsilon_{ij} = \frac{\partial U_0^*}{\partial \sigma_{ij}}.$$

Example 4.2 Consider the pin-connected structure shown in Fig. 4.6. The members of the truss are made of an elastic material whose uniaxial stress–strain behavior in tension and compression are given by

$$\sigma = \begin{cases} E\sqrt{\varepsilon}, & \varepsilon \geq 0, \\ -E\sqrt{-\varepsilon}, & \varepsilon \leq 0. \end{cases} \tag{4.20}$$

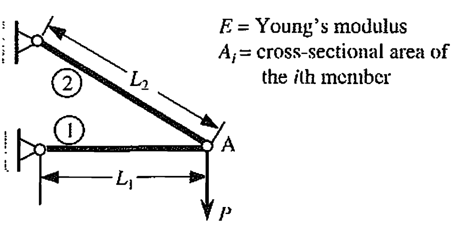

Figure 4.6 Pin-connected structure (truss).

We wish to compute the strain energy density U_0, the complementary strain energy density U_0^*, the strain energy U, and the complementary strain energy U^* of the structure. To compute U_0, we must find the strains in each member of the structure. On the other hand, U_0^* is computed using the stresses in each member. In both cases, the ultimate result will be expressed in terms of the applied (known) load P. Of course, because of the nonlinear stress–strain relation (4.20), $U_0 \neq U_0^*$. However, $U_{0i} + U_{0i}^*$ for the ith member should equal the product (no sum on i) $\sigma_i \varepsilon_i$.

In order to compute the stress and strain in each of the two members, we first compute the forces. From the free-body diagram of joint A, the axial forces in each member can be calculated as ($F_2 \sin 30° = P$ and $F_1 + F_2 \cos 30° = 0$)

$$F_1 = -\sqrt{3}P, \qquad F_2 = 2P. \tag{4.21}$$

Then the stresses in each member are calculated as (with $\sigma = F/A$)

$$\sigma_1 = -\frac{\sqrt{3}P}{A_1}, \qquad \sigma_2 = \frac{2P}{A_2}, \tag{4.22}$$

where A_1 and A_2 denote the cross-sectional areas of members 1 and 2, respectively. The strains are calculated using the stress–strain relation (4.20) (using $\varepsilon = \pm\sigma^2/E^2$):

$$\varepsilon_1 = -\frac{3P^2}{A_1^2E^2}, \qquad \varepsilon_2 = \frac{4P^2}{A_2^2E^2}. \tag{4.23}$$

Then the strain energy density of each member of the structure is

$$\begin{aligned} U_{01} &= \int_0^{\varepsilon_1} \sigma_1\, d\varepsilon_1 = \int_0^{\varepsilon_1} \left(-E\sqrt{-\varepsilon_1}\right) d\varepsilon_1 = -\frac{2E}{3}(-\varepsilon_1)^{3/2} = \frac{2\sqrt{3}P^3}{A_1^3E^2}, \\ U_{02} &= \int_0^{\varepsilon_2} \sigma_2\, d\varepsilon_2 = \int_0^{\varepsilon_2} \left(E\sqrt{\varepsilon_2}\right) d\varepsilon_2 = \frac{2E}{3}(\varepsilon_2)^{3/2} = \frac{2}{3}\left(\frac{8P^3}{A_2^3E^2}\right). \end{aligned} \tag{4.24}$$

The strain energy of the structure is

$$\begin{aligned} U &= \int_{V_1} U_{01}\, dV + \int_{V_2} U_{02}\, dV \\ &= U_{01}A_1L_1 + U_{02}A_2L_2 \\ &= \frac{2}{3}\left[\left(\frac{3\sqrt{3}P^3L_1}{A_1^2E^2}\right) + \left(\frac{8P^3L_2}{A_2^2E^2}\right)\right], \end{aligned} \tag{4.25}$$

where L_i is the length of the ith member.

The complementary strain energy density of each member of the structure is

$$U_{01}^* = \int_0^{\sigma_1} \varepsilon_1 \, d\sigma_1 = \int_0^{\sigma_1} \left(-\frac{\sigma_1^2}{E^2}\right) d\sigma_1 = \frac{1}{3E^2}(-\sigma_1^3) = \frac{\sqrt{3}P^3}{A_1^3 E^2},$$
$$U_{02}^* = \int_0^{\sigma_2} \varepsilon_2 \, d\sigma_2 = \int_0^{\sigma_2} \left(\frac{\sigma_2^2}{E^2}\right) d\sigma_2 = \frac{1}{3E^2}(\sigma_2^3) = \frac{1}{3}\left(\frac{8P^3}{A_2^3 E^2}\right). \tag{4.26}$$

The complementary strain energy of the structure is

$$\begin{aligned} U^* &= \int_{V_1} U_{01}^* \, dV + \int_{V_2} U_{02}^* \, dV \\ &= U_{01}^* A_1 L_1 + U_{02}^* A_2 L_2 \\ &= \frac{1}{3}\left[\left(\frac{3\sqrt{3}P^3 L_1}{A_1^2 E^2}\right) + \left(\frac{8P^3 L_2}{A_2^2 E^2}\right)\right]. \end{aligned} \tag{4.27}$$

Example 4.3 *(Euler–Bernoulli Beam Theory)* Consider bending of a straight beam according to the classical (Euler–Bernoulli) beam theory. We wish to compute the strain energy density, complementary strain energy density, strain energy, and complementary strain energy due to extension, bending, and transverse shear. For the strain energy density computation, the strain will be expressed in terms of the axial and transverse displacements, whereas for the complementary strain energy density computation, the stress will be expressed in terms of the axial and transverse forces, bending moment, and torque (valid only for circular cylindrical members). The three modes of deformation are separable for straight members with small strains. In particular, the following development treats bars as a subset of beams.

We begin with a review of the basic equations of bars and beams. The Euler–Bernoulli hypothesis concerning the kinematics of bending deformation assumes that straight lines perpendicular to the beam axis before deformation remain (a) straight, (b) perpendicular to the tangent line to the beam axis, and (c) inextensible after deformation. These assumptions lead to the following displacement field (see Fig. 4.7a):

$$u = u_0(x) - z\frac{dw_0}{dx}, \qquad v = 0, \qquad w = w_0(x), \tag{4.28}$$

where (u, v, w) are the displacements of a point (x, y, z) along the x, y, and z coordinates, respectively, and (u_0, w_0) are the displacements of the point $(x, 0, 0)$. Under the assumption of smallness of strains and rotations, the only nonzero strain is

$$\varepsilon_{xx} = \frac{du_0}{dx} - z\frac{d^2 w_0}{dx^2}. \tag{4.29}$$

Suppose that the beam is subjected to distributed axial force $f(x)$ and transverse load $q(x)$. Summing the forces and moments on an element of the beam (see Fig. 4.7b)

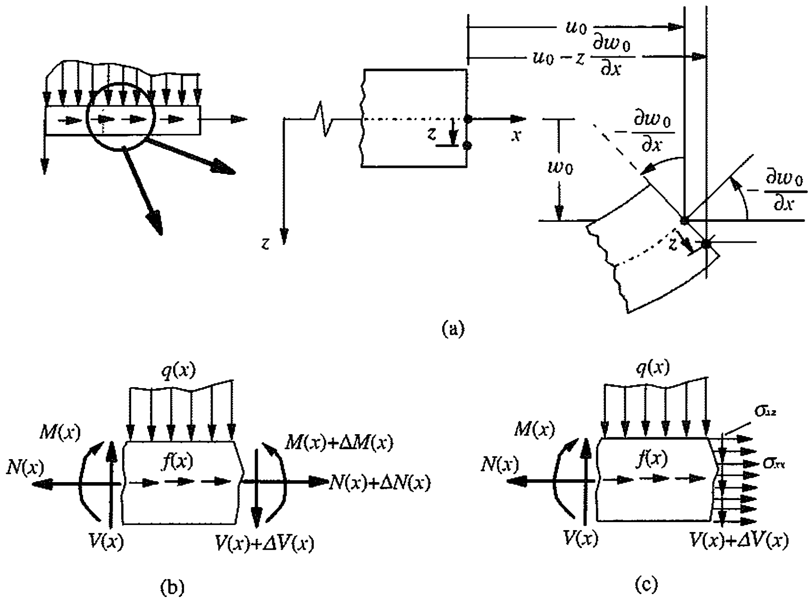

Figure 4.7 Bending of beams. (a) Kinematics of deformation of an Euler–Bernoulli beam. (b) Equilibrium of a beam element. (c) Definitions (or internal equilibrium) of stress resultants.

gives the following equilibrium equations:

$$\sum F_x = 0: \quad -\frac{dN}{dx} = f(x), \tag{4.30a}$$

$$\sum F_z = 0: \quad -\frac{dV}{dx} = q(x), \tag{4.30b}$$

$$\sum M_y = 0: \quad V - \frac{dM}{dx} = 0, \tag{4.30c}$$

where $N(x)$ is the axial force, $M(x)$ the bending moment, and $V(x)$ the shear force. These are known as the stress resultants and they can be defined in terms of the stresses σ_{xx} and σ_{xz} on a cross section as (see Fig. 4.7c)

$$N(x) = \int_A \sigma_{xx}\, dA, \qquad M(x) = \int_A \sigma_{xx} z\, dA, \qquad V(x) = \int_A \sigma_{xz}\, dA. \tag{4.31}$$

Here A denotes the area of cross section. Note that for linear strain measures, Eq. (4.30a), which governs the axial deformation of bars (i.e., axially loaded members), is uncoupled from Eqs. (4.30b,c). Hence it can be solved independently of beam equations (4.30b,c).

The stress resultants (N, M) can be related back to the stress σ_{xx} using the linear elastic constitutive relation for an isotropic material as

$$\sigma_{xx} = E\varepsilon_{xx} = E\left(\frac{du_0}{dx} - z\frac{d^2w_0}{dx^2}\right). \tag{4.32}$$

First, note that

$$N(x) = \int_A \sigma_{xx}\, dA = E\frac{du_0}{dx}\int_A dA = EA\frac{du_0}{dx}, \tag{4.33a}$$

$$M(x) = \int_A \sigma_{xx} z\, dA = E\int_A \left(\frac{du_0}{dx} - z\frac{d^2w_0}{dx^2}\right) z\, dA = -EI\frac{d^2w_0}{dx^2}, \tag{4.33b}$$

or

$$\frac{du_0}{dx} = \frac{N}{EA}, \qquad \frac{d^2w_0}{dx^2} = -\frac{M}{EI}. \tag{4.34}$$

Using the relations in Eq. (4.34) in Eq. (4.32), we obtain

$$\sigma_{xx} = \frac{N}{A} + \frac{Mz}{I}, \tag{4.35}$$

where I is the moment of inertia about the axis of bending (y-axis) and z is the transverse coordinate. Note that the x-axis is taken through the geometric centroid of the cross section so that $\int_A z\, dA = 0$.

The Euler–Bernoulli beam theory is inconsistent with the reality in the sense that the shear stress computed through the stress–strain relation ($\sigma_{xz} = 2G\varepsilon_{xz}$) is zero, since $\varepsilon_{xz} = 0$. Consequently, $V(x)$ is zero as per the third equation of (4.31). However, in computing the complementary strain energy, we assume that the shear strain is derived from $\gamma_{xz} = 2\varepsilon_{xz} = (\sigma_{xz}/G)$ and shear stress is computed, as V is not zero in reality but given by Eq. (4.30c), using the relation (see a book on mechanics of materials for its derivation)

$$\sigma_{xz} = \frac{VQ}{Ib}. \tag{4.36}$$

Here $Q(z)$ denotes the first moment of the hatched area (see Fig. 4.8a) about the centroidal axis y of the entire cross section, and b is the width (or thickness) of the cross section at z where the longitudinal shear stress (σ_{zx}) acts. The hatched area is only a portion of the total area of cross section that lies below the surface on which the shear force acts; Q is the maximum at the centroid (i.e., $z = 0$), and σ_{xz} is the maximum wherever Q/b is the maximum.

A circular cylindrical member subjected to torque T about its longitudinal axis develops shear stress $\sigma_{x\theta}$ in its cross section (see Fig. 4.8b). It can be shown that the stress is related to the torque applied by the relation

$$\sigma_{x\theta}(r) = \frac{Tr}{J}, \tag{4.37}$$

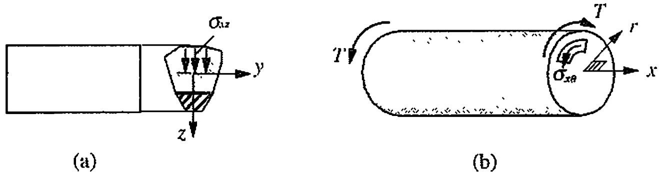

Figure 4.8 (a) Calculation of transverse shear stress on a cross section. (b) Torsion of a circular cylindrical member.

where r is the radial distance from the axis of the beam to the location where $\sigma_{x\theta}$ acts and J is the polar moment of area of the entire cross section. For a solid circular member of diameter d, we have $J = \pi d^4/32$. The stress is the maximum at the outer surface (i.e., $r = d/2$) of the member.

Returning to the computation of various energies, we shall compute them for all components of stresses in a beam. For noncircular members, the part involving T should be omitted. The strain energy density of the beam subjected to axial and bending loads is given by

$$\begin{aligned} U_0 &= \int_0^{\varepsilon_{xx}} \sigma_{xx}\, d\varepsilon_{xx} = \int_0^{\varepsilon_{xx}} E\varepsilon_{xx}\, d\varepsilon_{xx} \\ &= \frac{E}{2}\varepsilon_{xx}^2 = \frac{E}{2}\left(\frac{du_0}{dx} - z\frac{d^2 w_0}{dx^2}\right)^2, \end{aligned} \tag{4.38}$$

and the strain energy becomes

$$\begin{aligned} U &= \int_V U_0\, dV = \int_0^L \int_A \frac{E}{2}\left(\frac{du_0}{dx} - z\frac{d^2 w_0}{dx^2}\right)^2 dA dx \\ &= \frac{1}{2}\int_0^L \left[EA\left(\frac{du_0}{dx}\right)^2 + EI\left(\frac{d^2 w_0}{dx^2}\right)^2\right] dx. \end{aligned} \tag{4.39}$$

For pure extension we set $w_0 = 0$, and for pure bending we set $u_0 = 0$, to obtain the strain energy expression for bars and beams, respectively.

The strain energy density due to shearing by torque T is given by

$$U_0 = 2\int_0^{\varepsilon_{x\theta}} \sigma_{x\theta}\, d\varepsilon_{x\theta} = \int_0^{\varepsilon_{x\theta}} 4G\varepsilon_{x\theta}\, d\varepsilon_{x\theta} = 2G\varepsilon_{x\theta}^2. \tag{4.40}$$

The complementary strain energy density of a linear elastic beam is

$$\begin{aligned} U_0^* &= \int_0^{\sigma_{xx}} \varepsilon_{xx}\, d\sigma_{xx} + 2\int_0^{\sigma_{xz}} \varepsilon_{xz}\, d\sigma_{xz} + 2\int_0^{\sigma_{x\theta}} \varepsilon_{x\theta}\, d\sigma_{x\theta} \\ &= \frac{\sigma_{xx}^2}{2E} + \frac{\sigma_{xz}^2}{2G} + \frac{\sigma_{x\theta}^2}{2G} \end{aligned} \tag{4.41}$$

and the complementary strain energy becomes

$$
\begin{aligned}
U^* &= \int_V U_0^* \, dV \\
&= \int_0^L \int_A \left[\frac{1}{2E} \left(\frac{N}{A} + \frac{Mz}{I} \right)^2 + \frac{1}{2G} \left(\frac{VQ}{Ib} \right)^2 + \frac{1}{2G} \left(\frac{Tr}{J} \right)^2 \right] dA dx \\
&= \frac{1}{2} \int_0^L \left(\frac{N^2}{EA} + \frac{M^2}{EI} \right) dx + \frac{1}{2} \int_0^L \left(\frac{V^2 f_s}{GA} \right) dx + \frac{1}{2} \int_0^L \left(\frac{T^2}{GJ} \right) dx, \qquad (4.42)
\end{aligned}
$$

where

$$
f_s = \frac{A}{I^2 b^2} \int_A Q^2(z) \, dA. \qquad (4.43)
$$

Example 4.4 Consider the frame structure shown in Fig. 4.9. Each member of the structure has the same cross-sectional area A, moment of inertia I, and Young's modulus E. The material of the frame is assumed to be linearly elastic. We wish to compute the complementary strain energy of the frame structure.

The expressions for the axial and shear forces and bending moments for the two parts of the beam are

$$
\begin{aligned}
&N_{DB} = P_h, \qquad N_{BA} = P_v, \qquad V_{DB} = P_v, \qquad V_{BA} = P_h, \\
&M_{DB} = P_v \cdot x, \qquad M_{BA} = P_v \cdot b + P_h \cdot x.
\end{aligned}
$$

The complementary strain energy is $U^* = U_{DB}^* + U_{BA}^*$ [see Eq. (4.42)] with

$$
U_{DB}^* = \int_0^b \left[\frac{P_h^2}{2EA} + \frac{1}{2EI} (P_v x)^2 + \frac{f_s P_v^2}{2GA} \right] dx = \frac{P_h^2 b}{2EA} + \frac{P_v^2 b^3}{6EI} + \frac{f_s P_v^2}{2GA},
$$

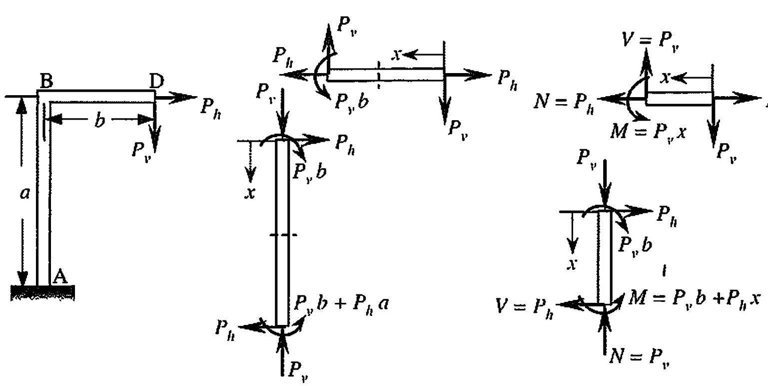

Figure 4.9 The frame structure of Example 4.4.

$$U_{BA}^{*} = \int_0^a \left[\frac{P_v^2}{2EA} + \frac{1}{2EI}(P_v b + P_h x)^2 + \frac{f_s P_h^2}{2GA} \right] dx$$

$$= \frac{P_v^2 a}{2EA} + \frac{1}{2EI}\left(P_v^2 ab^2 + P_v P_h a^2 b + \frac{1}{3} P_h^2 a^3 \right) + \frac{f_s P_h^2}{2GA}.$$

4.3 VIRTUAL WORK

We shall use the term *configuration* to mean the simultaneous position of all material points of a body. From purely geometrical considerations, a given mechanical system can take many possible configurations. The set of configurations that satisfy the geometric constraints of the system is called the *set of admissible configurations*. Of all admissible configurations, only one of them corresponds to the equilibrium configuration under the applied loads; it is this configuration that also satisfies Newton's second law (i.e., equilibrium of forces and moments). The admissible configurations are restricted to a neighborhood of the true configuration so that they are obtained from infinitesimal *variations* of the true configuration (i.e., infinitesimal movement of the material points). During such variations, the geometric constraints of the system are not violated and all applied forces are fixed at their actual equilibrium values. When a mechanical system experiences such variations in its equilibrium configuration, it is said to undergo *virtual displacements*. These displacements need not have any relationship with the actual displacements that might occur due to a change in the loads and/or boundary conditions. The displacements are called virtual because they are *imagined* to take place (i.e., hypothetical), with the actual loads acting at their fixed values.

For example, consider a beam fixed at $x = 0$ and subjected to any arbitrary loading (e.g., distributed as well as point loads), as shown in Fig. 4.10. The possible geometric configurations that the beam can take under the loads may be expressed in terms of the transverse deflection $w_0(x)$ and axial displacement $u_0(x)$. The support conditions require that

$$w_0(0) = \hat{w}_0, \qquad \left(\frac{dw_0}{dx}\right)_{x=0} = \hat{\theta}_0, \qquad u_0(0) = \hat{u}_0, \tag{4.44}$$

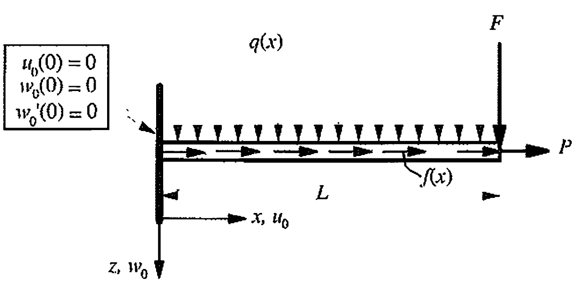

Figure 4.10 A cantilever beam with a set of arbitrary loads.

where $\hat{w}_0$, $\hat{\theta}_0$, and $\hat{u}_0$ are constants (which are zero for the beam shown in Fig. 4.10). These are called the *geometric boundary conditions*. Boundary conditions that involve specifying the forces applied on the beam are called *force boundary conditions*.

The set of all functions $w(x)$ and $u(x)$ that satisfy the geometric boundary conditions (4.44) is the space of admissible configurations for this case. This space consists of pairs of elements $\{(w_i, u_i)\}$ of the form

$$w_1(x) = \hat{w}_0 + \hat{\theta}_0 x + a_1 x^2, \qquad w_2(x) = \hat{w}_0 + \hat{\theta}_0 x + a_1 x^2 + a_2 x^3, \ldots,$$

$$u_1(x) = \hat{u}_0 + b_1 x, \qquad u_2(x) = \hat{u}_0 + b_1 x + b_2 x^2, \ldots.$$

The pair (w_0, u_0) that also satisfies, in addition to the geometric boundary conditions, the equilibrium equations and force boundary conditions (which require the precise nature of the applied loads) of the problem is the equilibrium solution we seek. The virtual displacements, $\delta w_0(x)$ and $\delta u_0(x)$, must be necessarily of the form

$$\delta w_0 = c_1 x^2, \quad \delta u_0 = d_1 x, \quad \delta w_0 = c_1 x^2 + c_2 x^3, \quad \delta u_0 = d_1 x + d_2 x^2, \ldots,$$

so that they satisfy the homogeneous form of the specified geometric boundary conditions:

$$\delta w_0(0) = 0, \qquad \left(\frac{d\delta w_0}{dx}\right)_{x=0} = 0, \qquad \delta u_0(0) = 0. \tag{4.45}$$

Thus, the virtual displacements at the boundary points at which the geometric conditions are specified, independent of the specified values, are necessarily zero.

The work done by the actual forces through a virtual displacement of the actual configuration is called *virtual work* done by actual forces. If we denote the virtual displacement by $\delta \mathbf{u}$, the virtual work done by a constant force $\mathbf{F}$ is given by

$$\delta W = \mathbf{F} \cdot \delta \mathbf{u}.$$

The virtual work done by actual forces in moving through virtual displacements in a deformable body consists of two parts: virtual work done by internal forces, δW_I, and virtual work done by external forces, δW_E. These may be computed as discussed next.

Consider a deformable body of volume V and closed surface area S. Suppose that the body is subjected to a body force $\mathbf{f}(\mathbf{x})$ per unit mass (or $\rho \mathbf{f}$ per unit volume) of the body, a specified surface force $\mathbf{t}(s)$ per unit area on a portion S_2 of the boundary, and specified displacement $\mathbf{u}$ on the remaining portion S_1 of the boundary such that $S = S_1 \cup S_2$ and $S_1 \cap S_2 = 0$. For this case, the virtual displacement $\delta \mathbf{u}(\mathbf{x})$ is any function, small in magnitude (to keep the system in equilibrium), that satisfies the requirement

$$\delta \mathbf{u} = \mathbf{0} \qquad \text{on } S_1.$$

The virtual work done by actual forces **f** and **t** in moving through the virtual displacement $\delta\mathbf{u}$ is

$$\delta W_E = -\left(\int_V \rho\mathbf{f}\cdot\delta\mathbf{u}\,dV + \int_{S_2}\mathbf{t}\cdot\delta\mathbf{u}\,dS\right). \tag{4.46}$$

The negative sign in front of the expression indicates that work is performed *on* the body.

As a result of the application of the loads, the body develops internal forces in the form of stresses. These stresses also perform work when the body is given a virtual displacement. In this study, we shall be concerned mainly with only *ideal systems*. An ideal system is one in which no work is dissipated by friction. We assume that the virtual displacement $\delta\mathbf{u}$ is applied slowly from zero to its final value. Associated with the virtual displacement is the virtual strain (for the linear case),

$$\delta\overleftrightarrow{\varepsilon} = \tfrac{1}{2}[\nabla(\delta\mathbf{u}) + (\nabla(\delta\mathbf{u}))^{\mathrm{T}}]. \tag{4.47}$$

The internal virtual work stored in the body per unit volume, analogous to the calculation of the strain energy, is the virtual strain energy density

$$\delta U_0 = \int_0^{\delta\overleftrightarrow{\varepsilon}} \sigma : d(\delta\overleftrightarrow{\varepsilon}) = \int_0^{\delta\varepsilon_{ij}} \sigma_{ij}d(\delta\varepsilon_{ij}) = \sigma_{ij}\delta\varepsilon_{ij}, \tag{4.48}$$

irrespective of the constitutive behavior. The total internal virtual work stored in the body is denoted by δW_I, and it is equal to

$$\delta W_I = \int_V \delta U_0\,dV = \int_V \sigma_{ij}\delta\varepsilon_{ij}\,dV. \tag{4.49}$$

Analogous to virtual displacements, we can also think of virtual forces. If a body is imagined to be subjected to a set of *self-equilibrating force system* $\delta\mathbf{F}$, the virtual work done by the virtual forces in moving through actual displacements **u** is given by

$$\delta W^* = \delta\mathbf{F}\cdot\mathbf{u},$$

and it is called *complementary virtual work*. The complementary internal virtual work and complementary external virtual work for a deformable body can be expressed, following the ideas already presented, as

$$\delta W_E^* = -\left(\int_V \rho\delta\mathbf{f}\cdot\mathbf{u}\,dV + \int_S \delta\mathbf{t}\cdot\mathbf{u}\,dS\right), \tag{4.50}$$

$$\delta U_0^* = \varepsilon_{ij}\delta\sigma_{ij}, \tag{4.51}$$

$$\delta W_I^* = \int_V \delta U_0^*\,dV = \int_V \varepsilon_{ij}\,\delta\sigma_{ij}\,dV. \tag{4.52}$$

Note that in selecting a virtual force system, one must make sure that they are in equilibrium among themselves.

Example 4.5 Consider the rigid link in Fig. 4.3 (Example 4.1). The initial, no-load position corresponds to $\theta = 0$, and the displacement to the equilibrium position is small (i.e., θ is small). We wish to determine the virtual work done by external forces as well as internal forces when the link is given an infinitesimal rotation $\delta\theta$ from its equilibrium position.

The virtual work done by actual external forces is given by

$$\begin{aligned} \delta W_E &= -\,(P\delta v + M\delta\theta) \\ &= -\,(PL + M)\,\delta\theta. \end{aligned}$$

The work done by internal force F_s in the extensional spring and internal moment M_s in the torsional spring is given by

$$\begin{aligned} \delta W_I &= F_s\delta v + M_s\delta\theta \\ &= k_1 v\delta v + k_2\theta\delta\theta \\ &\approx \left(k_1L^2 + k_2\right)\theta\delta\theta. \end{aligned}$$

Example 4.6 Consider the cantilever beam shown in Fig. 4.11a. Assume virtual displacements $\delta u_0(x)$ and $\delta w_0(x)$ such that

$$\delta w_0(0) = 0, \qquad \frac{d\delta w_0}{dx}(0) = 0, \qquad \delta u_0(0) = 0.$$

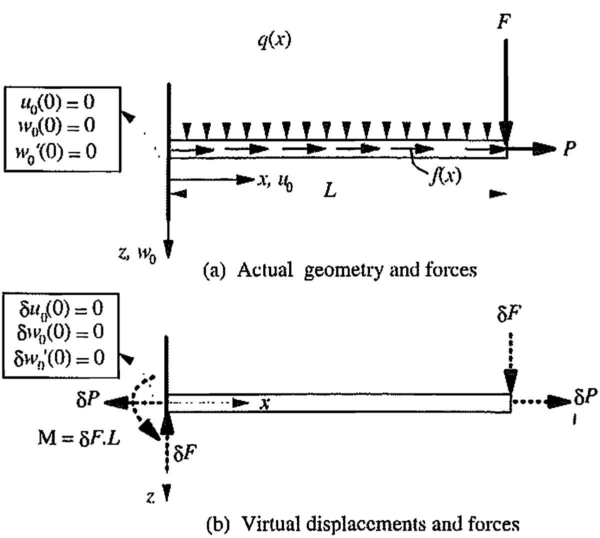

Figure 4.11 The cantilever beam discussed in Example 4.6.

Then the virtual work done by the actual forces in moving through virtual displacement $w_0(x)$ is given by

$$\delta W_E = -\left[\int_0^L (q_0\delta w_0 + f\delta u_0)\,dx + F\delta w_0(L) + P\delta u_0(L)\right]. \tag{4.53a}$$

The virtual work done by internal forces is

$$\begin{aligned}\delta W_I &= \int_V \delta U_0\,dV = \int_V \sigma_{ij}\delta\varepsilon_{ij}\,dV\\ &= \int_0^L\int_A \sigma_{11}\delta\varepsilon_{11}\,dAdx = \int_0^L\int_A \sigma_{xx}\delta\varepsilon_{xx}\,dAdx.\end{aligned} \tag{4.53b}$$

The virtual strains can be computed from the actual strains

$$\varepsilon_{xx} = \frac{du_0}{dx} - z\frac{d^2w_0}{dx^2}, \qquad \delta\varepsilon_{xx} = \frac{d\delta u_0}{dx} - z\frac{d^2\delta w_0}{dx^2}.$$

Then the virtual work is

$$\begin{aligned}\delta W_I &= \int_0^L\int_A \sigma_{xx}\left(\frac{d\delta u_0}{dx} - z\frac{d^2\delta w_0}{dx^2}\right)dAdx\\ &= \int_0^L\left(N\frac{d\delta u_0}{dx} - M\frac{d^2\delta w_0}{dx^2}\right)dx,\end{aligned} \tag{4.53c}$$

where

$$N = \int_A \sigma_{xx}\,dA, \qquad M = \int_A \sigma_{xx}z\,dA.$$

The total virtual work done is

$$\begin{aligned}\delta W &= \delta W_I + \delta W_E\\ &= \int_0^L\left[N\frac{d\delta u_0}{dx} + M\left(-\frac{d^2\delta w_0}{dx^2}\right) - q\delta w_0 - f\delta u_0\right]dx\\ &\quad - F\delta w_0(L) - P\delta u_0(L).\end{aligned} \tag{4.54a}$$

Note that no constitutive law is used in arriving at the result in Eq. (4.54a), but it is understood that N and M are known in terms of the kinematic variables u_0 and w_0. If we assume linear elastic behavior $\sigma_{xx} = E\varepsilon_{xx}$, then [see Eq. (4.34)]

$$N = EA\frac{du_0}{dx}, \qquad M = -EI\frac{d^2w_0}{dx^2}.$$

In this case the total virtual work done, which will be denoted by $\delta\Pi$ to distinguish it from δW, is dependent on the constitutive law. We have

$$\delta\Pi = \int_0^L \left(EA\frac{du_0}{dx}\frac{d\delta u_0}{dx} + EI\frac{d^2w_0}{dx^2}\frac{d^2\delta w_0}{dx^2} - q\delta w_0 - f\delta u_0 \right) dx - F\delta w_0(L) - P\delta u_0(L). \tag{4.54b}$$

Now suppose that the cantilever beam is subject to the virtual force system shown in Fig. 4.11b. The complementary virtual work done by external forces is

$$\delta W_E^* = -\left[\delta F w_0(L) + \delta P u_0(L) - \delta F w_0(0) + (\delta F \cdot L)\left(\frac{dw_0}{dx}\right)_{x=0} - \delta P u_0(0) \right] = -\left[\delta F w_0(L) + \delta P u_0(L)\right].$$

The complementary virtual strain energy is computed assuming that the strains are of the form

$$\varepsilon_{11} = \varepsilon_{xx}^{(0)}(x) + z\varepsilon_{xx}^{(1)}(x), \qquad \varepsilon_{13} = \varepsilon_{xz}^{(0)}(x).$$

Then the complementary virtual internal energy is

$$\begin{aligned}\delta W_I^* &= \int_V (\varepsilon_{11}\delta\sigma_{11} + 2\varepsilon_{13}\delta\sigma_{13})\, dV \\ &= \int_0^L \int_A \left[\left(\varepsilon_{xx}^{(0)} + z\varepsilon_{xx}^{(1)}\right)\delta\sigma_{xx} + 2\varepsilon_{xz}^{(0)}\delta\sigma_{xz} \right] dV \\ &= \int_0^L \left(\varepsilon_{xx}^{(0)}\delta N + \varepsilon_{xx}^{(1)}\delta M + 2\varepsilon_{xz}^{(0)}\delta V \right) dx.\end{aligned}$$

The total complementary virtual work is

$$\delta W^* = \int_0^L \left(\varepsilon_{xx}^{(0)}\delta N + \varepsilon_{xx}^{(1)}\delta M + 2\varepsilon_{xz}^{(1)}\delta V\right) dx - [\delta F w_0(L) + \delta P u_0(L)]. \tag{4.55a}$$

Once again, if we use linear constitutive relations

$$\varepsilon_{xx}^{(0)} = \frac{N}{EA}, \qquad \varepsilon_{xx}^{(1)} = \frac{M}{EI}, \qquad 2\varepsilon_{xz}^{(0)} = \frac{Vf_s}{GA},$$

Eq. (4.55a) becomes

$$\delta\Pi^* = \int_0^L \left(\frac{N}{EA}\delta N + \frac{M}{EI}\delta M + \frac{f_s V}{GA}\delta V \right) dx - [\delta F w_0(L) + \delta P u_0(L)]. \tag{4.55b}$$

Example 4.7 Consider the frame structure of Fig. 4.9. We wish to compute the total complementary virtual work, $\delta W^* = \delta W_I^* + \delta W_E^*$. Assuming a linear elastic constitutive relation, the virtual complementary strain energy can be written as

$$\delta W_I^* = \delta U^* = \int_0^L \left[\frac{N}{EA}\delta N + \frac{M}{EI}\delta M + \frac{f_s V}{GA}\delta V \right] dx, \tag{4.56}$$

where (N, M, V) are the axial force, bending moment, and shear force due to actual internal forces, and $(\delta N, \delta M, \delta V)$ are the virtual axial force, virtual moment, and virtual transverse force in the structure. The stress resultants (N, M, V) in the two segments of the structure can be expressed in terms of the applied external forces, P_v and P_h, as (see Fig. 4.9)

$$N_{DB} = P_h, \quad N_{BA} = P_v, \quad V_{DB} = P_v, \quad V_{BA} = P_h, \tag{4.57a}$$

$$M_{DB} = P_v \cdot x, \quad M_{BA} = P_v \cdot b + P_h \cdot x. \tag{4.57b}$$

If we apply self-equilibrating virtual forces δP_v and δP_h at point D as shown in Fig. 4.12, the axial force and bending moment expressions due to the *virtual* forces δP_v and δP_h for the two parts of the beam are

$$\delta N_{DB} = \delta P_h, \quad \delta N_{BA} = \delta P_v, \quad \delta V_{DB} = \delta P_v, \quad \delta V_{BA} = \delta P_h, \tag{4.58a}$$

$$\delta M_{DB} = \delta P_v \cdot x, \quad \delta M_{BA} = \delta P_v \cdot b + \delta P_h \cdot x, \tag{4.58b}$$

which can also be obtained by taking the first variation of the quantities in Eq. (4.57b). However, in general, the virtual forces need not be the same as the actual forces.

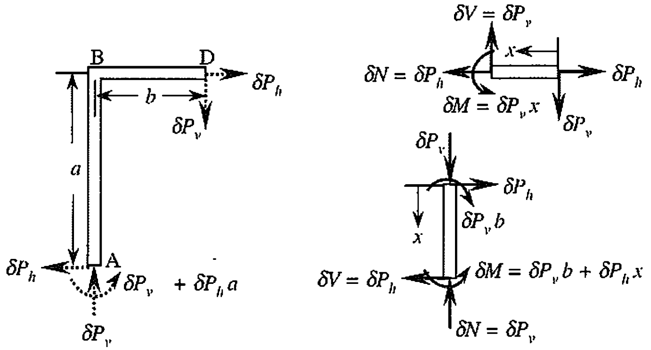

Figure 4.12 A frame structure with virtual forces.

The virtual complementary strain energies due to virtual forces δP_v and δP_h are

$$\delta U^*_{DB} = \int_0^b \left[\frac{N_{DB}}{EA}\delta N_{DB} + \frac{M_{DB}}{EI}\delta M_{DB} + \frac{f_s V_{DB}}{GA}\delta V_{DB} \right] dx$$

$$= \frac{P_h b}{EA}\delta P_h + \left(\frac{P_v b^3}{3EI} + \frac{f_s P_v b}{GA} \right) \delta P_v, \tag{4.59a}$$

$$\delta U^*_{BA} = \int_0^a \left[\frac{P_v}{EA}\delta P_v + \frac{1}{EI}\left(P_v b + P_h x\right)\left(\delta P_v b + \delta P_h x\right) + \frac{f_s P_h}{GA}\delta P_h \right] dx$$

$$= \left[\frac{P_v a}{EA} + \frac{1}{EI}\left(P_v b^2 a + P_h b \frac{a^2}{2} \right) \right] \delta P_v$$

$$+ \left[\frac{1}{EI}\left(P_v b \frac{a^2}{2} + P_h \frac{a^3}{3} \right) + \frac{f_s P_h a}{GA} \right] \delta P_h. \tag{4.59b}$$

The total virtual complementary strain energy of the beam is the sum of δU^*_{DB} and δU^*_{BA}:

$$\delta W^*_I = \delta U^*_{DB} + \delta U^*_{BA}. \tag{4.60}$$

The complementary virtual work done by external virtual forces is

$$\delta W^*_E = -(v \cdot \delta P_v + u \cdot \delta P_h). \tag{4.61}$$

We cannot express the displacements in terms of the forces unless we know the solution to the problem.

4.4 CALCULUS OF VARIATIONS

4.4.1 The Variational Operator

The delta operator δ used in conjunction with virtual quantities has special importance in variational methods. The operator is called the *variational operator* because it is used to denote a variation (or change) in a given quantity. In this section, we discuss certain operational properties of δ and elements of variational calculus. Using these tools, we can study the energy and variational principles of general problems.

Let $u = u(x)$ be the true configuration (i.e., the one corresponding to equilibrium) of a given mechanical system, and suppose that $u = \hat{u}$ on boundary S_1 of the total boundary S. Then an admissible configuration is of the form

$$\bar{u} = u + \alpha v \tag{4.62}$$

everywhere in the body, where v is an arbitrary function that satisfies the homogeneous geometric boundary condition of the system

$$v = 0 \qquad \text{on } S_1. \tag{4.63}$$

Here αv is a variation of the given configuration u. It should be understood that *the variations are small enough* (i.e., α is small) *not to disturb the equilibrium of the system, and the variation is consistent with the geometric constraint of the system.* Equation (4.62) defines a set of varied configurations; an infinite number of configurations $\bar{u}$ can be generated for a fixed v by assigning values to α. All of these configurations satisfy the specified geometric boundary conditions on boundary S_1, and therefore they constitute the set of admissible configurations. For any v, all configurations reduce to the actual one when α is zero. Therefore for any *fixed* x, αv can be viewed as a change or *variation* in the actual configuration u. This variation is often denoted by δu:

$$\delta u = \alpha v, \qquad \delta\left(\frac{du}{dx}\right) = \alpha\left(\frac{dv}{dx}\right) = \frac{d(\alpha v)}{dx} = \frac{d\delta u}{dx}, \tag{4.64}$$

and δu is called the *first variation* of u.

Next, consider a function of the dependent variable u and its derivative $u' \equiv du/dx$:

$$F = F(x, u, u'). \tag{4.65}$$

For fixed x, the change in F associated with a variation in u (and hence u') is

$$\begin{aligned}
\Delta F &= F(x, u + \alpha v, u' + \alpha v') - F(x, u, u') \\
&= F(x, u, u') + \frac{\partial F}{\partial u}\alpha v + \frac{\partial F}{\partial u'}\alpha v' \\
&\quad + \frac{(\alpha v)^2}{2!}\frac{\partial^2 F}{\partial u^2} + \frac{2(\alpha v)(\alpha v')}{2!}\frac{\partial^2 F}{\partial u \partial u'} + \cdots - F(x, u, u') \\
&= \frac{\partial F}{\partial u}\alpha v + \frac{\partial F}{\partial u'}\alpha v' + O(\alpha^2),
\end{aligned} \tag{4.66}$$

where $O(\alpha^2)$ denotes terms of order α^2 and higher. The first total variation of $F(x, u, u')$ is defined by

$$\begin{aligned}
\delta F &= \alpha\left[\lim_{\alpha \to 0} \frac{\Delta F}{\alpha}\right] \\
&= \alpha\left(\frac{\partial F}{\partial u}v + \frac{\partial F}{\partial u'}v'\right) \\
&= \frac{\partial F}{\partial u}\alpha v + \frac{\partial F}{\partial u'}\alpha v' \\
&= \frac{\partial F}{\partial u}\delta u + \frac{\partial F}{\partial u'}\delta u'.
\end{aligned} \tag{4.67a}$$

Alternatively, the first variation may be defined as

$$\begin{aligned}\delta F &= \alpha\left[\frac{dF(u+\alpha v, u'+\alpha v')}{d\alpha}\right]_{\alpha=0} \\ &= \frac{\partial F}{\partial u}\alpha v + \frac{\partial F}{\partial u'}\alpha v' \\ &= \frac{\partial F}{\partial u}\delta u + \frac{\partial F}{\partial u'}\delta u'.\end{aligned} \tag{4.67b}$$

There is an analogy between the first variation of F and the total differential of F. The total differential of F is

$$dF = \frac{\partial F}{\partial x}\,dx + \frac{\partial F}{\partial u}\,du + \frac{\partial F}{\partial u'}\,du'. \tag{4.68}$$

Since x is fixed during the variation of u to $u+\delta u$, $dx = 0$, and the analogy between δF in Eq. (4.67) and dF in Eq. (4.68) becomes apparent. That is, the variational operator δ acts like a differential operator with respect to the dependent variable. Indeed, the laws of variation of sums, products, ratios, powers, and so forth are completely analogous to the corresponding laws of differentiation; that is, the variational calculus resembles the differential calculus. For example, if $F_1 = F_1(u)$ and $F_2 = F_2(u)$, we have

$$\begin{aligned}&(1)\ \ \delta(F_1 \pm F_2) = \delta F_1 \pm \delta F_2. \\ &(2)\ \ \delta(F_1F_2) = \delta F_1 F_2 + F_1 \delta F_2. \\ &(3)\ \ \delta\left(\frac{F_1}{F_2}\right) = \frac{\delta F_1 F_2 - F_1 \delta F_2}{F_2^2}. \\ &(4)\ \ \delta(F_1)^n = n(F_1)^{n-1}\delta F_1.\end{aligned} \tag{4.69}$$

If $G = G(u, v, w)$ is a function of several dependent variables (and possibly their derivatives), the total variation is the sum of partial variations:

$$\delta G = \delta_u G + \delta_v G + \delta_w G, \tag{4.70}$$

where, for example, δ_u denotes the partial variation with respect to u. The variational operator can be interchanged with differential and integral operators:

$$\begin{aligned}&(1)\ \ \delta\left(\frac{du}{dx}\right) = \alpha\frac{dv}{dx} = \frac{d}{dx}(\alpha v) = \frac{d}{dx}(\delta u). \\ &(2)\ \ \delta\left(\int_0^a u\,dx\right) = \alpha\int_0^a v\,dx = \int_0^a \alpha v\,dx = \int_0^a \delta u\,dx.\end{aligned} \tag{4.71}$$

For example, the expressions in Eqs. (4.54b) and (4.55b) can be expressed as

$$\delta\Pi = \delta\left\{\int_0^L \left[\frac{EA}{2}\left(\frac{du_0}{dx}\right)^2 + \frac{EI}{2}\left(\frac{d^2w_0}{dx^2}\right)^2 - qw_0 - fu_0\right] dx \right.$$
$$\left. - Fw_0(L) - Pu_0(L)\right\}, \tag{4.72}$$

$$\delta\Pi^* = \delta\left[\int_0^L \left(\frac{N^2}{2EA} + \frac{M^2}{2EI} + \frac{f_s V^2}{2GA}\right) dx - Fw_0(L) - Pu_0(L)\right]. \tag{4.73}$$

All of the above discussion can be extended to two dimensions and to functions F that depend on more than one dependent variable in two or more dimensions. Let

$$F = F(x, y, u, v, u_x, v_x, u_y, v_y),$$

where $u = u(x, y)$ and $v = v(x, y)$ are dependent variables, and $u_x = \partial u/\partial x$, $u_y = \partial u/\partial y$, and so on. The first variation of F is given by

$$\delta F = \frac{\partial F}{\partial u}\delta u + \frac{\partial F}{\partial v}\delta v + \frac{\partial F}{\partial u_x}\delta u_x + \frac{\partial F}{\partial v_x}\delta v_x + \frac{\partial F}{\partial u_y}\delta u_y + \frac{\partial F}{\partial v_y}\delta v_y,$$

and

$$\delta(F_1(u, v)F_2(u, v)) = \delta F_1(u, v)F_2 + \delta F_2(u, v)F_1$$
$$= \left(\frac{\partial F_1}{\partial u}\delta u + \frac{\partial F_1}{\partial v}\delta v\right)F_2 + \left(\frac{\partial F_2}{\partial u}\delta u + \frac{\partial F_2}{\partial v}\delta v\right)F_1.$$

Example 4.8 Consider the following functions of dependent variables:

(a) $F(u, u', u'') = c_1 u^2 + c_2\left(\frac{du}{dx}\right)^2 + c_3\left(\frac{d^2u}{dx^2}\right)^2 + c_4 u\frac{du}{dx}$,

(b) $G(u, v, u', v') = c_1 u^2 + c_2 v^2 + c_3 uv + c_4\left(\frac{du}{dx}\right)^2 + c_5\left(\frac{dv}{dx}\right)^2 + c_6\frac{du}{dx}\frac{dv}{dx}$,

where c_i denote functions of x only. The first variations are

(a)
$$\delta F = \frac{\partial F}{\partial u}\delta u + \frac{\partial F}{\partial u'}\delta u' + \frac{\partial F}{\partial u''}\delta u''$$
$$= \left(2c_1 u + c_4\frac{du}{dx}\right)\delta u + \left(2c_2\frac{du}{dx} + c_4 u\right)\left(\frac{d\delta u}{dx}\right) + 2c_3\frac{d^2u}{dx^2}\left(\frac{d^2\delta u}{dx^2}\right).$$

(b)
$$\delta_u G = \frac{\partial G}{\partial u}\delta u + \frac{\partial G}{\partial u'}\delta u'$$
$$= (2c_1 u + c_3 v)\delta u + \left(2c_4\frac{du}{dx} + c_6\frac{dv}{dx}\right)\left(\frac{d\delta u}{dx}\right).$$

$$\delta_v G = \frac{\partial G}{\partial v}\delta v + \frac{\partial G}{\partial v'}\delta v'$$
$$= (2c_2 v + c_3 u)\delta v + \left(2c_5 \frac{dv}{dx} + c_6 \frac{du}{dx}\right)\left(\frac{d\delta v}{dx}\right).$$

4.4.2 Functionals

In the study of variational formulations of continuum problems, we encounter functions of dependent variables that are themselves functions of other parameters, such as position, time, and so on. In particular, integral expressions of dependent variables and their derivatives are of interest. Such integral expressions are termed *functionals*. A formal mathematical definition of a functional requires concepts from functional analysis. Here we attempt to present the formal definition without a detailed review of functional analysis, as it is outside the scope of the present study.

A functional I is a mapping (or operator) from a vector space U into the real number field $\Re$. Thus, if $u \in U$, then $I(u)$ is a real number:

$$I : U \rightarrow \Re. \tag{4.74}$$

Note that $I(\cdot)$ is an operator and $I(u)$ is a functional. For example, the integral expression

$$I(u) = \int_0^L [au(x) + bu'(x) + cu''(x)]dx, \qquad u' \equiv \frac{du}{dx}, \qquad u'' \equiv \frac{d^2u}{dx^2},$$

qualifies as a functional for all integrable and square-integrable functions $u(x)$ with their first and second derivatives. Note that the value of the functional $I(u)$ depends on the choice of u.

A functional is said to be *linear* if

$$I(\alpha u + \beta v) = \alpha I(u) + \beta I(v) \tag{4.75}$$

for all $\alpha, \beta \in \Re$ and dependent variables u and v. A *quadratic* functional is one which satisfies the relation

$$I(\alpha u) = \alpha^2 I(u) \tag{4.76}$$

for all $\alpha \in \Re$ and the dependent variable u.

4.4.3 The First Variation of a Functional

The first variation of a functional of u (and its derivatives) can be calculated as follows. Let $I(u)$ denote the integral defined in the interval (a, b):

$$I(u) = \int_a^b F(x, u, u')\,dx, \qquad u' = \frac{du}{dx}, \tag{4.77}$$

where F is a function, in general, of x, u, and du/dx. When the limits a and b are independent of u, the first variation of the functional $I(u)$ is

$$\delta I(u) = \delta \int_a^b F(x, u, u')\, dx = \int_a^b \delta F\, dx = \int_a^b \left(\frac{\partial F}{\partial u} \delta u + \frac{\partial F}{\partial u'} \delta u' \right) dx. \quad (4.78)$$

When the limits of integration depend on the dependent variable, the Leibniz rule is used to calculate the first variation. Suppose that the boundary points a and b depend on u. Then

$$\delta I(u) = \alpha \left[\frac{d}{d\alpha} \int_{a(u+\alpha v)}^{b(u+\alpha v)} F(x, u + \alpha v, u' + \alpha v')\, dx \right]_{\alpha=0}.$$

Using the Leibniz rule, we obtain

$$\delta I(u) = \int_a^b \delta F(x, u, u')\, dx + F(b, u(b), u'(b))\delta b - F(a, u(a), u'(a))\delta a. \quad (4.79)$$

Thus the variational operator can be used to compute the variation of any functional.

Example 4.9 We wish to determine the first variation of the following functionals:

$$\text{(a)} \quad I(u) = \int_0^L \left[\frac{a}{2} \left(\frac{du}{dx} \right)^2 + \frac{b}{2} u^2 - f u \right] dx.$$

$$\text{(b)} \quad I(u) = \int_\Omega \left(\frac{1}{2} \nabla u \cdot \nabla u - f u \right) d\mathbf{x} - \int_{\Gamma_2} q u\, ds.$$

(a) We have

$$\delta I(u) = \int_0^L \left(a \frac{du}{dx} \frac{d\delta u}{dx} + b u \delta u - f \delta u \right) dx.$$

(b) In this case, we have

$$\delta I(u) = \int_\Omega (\nabla u \cdot \nabla \delta u - f \delta u)\, d\mathbf{x} - \int_{\Gamma_2} q \delta u\, ds.$$

4.4.4 Fundamental Lemma of Variational Calculus

The *fundamental lemma of calculus of variations* is useful in obtaining differential equations from integral statements. The lemma can be stated as follows: *for any integrable function G, if the statement*

$$\int_a^b G \cdot \eta\, dx = 0 \quad (4.80)$$

holds for any arbitrary continuous function $\eta(x)$, for all x in (a, b), then it follows that $G = 0$ in (a, b). A mathematical proof of the lemma can be found in most books on variational calculus. A simple proof of the lemma follows. Since η is arbitrary, it can be replaced by G. We have

$$\int_a^b G^2\, dx = 0.$$

Since an integral of a positive function is positive, the above statement implies that $G = 0$.

A more general statement of the fundamental lemma is as follows: If η is arbitrary in $a < x < b$ and $\eta(a)$ is arbitrary, then the statement

$$\int_a^b G\eta\, dx + B(a)\eta(a) = 0 \tag{4.81a}$$

implies that

$$G = 0 \qquad \text{in } a < x < b \text{ and } B(a) = 0, \tag{4.81b}$$

because $\eta(x)$ is independent of $\eta(a)$.

4.4.5 Extremum of a Functional

From elementary calculus it is known that a differentiable function $f(x)$, $a < x < b$, has an extremum (i.e., minimum or maximum) at a point x_0 in the interval (a, b) only if (necessary condition)

$$\left.\frac{df}{dx}\right|_{x=x_0} = 0. \tag{4.82}$$

The sufficient condition for maximum (or minimum) is that

$$\frac{d^2 f}{dx^2} < 0 \quad \left(\text{or } \frac{d^2 f}{dx^2} > 0\right). \tag{4.83}$$

Similarly, for a differential function $f(x, y)$ in two dimensions, the necessary condition for an extremum at the point (x_0, y_0) is that its total differential be zero at (x_0, y_0):

$$df \equiv \frac{\partial f}{\partial x}\, dx + \frac{\partial f}{\partial y}\, dy = 0 \qquad \text{at } x = x_0 \text{ and } y = y_0. \tag{4.84}$$

If x and y are linearly independent (i.e., x and y are not constrained), Eq. (4.84) implies

$$\frac{\partial f}{\partial x} = 0 \qquad \text{and} \qquad \frac{\partial f}{\partial y} = 0 \tag{4.85}$$

at $x = x_0$ and $y = y_0$.

In the study of problems by variational principles and methods, we seek the extremum of integrals of functions of functions, $F = F(x, u(x), u'(x))$—or functionals. Consider a simple, but typical, variational problem. Find a function $u = u(x)$ such that $u(a) = u_a$ and $u(b) = u_b$, and

$$I(u) = \int_a^b F(x, u(x), u'(x))\, dx \tag{4.86}$$

is a minimum.

In analyzing the problem we are not interested in all functions u, but only in those functions that satisfy the stated boundary (or end) conditions. The set of all such functions is called, for obvious reasons, the *set of competing functions* (or set of admissible functions). We shall denote the set by $\mathcal{C}$. The problem is to seek an element u from $\mathcal{C}$ which renders I a minimum. If $u \in \mathcal{C}$, then $(u + \alpha v) \in \mathcal{C}$ for every v satisfying the conditions $v(a) = v(b) = 0$. The space of all such elements is called the space of admissible variations, as already mentioned. Figure 4.13 shows a typical competing function $\bar{u}(x) = u(x) + \alpha v(x)$ and a typical admissible variation $v(x)$.

Let $I(u)$ be a differentiable functional in the sense that

$$\frac{dI(u + \alpha v, u' + \alpha v')}{d\alpha}$$

exists, and let $\mathcal{C}$ denote the space of competing functions. Then an element $u \in \mathcal{C}$ is said to yield a *relative minimum (maximum)* for $I(\bar{u})$ in $\mathcal{C}$ if

$$I(\bar{u}) - I(u) \geq 0 \qquad (\leq 0). \tag{4.87}$$

If $I(\bar{u})$ assumes a relative minimum (maximum) at $u \in \mathcal{C}$ relative to elements $\bar{u} \in \mathcal{C}$, then it follows from the definition of the space of admissible variations $\mathcal{H}$ and Eq. (4.87) that

$$I(u + \alpha v) - I(u) \geq 0 \qquad (\leq 0) \tag{4.88}$$

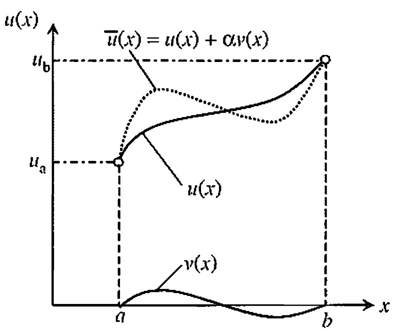

Figure 4.13 The variations of $u(x)$.

for all $v \in \mathcal{H}$, $\|v\| < \varepsilon$, and $\alpha \in \Re$. Since u is the minimizer, any other function $u \in C$ is of the form $\bar{u} = u + \alpha v$, and the actual minimizer is determined by setting $\alpha = 0$. Once $u(x)$ and $v(x)$ are assigned, $I(\bar{u})$ is a function of α alone, say $\bar{I}(\alpha)$. Now a necessary condition for $I(\bar{u}) = \bar{I}(\alpha)$ to attain a minimum is that

$$\frac{d\bar{I}(\alpha)}{d\alpha} = \frac{d}{d\alpha}[I(u + \alpha v)] = 0. \tag{4.89}$$

On the other hand, $I(\bar{u})$ attains its minimum at u, that is, $\alpha = 0$. These two conditions together imply $(d\bar{I}(\alpha)/d\alpha)|_{\alpha=0} = 0$, which is nothing but

$$\delta I(u) = 0. \tag{4.90}$$

Analogous to the second necessary condition for ordinary functions, the second necessary condition for a functional to assume a relative minimum (maximum) is that the second variation $\delta^2 I(u)$ is greater (less) than zero. The second variation $\delta^2 I(u)$ of a functional $I(u)$ is given by

$$\delta^2 I(u) \equiv \frac{\alpha^2}{2}\left[\frac{d^2}{d\alpha^2} I(u + \alpha v)\right]_{\alpha=0} \tag{4.91}$$

for all $v \in C$ and $\alpha \in \Re$.

4.4.6 Euler Equations

We now return to the problem of determining the minimum of a functional:

$$I(u) = \int_a^b F(x, u, u')\, dx, \tag{4.92}$$

subject to the end conditions

$$u(a) = u_a, \qquad u(b) = u_b. \tag{4.93}$$

It is clear that any candidate for the minimizer of the functional should satisfy the end conditions in Eq. (4.93), and be sufficiently differentiable (twice in the present case, as we shall see shortly). The set of all such functions is the set of admissible functions or competing functions for the present case. Functions from the admissible set can be viewed as smooth (i.e., differentiable twice) functions passing through points (a, u_a) and (b, u_b), as shown in Fig. 4.13. Clearly, any element $\bar{u} \in C$ (the set of competing functions) has the form

$$\bar{u} = u + \alpha v, \tag{4.94}$$

where α is a small number and v is a sufficiently differentiable function that satisfies the homogeneous form of the end conditions (because $\bar{u}$ must satisfy the specified end conditions) in Eq. (4.93)

$$v(a) = v(b) = 0, \tag{4.95}$$

and u is the function that minimizes the functional in Eq. (4.92). The set of all functions v is the set of admissible variations, $\mathcal{H}$. Now assuming that, for each admissible function $\bar{u}$, $F(x, \bar{u}, \bar{u}')$ exists and is continuously differentiable with respect to its arguments, and $I(\bar{u})$ takes one and only one real value, we seek the particular function $u(x)$ that makes the integral a minimum.

The necessary condition (4.89) for I to attain a minimum gives

$$\begin{aligned} 0 &= \left.\frac{dI(u+\alpha v)}{d\alpha}\right|_{\alpha=0} = \left[\frac{d}{d\alpha}\int_a^b F(x, \bar{u}, \bar{u}')\, dx\right]_{\alpha=0} \\ &= \int_a^b \left.\left(\frac{\partial F}{\partial \bar{u}}\frac{\partial \bar{u}}{\partial \alpha} + \frac{\partial F}{\partial \bar{u}'}\frac{\partial \bar{u}'}{\partial \alpha}\right)\right|_{\alpha=0} dx = \int_a^b \left(\frac{\partial F}{\partial u}v + \frac{\partial F}{\partial u'}v'\right) dx, \end{aligned} \tag{4.96}$$

where $\bar{u} = u + \alpha v$. Integrating the second term in the last equation by parts to transfer differentiation from v to u, we obtain

$$0 = \int_a^b v\left[\frac{\partial F}{\partial u} - \frac{d}{dx}\left(\frac{\partial F}{\partial u'}\right)\right] dx + \left.\left(\frac{\partial F}{\partial u'}v\right)\right|_a^b. \tag{4.97}$$

The boundary term vanished because v is zero at $x = a$ and $x = b$ [see Eq. (4.95)]. The fact that v is arbitrary inside the internal (a, b) and yet the equation should hold implies, by the fundamental lemma of the calculus of variations, that the expression in the square brackets is zero identically:

$$\frac{\partial F}{\partial u} - \frac{d}{dx}\left(\frac{\partial F}{\partial u'}\right) = 0 \qquad \text{in } a < x < b. \tag{4.98}$$

Equation (4.98) is called the *Euler equation* of the functional in Eq. (4.92). Of all the admissible functions, the one that satisfies (i.e., the solution of) Eq. (4.98) is the true minimizer of the functional I.

Next, consider the problem of finding (u, v), defined on a two-dimensional region Ω, such that the following functional is to be minimized:

$$I(u, v) = \int_\Omega F(x, y, u, v, u_x, v_x, u_y, v_y)\, dxdy \tag{4.99}$$

where $u_x = \partial u/\partial x$, $u_y = \partial u/\partial y$, and so on. For the moment we assume that u and v are specified on the boundary Γ of Ω. The vanishing of the first variation of $I(u, v)$ is written as

$$\delta I(u, v) = \delta_u I(u, v) + \delta_v I(u, v) = 0.$$

Here δ_u and δ_v denote (partial) variations with respect to u and v, respectively. We have [see also Eq. (4.72)]

$$\delta I = \int_\Omega \left\{\frac{\partial F}{\partial u}\delta u + \frac{\partial F}{\partial u_x}\delta u_x + \frac{\partial F}{\partial u_y}\delta u_y + \frac{\partial F}{\partial v}\delta v + \frac{\partial F}{\partial v_x}\delta v_x + \frac{\partial F}{\partial v_y}\delta v_y\right\} dxdy. \tag{4.100}$$

The next step in the development involves the use of integration by parts, or the gradient theorem on the second, third, fifth, and sixth terms in Eq. (4.100). Consider the second term. We have

$$\begin{aligned}\int_\Omega \frac{\partial F}{\partial u_x}\frac{\partial \delta u}{\partial x}\,dx\,dy &= \int_\Omega \left[\frac{\partial}{\partial x}\left(\frac{\partial F}{\partial u_x}\delta u\right) - \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial u_x}\right)\delta u\right] dx\,dy \\ &= \oint_\Gamma \frac{\partial F}{\partial u_x}\delta u n_x\,ds - \int_\Omega \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial u_x}\right)\delta u\,dx\,dy. \end{aligned} \tag{4.101}$$

Using a similar procedure on the other terms and collecting the coefficients of δu and δv separately, we obtain

$$\begin{aligned} 0 = \int_\Omega \Bigg\{ &\left[\frac{\partial F}{\partial u} - \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial u_x}\right) - \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial u_y}\right)\right]\delta u \\ &+ \left[\frac{\partial F}{\partial v} - \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial v_x}\right) - \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial v_y}\right)\right]\delta v \Bigg\} dx\,dy \\ &+ \oint_\Gamma \left[\left(\frac{\partial F}{\partial u_x}n_x + \frac{\partial F}{\partial u_y}n_y\right)\delta u + \left(\frac{\partial F}{\partial v_x}n_x + \frac{\partial F}{\partial v_y}n_y\right)\delta v\right] ds. \end{aligned} \tag{4.102}$$

Since (u, v) are specified on Γ, $\delta u = \delta v = 0$ and the boundary expressions vanish. Then, since δu and δv are arbitrary and independent of each other in Ω, the fundamental lemma yields the Euler equations

$$\delta u: \quad \frac{\partial F}{\partial u} - \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial u_x}\right) - \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial u_y}\right) = 0, \tag{4.103a}$$

$$\delta v: \quad \frac{\partial F}{\partial v} - \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial v_x}\right) - \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial v_y}\right) = 0. \tag{4.103b}$$

All of the foregoing discussion can be applied to the general case of a functional involving p dependent variables with mth-order partial derivatives with respect to n independent variables. In this case there will be p Euler equations involving $2m$th-order derivatives in n independent variables.

4.4.7 Natural and Essential Boundary Conditions

First consider the problem of minimizing the functional in Eq. (4.92) subject to no end conditions (hence $\delta v \neq 0$ at $x = a$ and $x = b$). The necessary condition for I to attain a minimum yields, as before, Eq. (4.97). Now suppose that $\partial F/\partial u'$ and v are selected such that

$$\frac{\partial F}{\partial u'}v = 0 \qquad \text{for } x = a \text{ and } x = b. \tag{4.104}$$

Then using the fundamental lemma of the calculus of variations, we obtain the Euler equation in Eq. (4.98).

Equation (4.104) is satisfied identically for any of the following combinations:

$$\begin{aligned}
&\text{(i)} \quad v(a) = 0, && v(b) = 0. \\
&\text{(ii)} \quad v(a) = 0, && \frac{\partial F}{\partial u'}(b) = 0. \\
&\text{(iii)} \quad \frac{\partial F}{\partial u'}(a) = 0, && v(b) = 0. \\
&\text{(iv)} \quad \frac{\partial F}{\partial u'}(a) = 0, && \frac{\partial F}{\partial u'}(b) = 0.
\end{aligned} \tag{4.105}$$

The requirement that $v = 0$ at an end point is equivalent to the requirement that u is specified (to be some value) at that point. The end conditions in (4.105) are classified into two types: *essential boundary conditions*, which require the variation of v and possibly its derivatives to vanish at the boundary, and *natural boundary conditions*, which require the specification of the coefficients of the variations of v and its derivatives. Thus we have

Essential Boundary Conditions

$$\text{specify } v = 0 \text{ or } u = \hat{u} \text{ on the boundary.} \tag{4.106a}$$

Natural Boundary Conditions

$$\frac{\partial F}{\partial u'} = 0 \text{ on the boundary.} \tag{4.106b}$$

In a given problem, only one of the four combinations given in Eq. (4.105) can be specified. Problems in which all of the boundary conditions are of essential type are called *Dirichlet boundary-value problems*, and those in which all of the boundary conditions are of natural type are called *Neumann boundary-value problems*. *Mixed boundary-value problems* are those in which both essential and natural boundary conditions are specified. Essential boundary conditions are also known as *Dirichlet* or *geometric* boundary conditions, and natural boundary conditions are known as *Neumann* or *dynamic* boundary conditions.

As a general rule, the vanishing of the variation v (or δu)—equivalently, the specification of u—on the boundary constitutes the essential boundary condition, and vanishing of the coefficient of the variation constitutes the natural boundary condition. This rule applies to any functional in one, two, and three dimensions, and integrands that are functions of one or more dependent variables and their derivatives of any order.

Next consider the functional in Eq. (4.99), and suppose that (u, v) are arbitrary on Γ for the moment. It is easy to identify the natural and essential boundary conditions of the problem from Eq. (4.102): in each of the pairings on boundary Γ, specifying the first element (which contains no variations of the dependent variables) constitutes the natural boundary condition, and vanishing of the second element (or, equivalently, specifying the quantity in front of the variational operator) constitutes the essential boundary condition. Thus we have either

$$u = \hat{u} \text{ (specified) so that } \delta u = 0 \quad \text{on } \Gamma, \tag{4.107a}$$

$$v = \hat{v} \text{ (specified) so that } \delta v = 0 \quad \text{on } \Gamma, \tag{4.107b}$$

or

$$\frac{\partial F}{\partial u_x} n_x + \frac{\partial F}{\partial u_y} n_y = 0 \quad \text{on } \Gamma, \tag{4.108a}$$

$$\frac{\partial F}{\partial v_x} n_x + \frac{\partial F}{\partial v_y} n_y = 0 \quad \text{on } \Gamma. \tag{4.108b}$$

Equations (4.107a,b) represent the essential boundary conditions and Eqs. (4.108a,b) the natural boundary conditions. The pair of elements (u, v) are called the *primary variables* and

$$Q_x \equiv \frac{\partial F}{\partial u_x} n_x + \frac{\partial F}{\partial u_y} n_y \quad \text{and} \quad Q_y \equiv \frac{\partial F}{\partial v_x} n_x + \frac{\partial F}{\partial v_y} n_y$$

are called the *secondary variables*. Thus, specification of the primary variables constitute essential boundary conditions and specification of the secondary variables constitute natural boundary conditions. In general, one element of the each pair (u, Q_x) and (v, Q_y) (but not both elements of the same pair) may be specified at any point of the boundary. Thus there are four possible combinations of natural and essential boundary conditions for the problem under discussion.

Example 4.10 Consider an elastic bar of length L, modulus of elasticity E, and area of cross section A. Assume that it is fixed at the left end and subjected to distributed axial load $f(x)$ and point load P at the right end. The total potential energy functional $\Pi = U + V$ of the bar is [see Eq. (4.39)]

$$\Pi(u) = \int_0^L \left[\frac{EA}{2} \left(\frac{du}{dx} \right)^2 - fu \right] dx - Pu(L), \tag{4.109}$$

where u denotes the axial displacement of the bar. The first term in $\Pi(u)$ represents the strain energy U stored in the bar, and the second and third terms denote the work done V on the bar by the distributed load f and point load P. We wish to determine the Euler equation of the bar by requiring that $\Pi(u)$ be a minimum subject to the geometric boundary condition $u(0) = 0$. As we shall see later, this is known as the principle of minimum total potential energy.

The first variation of Π is given by (see Example 4.9)

$$\delta\Pi(u) = \int_0^L \left(EA \frac{du}{dx} \frac{d\delta u}{dx} - f\delta u \right) dx - P\delta u(L).$$

where δu is arbitrary in $0 < x < L$ and at $x = L$, but satisfies the condition $\delta u(0) = 0$. To use the fundamental lemma of variational calculus, we must relieve δu of any differentiation. Integrating the first term by parts, we get

$$\begin{aligned}\delta\Pi(u) &= \int_0^L \left[-\frac{d}{dx}\left(EA\frac{du}{dx}\right) - f\right]\delta u\, dx + \left[EA\frac{du}{dx}\delta u\right]_0^L - P\delta u(L)\\ &= \int_0^L \delta u\left[-\frac{d}{dx}\left(EA\frac{du}{dx}\right) - f\right]dx + \delta u(L)\left[\left(EA\frac{du}{dx}\right)_{x=L} - P\right]\\ &\quad - \delta u(0)\left(EA\frac{du}{dx}\right)_{x=0}.\end{aligned}$$

The last term is zero because $\delta u(0) = 0$. Setting the coefficients of δu in $(0, L)$ and δu at $x = L$ to zero separately, we obtain the Euler equation and the natural (or force) boundary condition of the problem:

Euler Equation

$$-\frac{d}{dx}\left(EA\frac{du}{dx}\right) - f = 0, \qquad 0 < x < L. \tag{4.110a}$$

Natural Boundary Condition

$$EA\frac{du}{dx} - P = 0 \qquad \text{at } x = L. \tag{4.110b}$$

Thus, the solution u of Eqs. (4.110a,b) that satisfies $u(0) = 0$ is the minimizer of the energy functional $\Pi(u)$ in Eq. (4.109).

Equation (4.110a) can be obtained directly from Eqs. (4.98) by substituting

$$F(x, u, u') = \frac{EA}{2}\left(\frac{du}{dx}\right)^2 - fu, \qquad \frac{\partial F}{\partial u} = -f, \qquad \frac{\partial F}{\partial u'} = EA\frac{du}{dx}.$$

Example 4.11 The total potential energy of a linear elastic body in three dimensions, subjected to body force $\mathbf{f}$ (measured per unit volume) and surface traction $\hat{\mathbf{t}}$ on portion S_2 of the surface, is given by (summation on repeated indices is implied)

$$\Pi(\mathbf{u}) = \int_V \left(\frac{1}{2}\sigma_{ij}e_{ij} - f_i u_i\right)dV - \int_{S_2} \hat{t}_i u_i\, dS, \tag{4.111a}$$

where u_i denote the displacement components. It is assumed that the body is subjected to specified displacements on the remaining portion S_1 of the surface, and therefore the virtual displacements vanish there:

$$u_i = \hat{u}_i \qquad \text{and} \qquad \delta u_i = 0 \quad \text{on } S_1. \tag{4.111b}$$

The first term under the volume integral represents the strain energy density of the elastic body, the second term represents the work done by the body force $\mathbf{f}$, and the surface integral denotes the work done by the specified traction $\hat{\mathbf{t}}$.

For an isotropic body the stress–strain relations are given by

$$\sigma_{ij} = 2\mu e_{ij} + \lambda \delta_{ij} e_{kk}, \tag{4.112}$$

where μ and λ are the Lamé constants. Hence,

$$\sigma_{ij} e_{ij} = 2\mu e_{ij} e_{ij} + \lambda e_{ii} e_{kk}, \tag{4.113a}$$

where in Eq. (4.113a) and in the following discussion, sum on repeated indices is assumed. The linear strain–displacement relations of the linear theory are given by

$$e_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}), \tag{4.113b}$$

where $u_{i,j} = (\partial u_i / \partial x_j)$. Substituting Eqs. (4.113a,b) into Eq. (4.111a), we obtain

$$\Pi(\mathbf{u}) = \int_V \left[\frac{\mu}{4}(u_{i,j} + u_{j,i})(u_{i,j} + u_{j,i}) + \frac{\lambda}{2} u_{i,i} u_{k,k} - f_i u_i \right] dV - \int_{S_2} \hat{t}_i u_i \, dS. \tag{4.114}$$

Now we wish to derive the Euler equations associated with the functional in Eq. (4.114) using the requirement $\delta \Pi = 0$ and $\delta u_i = 0$ on S_1.

Setting the first variation of Π to zero, we obtain

$$0 = \int_V \left[\frac{\mu}{2}(\delta u_{i,j} + \delta u_{j,i})(u_{i,j} + u_{j,i}) + \lambda \delta u_{i,i} u_{k,k} - f_i \delta u_i \right] dV - \int_{S_2} \hat{t}_i \delta u_i \, dS, \tag{4.115}$$

wherein the product rule of variation is used and similar terms are combined. Using integration by parts,

$$\int_V \delta u_{i,j}(u_{i,j} + u_{j,i}) dV = -\int_V \delta u_i (u_{i,j} + u_{j,i})_{,j} \, dV + \oint_S \delta u_i (u_{i,j} + u_{j,i}) n_j \, dS,$$

where n_j denotes the jth direction cosine of the unit normal to the surface, we obtain

$$\begin{aligned}
0 &= \int_V \left[-\frac{\mu}{2}(u_{i,j}+u_{j,i})_{,j}\delta u_i - \frac{\mu}{2}(u_{i,j}+u_{j,i})_{,i}\delta u_j - \lambda u_{k,ki}\delta u_i - f_i\delta u_i\right] dV \\
&\quad + \oint_S \left[\frac{\mu}{2}(u_{i,j}+u_{j,i})(n_j\delta u_i + n_i\delta u_j) + \lambda u_{k,k}n_i\delta u_i\right] dS - \int_{S_2} \delta u_i \hat{t}_i\, dS \\
&= \int_V \left[-\mu(u_{i,j}+u_{j,i})_{,j} - \lambda u_{k,ki} - f_i\right]\delta u_i\, dV \\
&\quad + \oint_S \left[\mu(u_{i,j}+u_{j,i}) + \lambda u_{k,k}\delta_{ij}\right] n_j \delta u_i\, dS - \int_{S_2} \delta u_i \hat{t}_i\, dS.
\end{aligned} \tag{4.116}$$

In arriving at the last step, a change of dummy indices is made to combine terms. Recognizing that the expression inside the square brackets of the closed surface integral is nothing but σ_{ij} and $\sigma_{ij}n_j = t_i$ by Cauchy's formula, we can write

$$\oint_S t_i\delta u_i\, dS = \int_{S_1} t_i\delta u_i\, dS + \int_{S_2} t_i\delta u_i\, dS = \int_{S_2} t_i\delta u_i\, dS.$$

The integral over S_1 is zero by virtue of Eq. (4.111b). Hence, we have

$$0 = \int_V \left[-\mu(u_{i,j}+u_{j,i})_{,j} - \lambda u_{k,ki} - f_i\right]\delta u_i\, dV + \int_{S_2} \delta u_i (t_i - \hat{t}_i)\, dS.$$

Using the fundamental lemma of calculus of variations, we set the coefficients of δu_i in V and δu_i on S_2 to zero separately and obtain

$$\mu(u_{i,jj} + u_{j,ij}) + \lambda u_{k,ki} + f_i = 0 \qquad \text{in } V, \tag{4.117a}$$

$$t_i - \hat{t}_i = 0 \qquad \text{on } S_2, \tag{4.117b}$$

for $i = 1, 2, 3$. Equations (4.117a) are the well-known Navier's equations of elasticity.

4.4.8 Minimization of Functionals with Equality Constraints

In the preceding sections, we devoted our attention to the determination of a function that minimizes a given functional. The necessary condition leads to a differential equation (the Euler equation) and (natural) boundary conditions governing the function. It turns out that it is also a sufficient condition for linear problems, not requiring the evaluation of the second variation of the functional. In the present section, we consider ways to determine the minimum of quadratic functionals (i.e., functionals that involve quadratic terms of the dependent variables) with linear equality constraints.

Consider the problem of finding the minimum of a functional (assumed to be quadratic in u and v),

$$I(u, v) = \int_a^b F(x, u, u', v, v')\, dx \tag{4.118}$$

subject to the constraint

$$G(u, u', v, v') = 0. \tag{4.119}$$

In addition, u and v should satisfy, as before, certain end conditions (or essential boundary conditions) of the problem.

In the constrained minimization problems, the admissible functions should not only satisfy the specified end conditions and be sufficiently continuous, but they should satisfy the constraint conditions. Furthermore, the admissible variations should be such that the constraint conditions are not violated. Here we consider two separate methods of including the constraints in the (modified) functional.

Lagrange Multiplier Method The necessary condition for the minimum of $I(u, v)$ in Eq. (4.118) is $\delta I = 0$. We have

$$0 = \delta I = \int_a^b \left(\frac{\partial F}{\partial u}\delta u + \frac{\partial F}{\partial u'}\delta u' + \frac{\partial F}{\partial v}\delta v + \frac{\partial F}{\partial v'}\delta v' \right) dx. \tag{4.120}$$

Since u and v must satisfy the constraint condition (4.119), the variations δu and δv are related by

$$0 = \delta G = \frac{\partial G}{\partial u}\delta u + \frac{\partial G}{\partial u'}\delta u' + \frac{\partial G}{\partial v}\delta v + \frac{\partial G}{\partial v'}\delta v'. \tag{4.121}$$

The Lagrange multiplier method consists of multiplying Eq. (4.121) with an arbitrary parameter λ, integrating over the interval (a, b), and adding the result to Eq. (4.120). The multiplier λ is called the *Lagrange multiplier.* We have

$$\begin{aligned} 0 &= \int_a^b \Bigg[\frac{\partial F}{\partial u}\delta u + \frac{\partial F}{\partial u'}\delta u' + \frac{\partial F}{\partial v}\delta v + \frac{\partial F}{\partial v'}\delta v' \\ &\quad + \lambda \left(\frac{\partial G}{\partial u}\delta u + \frac{\partial G}{\partial u'}\delta u' + \frac{\partial G}{\partial v}\delta v + \frac{\partial G}{\partial v'}\delta v' \right) \Bigg] dx \\ &= \int_a^b \Bigg\{ \left[\frac{\partial F}{\partial u} + \lambda\frac{\partial G}{\partial u} - \frac{d}{dx}\left(\frac{\partial F}{\partial u'} + \lambda\frac{\partial G}{\partial u'} \right) \right] \delta u \\ &\quad + \left[\frac{\partial F}{\partial v} + \lambda\frac{\partial G}{\partial v} - \frac{d}{dx}\left(\frac{\partial F}{\partial v'} + \lambda\frac{\partial G}{\partial v'} \right) \right] \delta v \Bigg\} dx. \end{aligned} \tag{4.122}$$

The boundary terms vanish because $\delta v(a) = \delta v(b) = \delta u(a) = \delta u(b) = 0$. Since the variations δu and δv are not both independent, we cannot conclude that their coefficients are zero. Suppose the δu is independent and δv is related to δu by Eq. (4.121). We choose λ such that the coefficient of δv is zero. Then by the fundamental lemma of variational calculus, it follows that (because δu is arbitrary) the coefficient of δu is also zero. Thus we have

$$\frac{\partial}{\partial u}(F + \lambda G) - \frac{d}{dx}\left[\frac{\partial}{\partial u'}(F + \lambda G)\right] = 0, \tag{4.123a}$$

$$\frac{\partial}{\partial v}(F + \lambda G) - \frac{d}{dx}\left[\frac{\partial}{\partial v'}(F + \lambda G)\right] = 0. \tag{4.123b}$$

Equations (4.123a,b) and (4.119) furnish three equations for the determination of u, v, and λ.

It is clear from Eq. (4.122) that Lagrange's method can be viewed as one of determining u, v, and λ by setting the first variation of the *modified* functional

$$\begin{aligned} L(u, v, \lambda) &\equiv I(u, v) + \int_a^b \lambda G(u, u', v, v')\, dx \\ &= \int_a^b (F + \lambda G)\, dx \end{aligned} \tag{4.124}$$

to zero. The Euler equations of the functional are precisely Eqs. (4.123a,b) and (4.119). Indeed, we have

$$\begin{aligned} 0 = \delta L &= \int_a^b \delta(F + \lambda G)\, dx \\ &= \int_a^b (\delta F + \delta\lambda G + \lambda \delta G)\, dx \\ &= \int_a^b \left[\left(\frac{\partial F}{\partial u} + \lambda\frac{\partial G}{\partial u}\right)\delta u + \left(\frac{\partial F}{\partial u'} + \lambda\frac{\partial G}{\partial u'}\right)\delta u' \right. \\ &\quad \left. + \left(\frac{\partial F}{\partial v} + \lambda\frac{\partial G}{\partial v}\right)\delta v + \left(\frac{\partial F}{\partial v'} + \lambda\frac{\partial G}{\partial v'}\right)\delta v' + \delta\lambda G\right] dx \\ &= \int_a^b \left\{\left[\frac{\partial F}{\partial u} + \lambda\frac{\partial G}{\partial u} - \frac{d}{dx}\left(\frac{\partial F}{\partial u'} + \lambda\frac{\partial G}{\partial u'}\right)\right]\delta u \right. \\ &\quad \left. + \left[\frac{\partial F}{\partial v} + \lambda\frac{\partial F}{\partial v} - \frac{d}{dx}\left(\frac{\partial F}{\partial v'} + \lambda\frac{\partial G}{\partial v'}\right)\right]\delta v + G\delta\lambda\right\} dx, \end{aligned} \tag{4.125}$$

from which we obtain Eqs. (4.123a,b) and (4.119) by setting, respectively, the coefficients of δu and δv, and $\delta\lambda$, to zero.

Penalty Function Method The penalty function method involves the reduction of conditional extremum problems to extremum problems without constraints by the introduction of a penalty function associated with the constraints. As applied to the problem in (4.118) and (4.119), the technique involves seeking the minimum of a modified functional obtained by adding a quadratic term associated with the constraint in (4.119) (i.e., the constraint is approximately satisfied in the least-squares sense):

$$P(u, v) = I(u, v) + \frac{\gamma}{2}\int_a^b [G(u, u', v, v')]^2 dx, \tag{4.126}$$

where γ is the *penalty parameter* (a preassigned positive parameter). Setting the first variation of P to zero, we obtain the Euler equations for the functional:

$$\frac{\partial F}{\partial u} - \frac{d}{dx}\left(\frac{\partial F}{\partial u'}\right) + \gamma\left[G\frac{\partial G}{\partial u} - \frac{d}{dx}\left(G\frac{\partial G}{\partial u'}\right)\right] = 0, \tag{4.127a}$$

$$\frac{\partial F}{\partial v} - \frac{d}{dx}\left(\frac{\partial F}{\partial v'}\right) + \gamma\left[G\frac{\partial G}{\partial v} - \frac{d}{dx}\left(G\frac{\partial G}{\partial v'}\right)\right] = 0. \tag{4.127b}$$

For successively large values of γ, the solution of Eq. (4.127a,b) gets closer to the true solution. We note that in the penalty function method the constraint is satisfied only approximately, and that no additional unknowns are introduced into the variational formulation. Further, in the penalty function method an approximation to the Lagrange multiplier can be computed from the solution $(u(\gamma), v(\gamma))$ of Eqs. (4.127a,b) by the formula [compare Eqs. (4.127a,b) with Eq. (4.123a,b)]

$$\lambda = \gamma G(u, u', v, v'). \tag{4.128}$$

Both the Lagrange multiplier method and the penalty function method can be generalized for functionals with many variables and constraints. Note that when the constraint equation is a relationship involving constants, the Lagrange multiplier is also a constant. For example,

$$c_1 u - c_2 v = c_3, \qquad \int_a^b F(x, u(x), u'(x))\, dx = c_1,$$

where c_i $(i = 1, 2, 3)$ are constants, are constraint equations among numbers (note that a definite integral of a function is a number).

Example 4.12 Given the problem of minimizing the functional

$$I = \frac{1}{2}\int_{t_1}^{t_2} (x\dot{y} - y\dot{x})\, dt, \tag{a}$$

subject to the constraint

$$L = \int_{t_1}^{t_2} (\dot{x}^2 + \dot{y}^2)^{1/2} dt, \tag{b}$$

where t is a parameter, $\dot{x} = dx/dt$, $\dot{y} = dy/dt$, L is a constant, and x and y are dependent variables, we wish to determine the Euler equations using the Lagrange multiplier method.

The constraint condition can be viewed as an algebraic one in the sense that the integral quantity is a number. Hence, the Lagrange multiplier is also a number. Thus, we have

$$\begin{aligned} I_L &= \frac{1}{2}\int_{t_1}^{t_2} (x\dot{y} - y\dot{x})\, dt + \lambda\left(\int_{t_1}^{t_2} \sqrt{\dot{x}^2 + \dot{y}^2}\, dt - L\right) \\ &= \int_{t_1}^{t_2} F(x, y, \lambda, \dot{x}, \dot{y})\, dt, \end{aligned} \tag{c}$$

where λ is the Lagrange multiplier, independent of t.

The two Euler equations, in addition to Eq. (b), are given by

$$\delta x\colon\quad \frac{\partial F}{\partial x} - \frac{d}{dt}\left(\frac{\partial F}{\partial \dot{x}}\right) = 0,$$

$$\delta y\colon\quad \frac{\partial F}{\partial y} - \frac{d}{dt}\left(\frac{\partial F}{\partial \dot{y}}\right) = 0,$$

or

$$\delta x\colon\quad \frac{1}{2}\dot{y} - \frac{d}{dt}\left[-\frac{1}{2}y + \frac{\lambda\dot{x}}{\sqrt{\dot{x}^2 + \dot{y}^2}}\right] = 0, \tag{d}$$

$$\delta y\colon\quad -\frac{1}{2}\dot{x} - \frac{d}{dt}\left[\frac{1}{2}x + \frac{\lambda\dot{y}}{\sqrt{\dot{x}^2 + \dot{y}^2}}\right] = 0. \tag{e}$$

Example 4.13 Consider the problem of minimizing the functional

$$\Pi(w_0, \phi) = \int_0^L \left[\frac{EI}{2}\left(\frac{d\phi}{dx}\right)^2 + qw_0\right] dx - Fw_0(L), \tag{4.129}$$

subject to the constraint

$$G(w_0', \phi) \equiv \frac{dw_0}{dx} + \phi = 0. \tag{4.130}$$

The functional represents the total potential energy of a cantilever beam subjected to uniformly distributed load q and point load F at its free end. The constraint in Eq. (4.130) represents the relation between the transverse deflection w_0 and rotation ϕ of a transverse normal line. The essential boundary conditions for the problem are (see Fig. 4.10 for the geometry and coordinates)

$$w_0(0) = \phi(0) = 0. \tag{4.131}$$

In the Lagrange multiplier method, the modified functional for the problem is given by

$$\Pi_L(w_0, \phi, \lambda) = \Pi(w_0, \phi) + \int_0^L \lambda \left(\frac{dw_0}{dx} + \phi\right) dx, \tag{4.132}$$

where $\lambda(x)$ denotes the Lagrange multiplier. The Euler equations of the problem are obtained by setting the partial variations of Π_L with respect to w_0, ϕ, and λ to zero:

$$\delta_w \Pi_L = 0: \qquad -\frac{d\lambda}{dx} + q = 0, \quad 0 < x < L. \tag{4.133a}$$

$$\lambda - F = 0, \quad \text{at } x = L. \tag{4.133b}$$

$$\delta_\phi \Pi_L = 0: \qquad -\frac{d}{dx}\left(EI\frac{d\phi}{dx}\right) + \lambda = 0, \quad 0 < x < L. \tag{4.134a}$$

$$EI\frac{d\phi}{dx} = 0, \quad \text{at } x = L. \tag{4.134b}$$

$$\delta_\lambda \Pi_L = 0: \qquad \frac{dw_0}{dx} + \phi = 0, \quad 0 < x < L. \tag{4.135}$$

The Lagrange multiplier $\lambda(x)$ in the present case turns out to be the shear force, $V(x)$.

In the penalty function method, the modified functional is given by

$$\Pi_p(w, \phi) = \Pi(w_0, \phi) + \frac{\gamma}{2}\int_0^L \left(\frac{dw_0}{dx} + \phi\right)^2 dx. \tag{4.136}$$

The Euler equations are given by

$$\delta_w \Pi_p = 0: \qquad -\gamma\frac{d}{dx}\left(\frac{dw_0}{dx} + \phi\right) + q = 0, \quad 0 < x < L. \tag{4.137a}$$

$$\gamma\left(\frac{dw_0}{dx} + \phi\right) - F = 0, \quad \text{at } x = L. \tag{4.137b}$$

$$\delta_\phi \Pi_p = 0: \qquad -\frac{d}{dx}\left(EI\frac{d\phi}{dx}\right) + \gamma\left(\frac{dw_0}{dx} + \phi\right) = 0, \quad 0 < x < L. \tag{4.138a}$$

$$EI\frac{d\phi}{dx} = 0, \quad \text{at } x = L. \tag{4.138b}$$

A comparison of Eq. (4.133b) with Eq. (4.137b) shows that the Lagrange multiplier is related to ϕ and w_0 by

$$\lambda(x) = \gamma\left(\frac{dw_0}{dx} + \phi\right). \tag{4.139}$$

For the choice of $\gamma = K_s GA$, Eq. (4.139) represents the shear force and shear–strain relation in the Timoshenko beam theory (see Exercise 4.24):

$$\lambda(x) \sim V(x) = K_s GA\gamma_{xz} = K_s GA \left(\phi + \frac{dw_0}{dx}\right), \tag{4.140}$$

where K_s is the shear correction factor, G the shear modulus, and A the area of cross section of the beam.

EXERCISES

4.1 Consider the double pendulum shown in Fig. E4.1 at some instant of time. Write the potential energy (only due to gravity) of the system assuming that the energy is zero when $\theta_1 = \theta_2 = 0$. Neglect the weight of the pendulum bars and assume that they are rigid.

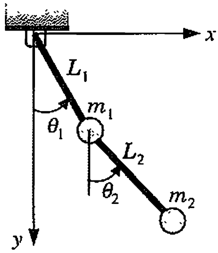

Figure E4.1

4.2 Consider the rigid-body assemblage shown in Fig. E4.2 at some instant of time. The bars are interconnected by hinges, and their relative rotations are resisted by torsional springs located at each hinge. Write the potential energy of the system assuming that the energy is zero when $\theta_1 = \theta_2 = \theta_3 = 0$. Include the weight W of the rigid bars and assume that the displacements are small and the torsional springs are linear.

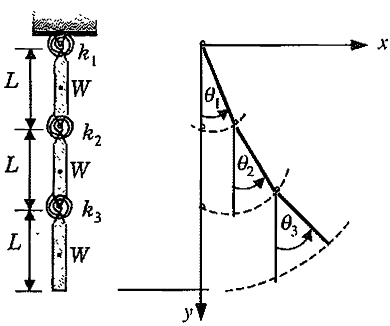

Figure E4.2

4.3 The rigid link shown in Fig. E4.3 is in equilibrium when pinned at one end and subjected to forces F_x and F_y and moment M as shown. Find the incremental work done, ΔW^*, when the link is subjected to increments ΔF_x, ΔF_y, and ΔM. Express the answer only in terms of the increments ΔF_x and ΔF_y.

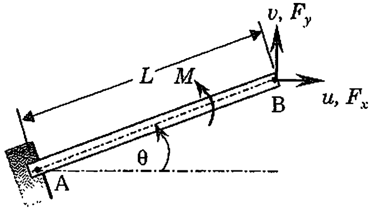

Figure E4.3

4.4 Determine the strain energy and external work done for the bar with an end linear elastic spring shown in Fig. E4.4.

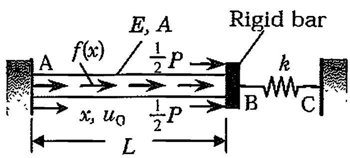

Figure E4.4

4.5 Determine the strain energy and external work done for the beam with an end linear elastic spring shown in Fig. E4.5. Neglect the energy due to shear.

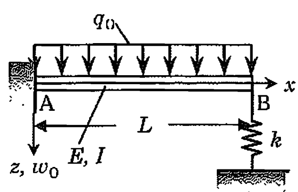

Figure E4.5

4.6 Determine the total complementary strain energy due to bending as well as shear for the beam shown in Fig. E4.6.

4.7 Repeat Exercise 4.5 when the spring behaves according to the relation $F_s = kw_0^n$, w_0 being the elongation of the spring.

4.8 Determine the complementary strain energy of the structure shown in Fig. E4.8. Include energies due to bending as well as torsion but neglect that due to transverse shear forces.

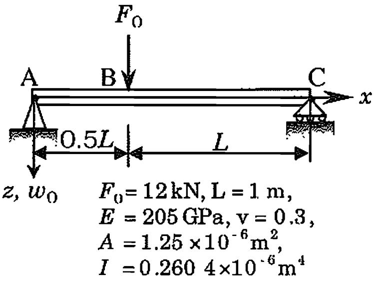

Figure E4.6

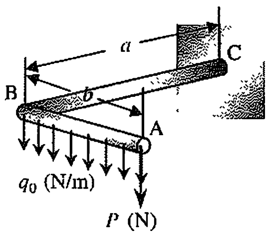

Figure E4.8

4.9 Consider the pin-connected structure shown in Fig. E4.9. Suppose that the material of all members obeys the following stress–strain relation:

$$\sigma = \begin{cases} K\sqrt{\varepsilon}, & \varepsilon \geq 0, \\ -K\sqrt{-\varepsilon}, & \varepsilon \leq 0, \end{cases}$$

where K is a constant. All members have the same cross-sectional area, A. Write the complementary strain energy densities of the members, and the total complementary strain energy of the structure.

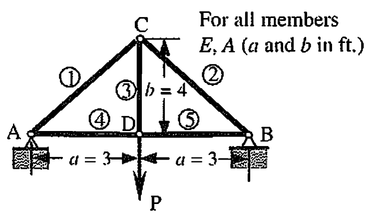

Figure E4.9

4.10 Determine (a) the strain energy and (b) the complementary strain energy of the truss shown in Fig. E4.10. Assume elastic behavior of the form $\sigma = E\sqrt{\varepsilon}$. Express your answer in terms of displacements in the former case and applied loads in the latter case. *Hint:* You may have to use a displacement constraint to determine the reactions in the truss members.

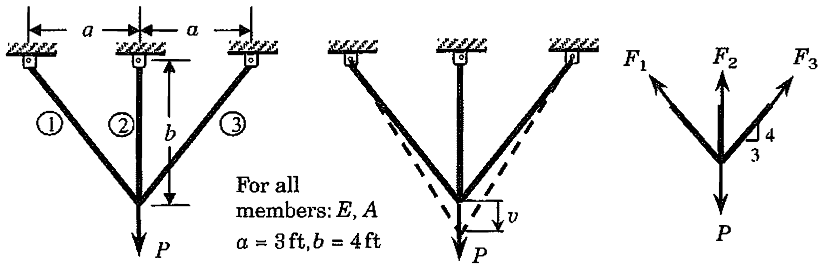

Figure E4.10

4.11 Identify admissible virtual displacements for the problem in Exercise 4.5.

4.12 Identify admissible virtual displacements for the problem in Exercise 4.8.

4.13 Identify admissible virtual forces for the problem in Exercise 4.4.

4.14 Identify admissible virtual forces for the problem in Exercise 4.6.

4.15 Identify admissible virtual forces for the problem in Exercise 4.9.

4.16 Write the total virtual work expression for the problem in Exercise 4.4.

4.17 Write the total virtual work expression for the problem in Exercise 4.5.

4.18 Write the total complementary virtual work expression for the problem in Exercise 4.3.

4.19 Write the total complementary virtual work expression for the problem in Exercise 4.4.

4.20 Write the total complementary virtual work expression for the problem in Exercise 4.5.

4.21 Write the total complementary virtual work expression for the problem in Exercise 4.8.

4.22 Write the total complementary virtual work expression for the problem in Exercise 4.9.

4.23 Two rigid links are pin-connected at point A and to a rigid support at point O as shown in Fig. E4.23. Compute the virtual work done by the forces due to the virtual changes $\delta\theta_1$ and $\delta\theta_2$ from the equilibrium configuration shown in the figure.

4.24 A rigid bar supported by three linear elastic springs and subjected to load P and moment M occupies the equilibrium configuration shown in Fig. E4.24. Write the total complementary virtual work done, δW^*, and write the equilibrium equations that the virtual forces δP, δM, and δF_3 (virtual force in spring k_3) must satisfy.

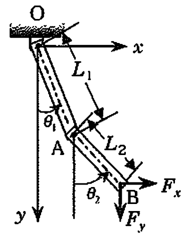

Figure E4.23

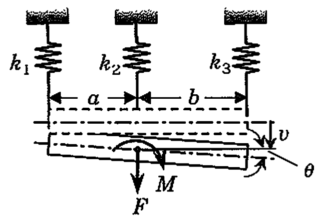

Figure E4.24

4.25 Write the virtual strain energy expression for the Euler–Bernoulli beam theory with the following nonlinear strain–displacement relation:

$$\varepsilon_{xx} = \frac{du_0}{dx} + \frac{1}{2}\left(\frac{dw_0}{dx}\right)^2 - z\frac{d^2w_0}{dx^2}.$$

4.26 The *Timoshenko beam theory* is based on the displacement field

$$u(x, z) = z\phi(x),$$
$$w(x, z) = w_0(x), \tag{a}$$

where w_0 is the transverse deflection of a point on the midplane (i.e., $z = 0$) of the plate and ϕ is the rotation of a transverse normal line about the y-axis.

(i) Compute linear strains using the displacement field (a).

(ii) Write the expression for the total virtual work done by actual forces in moving through the virtual displacements (δw_0, $\delta\phi$), assuming that the beam is loaded with a distributed transverse force $q(x)$. Express the internal virtual work in terms of the stress resultants

$$M = \int_A z\sigma_{xx}\, dA, \qquad V = \int_A \sigma_{xz}\, dA. \tag{b}$$

4.27 The *classical theory* of circular plates is based on the displacement field

$$u_r(r,z) = -z\frac{dw_0}{dr},\qquad u_z(r,z) = w_0(r), \tag{a}$$

where w_0 is the transverse deflection of a point on the midplane (i.e., $z = 0$) of the plate, the r-coordinate is taken radially outward from the center of the plate, the z-coordinate along the thickness (or height) of the plate, and the θ-coordinate is taken along a circumference of the plate (see Fig. E4.27). In a general case where applied loads and geometric boundary conditions are not axisymmetric, the displacements (u_r, u_θ, u_z) along the coordinates (r, θ, z) are functions of $r, \theta,$ and z-coordinates. Assume that the applied loads and boundary conditions are independent of the θ-coordinate (i.e., axisymmetric) so that the displacement u_θ is identically zero and (u_r, u_z) are only functions of r and z.

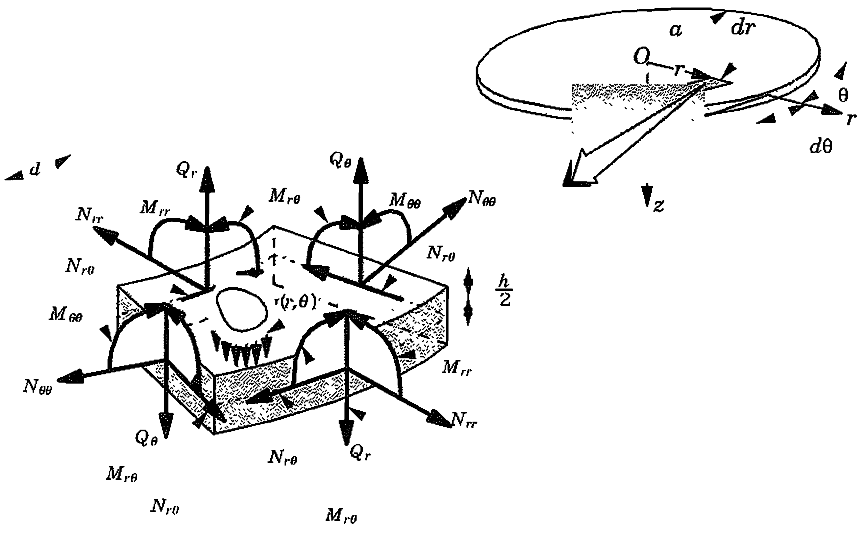

Figure E4.27

(i) Compute linear strains using the displacement field in (a).

(ii) Write the expression for the total virtual work done, $\delta W = \delta W_I + \delta W_E$, assuming that the plate is subjected to transversely distributed load $q = q(r)$. Express the internal virtual work in terms of the moments (see Fig. E4.27)

$$M_{rr} = \int_{-h/2}^{h/2} \sigma_{rr} z\, dz, \qquad M_{\theta\theta} = \int_{-h/2}^{h/2} \sigma_{\theta\theta} z\, dz, \tag{b}$$

where σ_{rr} and $\sigma_{\theta\theta}$ are the radial and circumferential stresses, respectively.

Calculate the first variation of the functionals in Exercises 4.28–4.36. The following notation is used:

$$u' \equiv \frac{du}{dx}, \qquad u'' \equiv \frac{d^2u}{dx^2}, \qquad u_x \equiv \frac{\partial u}{\partial x}, \qquad u_{i,j} \equiv \frac{\partial u_i}{\partial x_j}.$$

All variables other than those listed in the argument of the functional are assumed to be functions of position.

4.28 $I(u) = \int_a^b \sqrt{1 + (u')^2}\, dx.$

4.29 $I(u) = \int_a^b u\sqrt{1 + (u')^2}\, dx.$

4.30 $I(u) = \int_0^1 \sqrt{(1 + (u')^2)/u}\, dx.$

4.31 $I(u) = \frac{1}{2}\int_a^b \{(u'')^2 + 2u'u'' + (u')^2 - 2u\}\, dx.$

4.32 $$\Pi(u_0, w_0) = \int_0^L \left\{ \frac{EA}{2}\left[\frac{du_0}{dx} + \frac{1}{2}\left(\frac{dw_0}{dx}\right)^2\right]^2 + \frac{EI}{2}\left(\frac{d^2w_0}{dx^2}\right)^2 + qw_0 \right\} dx$$
$$- Pu_0(L) - M_0\frac{dw_0}{dx}(L).$$

The boundary conditions are: $u_0(0) = 0$, $w_0(0) = 0$, and $\left(\frac{dw_0}{dx}\right)_0 = 0.$

4.33 $I(u, v) = \frac{1}{2}\int_\Omega \left(u_x^2 + 2u_x v_x + 2u_y v_y + v_y^2 + 2fu + 2gv\right) dxdy.$

4.34 $$I(u, \mathbf{v}) = \int_\Omega \left\{ \frac{1}{2k}\mathbf{v}\cdot\mathbf{v} + \mathbf{v}\cdot\nabla u + Qu \right\} d\mathbf{x} - \int_{S_1} qu\, ds.$$

4.35 $$I(u_1, u_2, P) = \int_\Omega \left\{ \frac{\mu}{2}(u_{i,j} + u_{j,i})u_{i,j} - Pu_{i,i} \right\} dx_1dx_2 - \int_\Omega f_i u_i\, dx_1dx_2$$
$$- \int_{S_2} \hat{t}_i u_i\, ds.$$

$u_i = 0$ on the boundary S_1 and $S_1 \cup S_2 = S$, the total boundary.

4.36 $$\Pi(w) = \frac{D}{2}\int_\Omega \left[\left(\frac{\partial^2 w}{\partial x^2}\right)^2 + \left(\frac{\partial^2 w}{\partial y^2}\right)^2 + 2\nu\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2} + 2(1-\nu)\left(\frac{\partial^2 w}{\partial x \partial y}\right)^2 \right]$$
$$\times dxdy + \frac{1}{2}\int_\Omega kw^2\, dxdy - \int_\Omega qw\, dxdy.$$

$w = 0$ and $\partial w/\partial n = 0$ on the boundary of the domain Ω.

Express the next three problems in mathematical terms as one of minimizing appropriate quantities, subjected to certain end conditions, and possibly some constraints.

4.37 *The Brachistochrone problem.* Determine the curve $y = y(x)$ connecting two points, A: $(0, 0)$ and B: (x_b, y_b), in a vertical plane such that a material particle, sliding without friction under its own weight, travels from point A to point B along the curve in the *shortest time* (see Fig. E4.37).

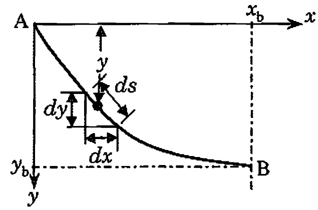

Figure E4.37

4.38 *Geodesic problem.* Find the curve $y = y(x)$ of *minimum length* joining two given points, A: (a, y_a) and B: (b, y_b), in the xy-plane (see Fig. E4.38).

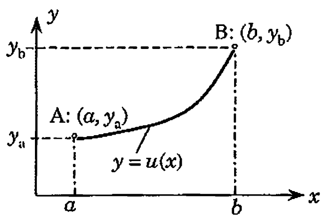

Figure E4.38

4.39 *Isoperimetric problem.* Among all curves with a continuous derivative that join two given points, A: (a, y_a) and B: (b, y_b), and have the given length L, find the one that encompasses the *largest possible area.*

Determine (a) the Euler equations and (b) the natural boundary conditions associated with the functionals identified in Exercises 4.40–4.43. Specified essential boundary conditions are shown if there are any.

4.40 Functional in Exercise 4.31.

4.41 Functional in Exercise 4.32.

4.42 Functional in Exercise 4.33.

4.43 Functional in Exercise 4.35.

4.44 Functional in Exercise 4.36.

4.45 Obtain the Euler equations and the natural boundary conditions associated with the functional

$$\Pi(w_0) = \frac{\pi}{2}\int_{r_i}^{r_0}\left\{D_{11}\left(\frac{d^2w_0}{dr^2}\right)^2 + \frac{2D_{12}}{r}\frac{dw_0}{dr}\frac{d^2w_0}{dr^2} + D_{22}\left(\frac{1}{r}\frac{dw_0}{dr}\right)^2\right\} r\,dr,$$

which arises in connection with axisymmetric bending of polar orthotropic annular plates with inner radius r_i and outer radius r_0. Assume that the deflection $w_0 = 0$ at $r = r_0$, the outer radius of the plate.

4.46 Derive the Euler equations and the natural and essential boundary conditions associated with the Lagrange multiplier functional of the following problem: Minimize the functional

$$\Pi(w_0, \phi_x, \phi_y) = \frac{D}{2}\int_\Omega \left[\left(\frac{\partial \phi_x}{\partial x}\right)^2 + \left(\frac{\partial \phi_y}{\partial y}\right)^2 + 2\nu\frac{\partial \phi_x}{\partial x}\frac{\partial \phi_y}{\partial y} + (1-\nu)\left(\frac{\partial \phi_x}{\partial y} + \frac{\partial \phi_y}{\partial x}\right)^2\right] dxdy - \int_\Omega q w_0 \, dxdy$$

subject to the constraints

$$\frac{\partial w_0}{\partial x} + \phi_x = 0, \qquad \frac{\partial w_0}{\partial y} + \phi_y = 0.$$

Use two different Lagrange multipliers, say λ_x and λ_y, with the two constraints.

4.47 Repeat Exercise 4.46 with the penalty functional of the problem (i.e., construct the penalty functional and derive the Euler equations). Use two different penalty parameters, say γ_x and γ_y, with the two constraints. Note that the penalty functional represents the total potential energy of the first-order shear deformation plate theory (FSDT) discussed in Chapter 8 (for $\gamma_X = \gamma_y = K_s Gh$, where K_s is the shear correction coefficient, G the shear modulus, and h the thickness of the plate).

4.48 Consider the problem of maximizing the functional

$$I(u) = \int_a^b u(x)\, dx, \qquad u(a) = u_a, \qquad u(b) = u_b,$$

subject to the constraint

$$G(u') \equiv \int_a^b \sqrt{1 + \left(\frac{du}{dx}\right)^2}\, dx - L = 0.$$

Use (a) the Lagrange multiplier method and (b) the penalty method to introduce the constraint into $I(u)$, and derive associated Euler equations. Note that the constraint is an integral constraint.

4.49 Consider the problem of minimizing the functional

$$I(u) = \int_a^b u\sqrt{1 + \left(\frac{du}{dx}\right)^2}\, dx, \qquad u(a) = u_a, \qquad u(b) = u_b,$$

subject to the constraint

$$\int_a^b \sqrt{1 + \left(\frac{du}{dx}\right)^2}\, dx = L.$$

Use (a) the Lagrange multiplier method and (b) the penalty function method to introduce the constraint into the functional, and derive the Euler equations.

4.50 Consider the problem of minimizing the functional

$$\Pi(w_0, \kappa_{xx}) = \frac{EI}{2}\int_0^L \kappa_{xx}^2\, dx - \int_0^L q w_0\, dx,$$

subject to the constraint

$$\kappa_{xx} - \frac{d^2 w_0}{dx^2} = 0,$$

where w_0 is the tranverse deflection, q the distributed load, EI is the flexural stiffness, and L is the length of the beam. Construct a functional using the Lagrange multiplier method and determine the Euler equations.

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5

ENERGY PRINCIPLES OF STRUCTURAL MECHANICS

5.1 VIRTUAL WORK PRINCIPLES

5.1.1 Introduction

In this chapter we study the virtual work and energy principles of solid mechanics. These include the principles of virtual displacements and virtual forces, the principle of minimum total potential energy, and the principle of maximum total complementary energy. These principles will be used to derive the unit-dummy-displacement and unit-dummy-load methods and Castigliano's theorems I and II of structural mechanics.

The variational principles will be used to derive the equations of equilibrium or motion of deformable solids, including bars and beams. The use of energy methods (i.e., unit-dummy-displacement and unit-dummy-load methods and Castigliano's theorems) in the determination of point displacements and forces will be illustrated with bars, beams, trusses, and frames. Betti's and Maxwell's reciprocity theorems will also be discussed. Both direct variational methods (i.e., Ritz, Galerkin, and weighted-residual methods) and the finite element method that make use of variational methods will be discussed in subsequent chapters.

5.1.2 The Principle of Virtual Displacements

Recall from Section 4.2 that virtual work is the work done on a particle or a deformable body by actual forces in moving through a hypothetical or virtual displacement that is consistent with the geometric constraints. The applied forces are kept constant during the virtual displacement. The principle of virtual displacements states that the virtual work done by actual forces is zero if and only if the body is in equilibrium.

It is informative to consider the principle of virtual displacements for a particle in static equilibrium. Suppose that the particle is in equilibrium under the action of

n concurrent forces $\mathbf{F}_1, \mathbf{F}_2, \ldots, \mathbf{F}_n$. Now suppose that the particle is given an arbitrary virtual displacement $\delta\mathbf{u}$ during which all forces along with their directions are fixed. The total virtual work done by all forces is given by

$$\begin{aligned}\delta W &= \mathbf{F}_1 \cdot \delta\mathbf{u} + \mathbf{F}_2 \cdot \delta\mathbf{u} + \cdots + \mathbf{F}_n \cdot \delta\mathbf{u} \\ &= \left(\sum_{i=1}^{n} \mathbf{F}_i\right) \cdot \delta\mathbf{u}.\end{aligned} \tag{5.1}$$

Note that the expression in parentheses is the vector sum of all forces acting on the particle. From vector mechanics we know that the sum is zero if the particle is in equilibrium, giving $\delta W = 0$. Conversely, if $\delta W = 0$ and $\delta\mathbf{u}$ is arbitrary, it follows that $\sum_i^n \mathbf{F}_i = 0$, i.e., the particle is in equilibrium. In other words, the particle is in equilibrium if and only if $\delta W = 0$ for *any* choice of $\delta\mathbf{u}$. The statement $\delta W = 0$ is the mathematical statement of the principle of virtual displacements for a particle. The discussion can be extended to rigid bodies because a rigid body is merely a collection of particles.

Next, we consider a generalization of the principle of virtual displacements to deformable bodies. In deformable bodies, material points can move relative to one another and do internal work in addition to the work done by external forces. Thus we should consider the virtual work done by internal forces (i.e., stresses) as well as that done by external forces.

Consider a continuous material occupying the volume V and in equilibrium under the action of body forces $\mathbf{f}$ and surface tractions $\mathbf{t}$. Suppose that over portion S_1 of the boundary displacements are specified to be $\hat{\mathbf{u}}$, and on portion S_2 tractions are specified to be $\hat{\mathbf{t}}$. The boundary portions S_1 and S_2 are disjoint (i.e., do not overlap), and their sum is the total boundary, S. Let $\mathbf{u} = (u_1, u_2, u_3)$ be the displacement vector corresponding to the equilibrium configuration of the body, and let σ_{ij} and ε_{ij} be the associated stress and strain components, respectively. We make no assumption concerning the constitutive behavior of the material body.

The set of admissible configurations is defined by sufficiently differentiable displacement fields that satisfy the geometric boundary conditions: $\mathbf{u} = \hat{\mathbf{u}}$ on S_1. Of all such admissible configurations, the actual one corresponds to the equilibrium configuration with the prescribed loads. In order to determine the displacement field $\mathbf{u}$ corresponding to the equilibrium configuration, we let the body experience a virtual displacement $\delta\mathbf{u}$ from the equilibrium configuration. The virtual displacements are arbitrary continuous functions except that they satisfy the homogeneous form of the specified geometric boundary conditions, i.e., $\delta\mathbf{u} = \mathbf{0}$ on S_1. The principle of virtual work states that *a continuous body is in equilibrium if and only if the virtual work of all forces, internal and external, acting on the body is zero in a virtual displacement*:

$$\delta W = \delta W_I + \delta W_E = 0, \tag{5.2}$$

where δW_I is the virtual work due to the internal forces and δW_E is the virtual work due to the external forces.

The principle of virtual work is independent of any constitutive law. The principle may be used to derive the equilibrium equations of deformable solids, as already illustrated by Examples 4.10 and 4.11. Here we consider additional examples.

Example 5.1 *(Euler–Bernoulli beam theory)* In Example 4.3, we have discussed the kinematic hypothesis of Euler–Bernoulli beam theory, and used a displacement field consistent with the hypothesis (see also Fig. 4.7a). Here we use the principle of virtual displacements to derive the governing equations of the Euler–Bernoulli beam theory for the case of large deflections but small strains.

The displacement field of a beam under the Euler–Bernoulli kinematic hypothesis is given by [see Eq. (4.28)]

$$u(x,y,z) = u_0(x) - z\frac{dw_0}{dx}, \qquad v = 0, \qquad w(x,y,z) = w_0(x). \tag{5.3}$$

If we assume that the strains are small in the sense that the following terms are negligible compared to du/dx,

$$\left(\frac{du}{dx}\right)^2, \quad \varepsilon_{zz}, \quad \frac{du}{dx}\frac{du}{dz}, \tag{5.4}$$

the only nonzero nonlinear strain is given by

$$\varepsilon_{xx} = \frac{du_0}{dx} + \frac{1}{2}\left(\frac{dw_0}{dx}\right)^2 - z\frac{d^2w_0}{dx^2}. \tag{5.5}$$

We shall consider the bending and stretching of a beam on a linear elastic foundation with foundation modulus k, and subjected to a distributed longitudinal load $f(x)$ and distributed transverse load $q(x)$ at the top (see Fig. 4.7b). Then the external and internal virtual works due to the virtual displacements δu_0 and δw_0 are given by

$$\begin{aligned}\delta W_E &= -\left[\int_0^L \left(f(x)\delta u_0 + q(x)\delta w\left(x, -\frac{h}{2}\right)\right) dx + \int_0^L F_s\delta w\left(x, \frac{h}{2}\right) dx\right] \\ &= -\left[\int_0^L (f(x)\delta u_0 + q(x)\delta w_0(x))\, dx - \int_0^L kw_0(x)\delta w_0(x)\, dx\right], \end{aligned}\tag{5.6}$$

$$\begin{aligned}\delta W_I &= \int_0^L \int_A \sigma_{xx}\delta\varepsilon_{xx}\, dx dA \\ &= \int_0^L \int_A \sigma_{xx}\left(\frac{d\delta u_0}{dx} + \frac{dw_0}{dx}\frac{d\delta w_0}{dx} - z\frac{d^2\delta w_0}{dx^2}\right) dx dA, \end{aligned}\tag{5.7}$$

where L is the length and A the cross-sectional area of the beam. The foundation reaction force F_s is replaced with $F_s = -kw_0(x)$ using the linear elastic constitutive equation for the foundation.

The principle of virtual displacements requires that $\delta W = \delta W_I + \delta W_E = 0$, which gives

$$
\begin{aligned}
0 &= \int_0^L \int_A \sigma_{xx} \left(\frac{d\delta u_0}{dx} + \frac{dw_0}{dx}\frac{d\delta w_0}{dx} - z\frac{d^2\delta w_0}{dx^2} \right) dA dx \\
&\quad - \int_0^L \left[f\delta u_0 + (q - kw_0)\,\delta w_0 \right] dx \\
&= \int_0^L \left[N\left(\frac{d\delta u_0}{dx} + \frac{dw_0}{dx}\frac{d\delta w_0}{dx} \right) - M\frac{d^2\delta w_0}{dx^2} \right] dx \\
&\quad - \int_0^L \left[f\delta u_0 + (q - kw_0)\,\delta w_0 \right] dx \\
&= \int_0^L \left[-\frac{dN}{dx}\delta u_0 - \frac{d}{dx}\left(\frac{dw_0}{dx}N \right)\delta w_0 - \frac{d^2 M}{dx^2}\delta w_0 \right] dx \\
&\quad - \int_0^L \left[f\delta u_0 + (q - kw_0)\,\delta w_0 \right] dx \\
&\quad + \left[N\delta u_0 + \left(\frac{dw_0}{dx}N + \frac{dM}{dx} \right)\delta w_0 - M\frac{d\delta w_0}{dx} \right]_0^L,
\end{aligned}
\tag{5.8}
$$

where N and M are the stress resultants defined by

$$
N = \int_A \sigma_{xx}\, dA, \qquad M = \int_A \sigma_{xx} z\, dA. \tag{5.9}
$$

Hence, the Euler equations are

$$
\delta u_0: \quad -\frac{dN}{dx} = f(x), \tag{5.10}
$$

$$
\delta w_0: \quad -\frac{d^2 M}{dx^2} - \frac{d}{dx}\left(\frac{dw_0}{dx}N \right) + kw_0 - q = 0, \tag{5.11}
$$

in $0 < x < L$.

Note that δu_0, δw_0, and $d\delta w_0/dx$ appear in the boundary terms. Hence, u_0, w_0, and dw_0/dx are the primary variables of the theory, and their specification constitutes the essential boundary conditions. Since nothing is said about the primary variables being specified, δu_0, δw_0, and $d\delta w_0/dx$ are arbitrary at $x = 0$ and $x = L$. Hence, the natural boundary conditions are:

$$
N = 0, \qquad \frac{dM}{dx} + \frac{dw_0}{dx}N = 0, \qquad M = 0. \tag{5.12}
$$

5.1.3 Unit-Dummy-Displacement Method

The principle of virtual work can also be used, in addition to deriving governing equations of equilibrium, to directly determine reaction forces and displacements in structural problems, in particular, truss and frame structures. If $\mathbf{R}_0$ is the force at point O in a structure, we can prescribe a virtual displacement $\delta\mathbf{u}_0$ at the point. The virtual strains due to the virtual displacement are determined from kinematic considerations. Then the principle of virtual work reads as

$$\mathbf{R}_0 \cdot \delta\mathbf{u}_0 = \int_V \sigma_{ij} \delta\varepsilon_{ij}^0 \, dV, \tag{5.13}$$

where σ_{ij} are the actual stresses, and $\delta\varepsilon_{ij}^0$ are the virtual strains in the structure derived from the virtual displacement $\delta\mathbf{u}_0$, consistent with the geometric constraints. Since $\delta\mathbf{u}_0$ is arbitrary, one can take $\delta\mathbf{u}_0 = \hat{\mathbf{e}}_R$, the unit vector along the force. If $\delta\varepsilon_{ij}^0$ are the strains due to the unit virtual displacement at point O, then

$$R_0 = \int_V \sigma_{ij} \delta\varepsilon_{ij}^0 \, dV. \tag{5.14}$$

This procedure is called the *unit-dummy-displacement method*. The word "dummy" is used historically to mean "virtual." Equation (5.14) is also valid for the case in which R_0 is replaced with a bending moment M_0:

$$M_0 = \int_V \sigma_{ij} \delta\varepsilon_{ij}^0 \, dV. \tag{5.15}$$

Equation (5.13) is more general than Eq. (5.14) in the sense that it is valid for an arbitrary virtual displacement vector (i.e., it need not be a unit vector).

Equations (5.14) and (5.15) can be used to determine point loads and moments or deflections and rotations of structures. If we were to use the unit-dummy-displacement method for the determination of displacements, rotations, forces, and/or moments in a bar, truss, beam, or frame structure, it is necessary that the displacement field is represented in terms of displacements and/or rotations of discrete points in the structure. In other words, we must identify generalized coordinates that can be used to represent the deformation of the structure. Suppose that $q_1, q_2, \ldots, q_n$ denote the generalized coordinates of a structure. Then the principle of virtual displacements can be stated as

$$\begin{aligned} 0 = \delta W(q_1, q_2, \ldots, q_n) &= \delta W_E + \delta W_I \\ &= \left(\frac{\partial W_E}{\partial q_i} + \frac{\partial W_I}{\partial q_i} \right) \delta q_i. \end{aligned} \tag{5.16}$$

Since

$$-\frac{\partial W_E}{\partial q_i} = F_i, \tag{5.17}$$

we have

$$F_i = \frac{\partial W_I}{\partial q_i} = \int_V \frac{\partial U_0}{\partial q_i}\, dV,$$

where F_i is a generalized force causing the generalized displacement q_i. The generalized coordinate q_i can be a displacement or a rotation, implying that F_i can be a force or a moment. For linear elastic beams under axial and bending deformation, we can write

$$F_i = \int_0^L \left[EA\frac{du_0}{dx}\frac{\partial}{\partial q_i}\left(\frac{du_0}{dx}\right) + EI\frac{d^2 w_0}{dx^2}\frac{\partial}{\partial q_i}\left(\frac{d^2 w_0}{dx^2}\right)\right] dx, \tag{5.18}$$

where it is understood that the axial displacement u_0 and transverse displacement w_0 can be expressed in terms of the generalized coordinates q_i.

Two examples of application of the unit-dummy-displacement method are presented next.

Example 5.2 Consider the truss shown in Fig. 5.1a. We wish to determine the vertical displacement v and horizontal displacement u of point O using the unit-dummy-displacement method. We shall assume linear elastic behavior and small strains.

Assume virtual (or dummy) displacements of δv_0 downward and δu_0 horizontal (see Fig. 5.1b) at point O so that $\delta \mathbf{u}_0 = \delta u_0 \hat{\mathbf{e}}_x + \delta v_0 \hat{\mathbf{e}}_y$. The load in this case is $\mathbf{R} = 0\hat{\mathbf{e}}_x + P\hat{\mathbf{e}}_y$. Then we have

$$\begin{aligned}\mathbf{R}_0 \cdot \delta \mathbf{u}_0 = 0 \cdot \delta u_0 + P \cdot \delta v_0 &= \int_V \sigma_{ij}\delta\varepsilon_{ij}\, dV \\ &= \int_0^{h_1} A^{(1)}\sigma_{11}^{(1)}\delta\varepsilon_{11}^{(1)}\, dx + \int_0^{h_2} A^{(2)}\sigma_{11}^{(2)}\delta\varepsilon_{11}^{(2)}\, dx,\end{aligned}$$

where $A^{(1)} = A^{(2)} = A$, $h_1 = h$, and $h_2 = \sqrt{2}h$. The actual stresses are computed in terms of the actual displacements u_0 and v_0 of point O from the deformed geometry

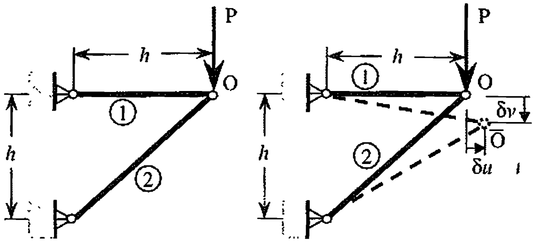

Figure 5.1 (a) Original truss. (b) Truss with virtual displacements.

of each member of the truss as follows:

$$\sigma_{11}^{(1)} = E^{(1)}\varepsilon_{11}^{(1)}, \qquad \sigma_{11}^{(2)} = E^{(2)}\varepsilon_{11}^{(2)}, \qquad E^{(1)} = E^{(2)} = E$$
$$\varepsilon_{11}^{(1)} = \frac{1}{h}\left(\sqrt{(h+u_0)^2 + v_0^2} - h\right) \approx \frac{u_0}{h} \tag{5.19}$$
$$\varepsilon_{11}^{(2)} = \frac{1}{\sqrt{2}h}\left(\sqrt{(h+u_0)^2 + (h-v_0)^2} - \sqrt{2}h\right) \approx \frac{u_0 - v_0}{2h},$$

where the squares of u_0/h and v_0/h were neglected in computing the strains (small strain assumption). Similarly, the virtual strains $\delta\varepsilon_{11}^{(1)}$ and $\delta\varepsilon_{11}^{(2)}$ can be computed as (alternatively, the virtual strains can be computed by taking the variation of the actual strains):

$$\delta\varepsilon_{11}^{(1)} = \frac{\delta u_0}{h}, \qquad \delta\varepsilon_{11}^{(2)} = \frac{\delta u_0 - \delta v_0}{2h}. \tag{5.20}$$

Substituting Eqs. (5.19) and (5.20) in (5.18), we obtain

$$\begin{aligned} 0 \cdot \delta u_0 + P \cdot \delta v_0 &= hA\sigma_{11}^{(1)}\frac{\delta u_0}{h} + \sqrt{2}hA\sigma_{11}^{(2)}\frac{\delta u_0 - \delta v_0}{2h} \\ &= \left(A\sigma_{11}^{(1)} + \frac{1}{\sqrt{2}}A\sigma_{11}^{(2)}\right)\delta u_0 + \left(-\frac{1}{\sqrt{2}}A\sigma_{11}^{(2)}\right)\delta v_0. \end{aligned} \tag{5.21}$$

Since the variations δu_0 and δv_0 are arbitrary, we can set them to (i) $\delta u_0 = 1$ and $\delta v_0 = 0$, and then to (ii) $\delta u_0 = 0$ and $\delta v_0 = 1$, to obtain the two relations for the two displacements. Alternatively, collecting the coefficients of δu and δv separately, we arrive at

$$0 = A\sigma_{11}^{(1)} + \frac{1}{\sqrt{2}}A\sigma_{11}^{(2)} = AE\frac{u_0}{h} + \frac{1}{\sqrt{2}}AE\left(\frac{u_0 - v_0}{2h}\right), \tag{5.22a}$$

$$P = -\frac{1}{\sqrt{2}}A\sigma_{11}^{(2)} = -\frac{1}{\sqrt{2}}AE\left(\frac{u_0 - v_0}{2h}\right), \tag{5.22b}$$

or

$$u_0 = \frac{Ph}{AE}, \qquad v_0 = \frac{2\sqrt{2}Ph}{AE} + u_0 = \frac{Ph}{AE}\left(1 + 2\sqrt{2}\right). \tag{5.22c}$$

Example 5.3 Consider a cantilever beam of length L and axial and bending stiffnesses EA and EI, respectively. We wish to determine the generalized deflections of the free end of the cantilever beam when it is subjected to transverse load F_0, bending moment M_0, and axial load P_0 at its free end ($x = L$; see Fig. 5.2). We shall use the Euler–Bernoulli beam theory (see Example 5.1).

First we must write the displacements u_0 and w_0 in terms of the generalized coordinates, which can be identified as the displacements of the free end, as discussed below. Then we apply the unit-dummy-displacement method to determine the generalized displacements in terms of the applied loads.

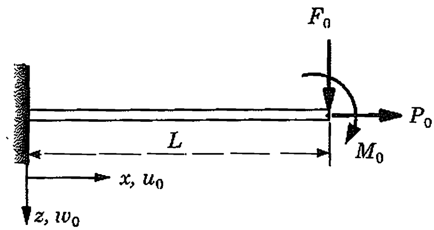

Figure 5.2 A cantilever beam with end loads.

The axial displacement $u_0(x)$ is governed by the equation

$$EA\frac{d^2u_0}{dx^2} = 0 \rightarrow u_0(x) = a_1 + a_2x. \tag{5.23}$$

Using the only displacement boundary condition $u_0(0) = 0$, we determine $a_1 = 0$. Then a_2 can be used as the generalized coordinate; it is useful to express a_2 in terms of the axial displacement at the free end, u_L: $u_0(L) = a_2L$ or $a_2 = u_0(L)/L = u_L/L$. Thus, we have ($c_1 = u_L$)

$$u_0(x) = \left(\frac{x}{L}\right)c_1.$$

Then the unit-dummy-displacement method gives

$$P_0 = \int_0^L EA\frac{du_0}{dx}\frac{d}{dc_1}\left(\frac{du_0}{dx}\right)dx = \int_0^L EA\frac{c_1}{L}\frac{1}{L}dx = \frac{EA}{L}c_1,$$

or

$$c_1 = u_L = u_0(L) = \frac{P_0L}{EA}.$$

Similarly, we proceed to express $w_0(x)$ in terms of generalized coordinates c_i, which must be identified. The equation governing $w_0(x)$ is

$$EI\frac{d^4w_0}{dx^4} = 0 \rightarrow w_0(x) = a_1 + a_2x + a_3x^2 + a_4x^3. \tag{5.24}$$

Using the two known boundary conditions $w_0(0) = 0$ and $(dw_0/dx)(0) = 0$, we obtain $a_1 = 0$ and $a_2 = 0$. Next, we express a_3 and a_4 in terms of the displacement and rotation at the free end, $x = L$:

$$w_0(L) = w_L \rightarrow w_L = a_3L^2 + a_4L^3; \quad \left(\frac{dw_0}{dx}\right)_{x=L} = \theta_L \rightarrow \theta_L = 2a_3L + 3a_4L^2.$$

Thus $w_L \equiv c_2$ and $\theta_L \equiv c_3$ can be used as the generalized coordinates. Solving the above equations for a_3 and a_4 in terms of w_L and θ_L, and substituting the result into Eq. (5.25), we obtain

$$w_0(x) = \left(3\frac{x^2}{L^2} - 2\frac{x^3}{L^3}\right)c_2 + L\left(-\frac{x^2}{L^2} + \frac{x^3}{L^3}\right)c_3.$$

Now using the unit-dummy-load method, we obtain

$$\begin{aligned} F_0 &= \int_0^L EI\frac{d^2w_0}{dx^2}\frac{d}{dc_2}\left(\frac{d^2w_0}{dx^2}\right)dx \\ &= \int_0^L EI\left[\left(\frac{6}{L^2} - \frac{12}{L^2}\frac{x}{L}\right)c_2 + L\left(-\frac{2}{L^2} + \frac{6}{L^2}\frac{x}{L}\right)c_3\right]\left(\frac{6}{L^2} - \frac{12}{L^2}\frac{x}{L}\right)dx, \\ M_0 &= \int_0^L EI\frac{d^2w_0}{dx^2}\frac{d}{dc_3}\left(\frac{d^2w_0}{dx^2}\right)dx \\ &= \int_0^L EI\left[\left(\frac{6}{L^2} - \frac{12}{L^2}\frac{x}{L}\right)c_2 + L\left(-\frac{2}{L^2} + \frac{6}{L^2}\frac{x}{L}\right)c_3\right]L\left(-\frac{2}{L^2} + \frac{6}{L^2}\frac{x}{L}\right)dx. \end{aligned}$$

Upon carrying out the integration, we obtain

$$F_0 = \frac{12EI}{L^3}c_2 - \frac{6EI}{L^2}c_3, \qquad M_0 = -\frac{6EI}{L^2}c_2 + \frac{4EI}{L}c_3,$$

whose solution gives the deflection and rotation of the free end:

$$c_2 = w_L = w_0(L) = \frac{F_0L^3}{3EI} + \frac{M_0L^2}{2EI}, \quad c_3 = \theta_L = \left(\frac{dw_0}{dx}\right)_{x=L} = \frac{F_0L^2}{2EI} + \frac{M_0L}{EI}.$$

The procedure discussed in the above example to express the displacements $u_0(x)$ and $w_0(x)$ in terms of unknown generalized displacement degrees of freedom holds for any bar and beam structure. When a distributed load $q(x) \neq 0$, one may convert it to a set of statically equivalent point loads F_i acting at the same points and directions as the generalized displacements c_i of the beam. For bars and beams with constant EA and EI but arbitrary load $q(x)$, this procedure results in exact values of the generalized displacements c_i. The procedure can be generalized to bars and beams with arbitrary boundary conditions, geometric properties, and material properties. However, in such cases the solutions obtained are approximate.

To further explain the idea, we express the displacement in terms of a linear combination of undetermined parameters c_i and known functions φ_i that satisfy the kinematic boundary conditions:

$$w_0(x) \approx \sum_{i=1}^{N} c_i\varphi_i(x). \tag{5.25}$$

For example, the deflection of a simply supported beam under any combination of loads can be expressed as

$$w_0(x) \approx \sum_{i=1}^{N} c_i \sin \frac{i\pi x}{L},$$

where L is the length of the beam. This choice of functions satisfies the boundary conditions $w_0(0) = w_0(L) = 0$. By substituting the expansion (5.25) into Eq. (5.18) with $u_0 = 0$, we obtain

$$\sum_{i=1}^{N} F_i \delta c_i = \int_0^L EI \frac{d^2 w_0}{dx^2} \frac{d^2 \delta w_0}{dx^2}\, dx$$

$$= \int_0^L EI \left(\sum_{j=1}^{N} \frac{d^2\varphi_j}{dx^2} c_j\right) \left(\sum_{i=1}^{N} \frac{d^2\varphi_i}{dx^2} \delta c_i\right) dx$$

$$0 = \sum_{i=1}^{N} \left[F_i - \sum_{j=1}^{N} \left(\int_0^L EI \frac{d^2\varphi_j}{dx^2} \frac{d^2\varphi_i}{dx^2}\, dx \right) c_j \right] \delta c_i$$

which gives, because of the independent nature of c_i, the necessary equations for the parameters c_i:

$$F_i = \sum_{j=1}^{N} \left(\int_0^L EI \frac{d^2\varphi_j}{dx^2} \frac{d^2\varphi_i}{dx^2}\, dx \right) c_j, \qquad i = 1, 2, \ldots, N. \tag{5.26}$$

Then the continuous (approximate) solution can be obtained from Eq. (5.25). These ideas will be further discussed in Chapter 7 in connection with the Ritz method.

5.2 PRINCIPLE OF TOTAL POTENTIAL ENERGY AND CASTIGLIANO'S THEOREM I

5.2.1 Principle of Minimum Total Potential Energy

The principle of virtual work discussed in the previous section is applicable to any continuous body with arbitrary constitutive behavior (i.e., elastic or inelastic). A special case of the principle of virtual work that deals with elastic (linear as well as nonlinear) bodies is known as the principle of minimum total potential energy.

Recall that for elastic bodies (in the absence of temperature variations) there exists a strain energy density function U_0 such that [see Eq. (3.34)]

$$\sigma_{ij} = \frac{\partial U_0}{\partial \varepsilon_{ij}}. \tag{5.27}$$

The strain energy density U_0 is a function of strains at a point and is assumed to be positive definite. The principle of virtual displacements, $\delta W = 0$, can be expressed in terms of the strain energy density U_0:

$$\begin{aligned} 0 = \delta W &= \int_\Omega \sigma_{ij}\delta\varepsilon_{ij}\, d\Omega - \left[\int_\Omega \mathbf{f}\cdot\delta\mathbf{u}\, d\Omega + \int_{S_2} \hat{\mathbf{t}}\cdot\delta\mathbf{u}\, dS\right] \\ &= \int_\Omega \frac{\partial U_0}{\partial \varepsilon_{ij}}\delta\varepsilon_{ij}\, d\Omega + \delta V \\ &= \int_\Omega \delta U_0\, d\Omega + \delta V = \delta(U+V) \equiv \delta\Pi, \end{aligned} \tag{5.28}$$

where

$$\delta V = -\left[\int_\Omega \mathbf{f}\cdot\delta\mathbf{u}\, d\Omega + \int_{S_2} \hat{\mathbf{t}}\cdot\delta\mathbf{u}\, dS\right],$$

or

$$V = -\left[\int_\Omega \mathbf{f}\cdot\mathbf{u}\, d\Omega + \int_{S_2} \hat{\mathbf{t}}\cdot\mathbf{u}\, dS\right], \tag{5.29a}$$

and U is the strain energy:

$$U = \int_\Omega U_0\, d\Omega. \tag{5.29b}$$

Note that we have used a constitutive equation (5.27) in arriving at Eq. (5.28). If the temperature changes are considered, the strain energy density U_0 should be replaced by $\rho\Psi$, where Ψ is the free-energy functional defined in Eqs. (4.14) and (4.15). The sum $V + U \equiv \Pi$ is called the *total potential energy* of the elastic body, and the statement

$$\Pi \equiv \delta(U + V) = 0 \tag{5.30}$$

is known as the *principle of minimum total potential energy.*

The principle of virtual displacements as well as the principle of minimum total potential energy give, when applied to an elastic body, the equilibrium equations (3.13) as the Euler equations. The main difference between them is that principle of virtual displacements gives the equilibrium equations in terms of stresses (or stress resultants), whereas the principle of minimum total potential energy gives them in terms of the displacements, because in the latter a constitutive relation is assumed to replace the stresses in terms of the displacements.

Example 5.4 Consider the bending of a beam according to the Euler–Bernoulli beam theory (see Example 5.1). We wish to construct the total potential energy functional and then determine the governing equation and boundary conditions.

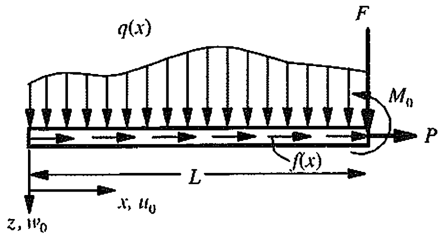

Figure 5.3 A beam with applied loads.

Recall from Example 4.3 that the strain energy of a beam under the assumption of small strains and displacements for the linear elastic case (i.e., obeying Hooke's law) is given by Eq. (4.39):

$$U = \frac{1}{2}\int_0^L \left[EA\left(\frac{du_0}{dx}\right)^2 + EI\left(\frac{d^2w_0}{dx^2}\right)^2\right] dx, \tag{5.31a}$$

where L is the length, A the cross-sectional area, I the second moment of area about the axis (y) of bending, and E is Young's modulus of the beam. Suppose that the beam is subjected to distributed axial force $f(x)$ (measured per unit length), distributed transverse load $q(x)$ (measured per unit length), horizontal point load P at $x = L$, transverse point load F at $x = L$, and bending moment M_0 (counterclockwise) at $x = L$, as shown in Fig. 5.3. Then the potential of applied forces is given by

$$V = -\left[\int_0^L (fu_0 + qw_0)\,dx + Pu_0(L) + Fw_0(L) + M_0\left(-\frac{dw_0}{dx}\right)_{x=L}\right]. \tag{5.31b}$$

Note that $(-dw_0/dx)$ is the rotation in the direction of the moment M_0. Hence, the total potential energy of the beam is

$$\Pi(u_0, w_0) = \frac{1}{2}\int_0^L \left[EA\left(\frac{du_0}{dx}\right)^2 + EI\left(\frac{d^2w_0}{dx^2}\right)^2 - 2(fu_0 + qw_0)\right] dx$$
$$- \left[Pu_0(L) + Fw_0(L) - M_0\left.\frac{dw_0}{dx}\right|_{x=L}\right]. \tag{5.32}$$

Applying the principle of minimum total potential energy, $\delta\Pi = 0$, and using the tools of variational calculus we obtain (see Example 4.6)

$$0 = \delta\Pi = \int_0^L \left(EA\frac{du_0}{dx}\frac{d\delta u_0}{dx} + EI\frac{d^2w_0}{dx^2}\frac{d^2\delta w_0}{dx^2}\right) dx$$
$$- \left[\int_0^L (f\delta u_0 + q\delta w_0)\,dx + P\delta u_0(L) + F\delta w_0(L) - M_0\left.\frac{d\delta w_0}{dx}\right|_{x=L}\right]$$

$$
\begin{aligned}
&= \int_0^L \left[-\frac{d}{dx}\left(EA\frac{du_0}{dx}\right)\delta u_0 + \frac{d^2}{dx^2}\left(EI\frac{d^2w_0}{dx^2}\right)\delta w_0 \right] dx \\
&\quad + \left[EA\frac{du_0}{dx}\delta u_0 \right]_0^L + \left[EI\frac{d^2w_0}{dx^2}\frac{d\delta w_0}{dx} - \frac{d}{dx}\left(EI\frac{d^2w_0}{dx^2}\right)\delta w_0 \right]_0^L \\
&\quad - \left[\int_0^L (f\delta u_0 + q\delta w_0)\, dx + P\delta u_0(L) + F\delta w_0(L) - M_0 \frac{d\delta w_0}{dx}\bigg|_{x=L} \right].
\end{aligned}
\tag{5.33}
$$

An examination of the boundary terms resulting from integration by parts shows that the primary and secondary variables of the theory are

$$
\text{Primary variables:}\quad u_0,\quad w_0,\quad \frac{dw_0}{dx}.
$$

$$
\text{Secondary variables:}\quad EA\frac{du_0}{dx},\quad \frac{d}{dx}\left(EI\frac{d^2w_0}{dx^2}\right),\quad EI\frac{d^2w_0}{dx^2}.
$$

It is clear that the secondary variables are nothing but the axial force $N(x)$, shear force $V(x) = dM/dx$, and bending moment $M(x)$. Only one element of each of the pairs (u_0, N), (w_0, V) and $(-dw_0/dx, M)$ may be specified at a point. Recall that specification of a primary variable is an essential boundary condition and that of a secondary variable is a natural boundary condition.

Returning to the expression in Eq. (5.33), first we collect the coefficients of δu_0 and δw_0 in $(0, L)$ together and set them to zero separately to obtain the Euler equations:

$$
\delta u_0:\quad -\frac{d}{dx}\left(EA\frac{du_0}{dx}\right) - f(x) = 0,\quad 0 < x < L, \tag{5.34}
$$

$$
\delta w_0:\quad \frac{d^2}{dx^2}\left(EI\frac{d^2w_0}{dx^2}\right) - q(x) = 0,\quad 0 < x < L. \tag{5.35}
$$

These equations are equivalent to those in Eqs. (5.10) and (5.11) (without the nonlinear and elastic foundation terms).

Now considering all boundary terms in (5.33), we conclude that

$$
\begin{aligned}
&\left(EA\frac{du_0}{dx}\right)_{x=0}\delta u_0(0) = 0, &&\left[EA\frac{du_0}{dx} - P\right]_{x=L}\delta u_0(L) = 0,\\
&\left[\frac{d}{dx}\left(EI\frac{d^2w_0}{dx^2}\right)\right]_{x=0}\delta w_0(0) = 0, &&\left[-\frac{d}{dx}\left(EI\frac{d^2w_0}{dx^2}\right) - F\right]_{x=L}\delta w_0(L) = 0,\\
&\left(EI\frac{d^2w_0}{dx^2}\right)_{x=0}\left(\frac{d\delta w_0}{dx}\right)_{x=0} = 0, &&\left(EI\frac{d^2w_0}{dx^2} + M_0\right)_{x=L}\left(\frac{d\delta w_0}{dx}\right)_{x=L} = 0.
\end{aligned}
\tag{5.36}
$$

If any of the quantities δu_0, δw_0, and $d\delta w_0/dx$ are zero at $x = 0$ or $x = L$, because of specified geometric boundary conditions there, the corresponding expressions vanish; the vanishing of the coefficients of δu_0, δw_0, and $d\delta w_0/dx$ at points where the geometric boundary conditions are not specified provides the natural boundary conditions. For example, suppose that the beam is clamped at $x = 0$ and free at $x = L$. Then $\delta u_0(0) = 0$, $\delta w_0(0) = 0$, and $d\delta w_0(0)/dx = 0$, and the natural boundary conditions become

$$\begin{aligned} \left[EA\frac{du_0}{dx} - P\right]_{x=L} &= 0, \\ \left[-\frac{d}{dx}\left(EI\frac{d^2w_0}{dx^2}\right) - F\right]_{x=L} &= 0, \\ \left(EI\frac{d^2w_0}{dx^2} + M_0\right)_{x=L} &= 0. \end{aligned} \tag{5.37}$$

5.2.2 Castigliano's Theorem I

Like the unit-dummy-displacement method, which can be used to determine the unknown point loads and displacements of structural systems composed of discrete structural members, Castigliano's theorem allows one to compute displacements or loads. Castigliano (1847–1884), an Italian mathematician and railroad engineer, was mainly concerned with linear elastic materials. The generalization of Castigliano's original Theorem I to the case in which displacements are nonlinear functions of external forces is attributed to Engesser. In the present study we consider Castigliano's theorems in a generalized form that are applicable to both linear and nonlinear elastic materials.

Suppose that the displacement field (or geometric configuration) of a structure can be expressed in terms of the displacements (and possibly rotations) of a finite number of points $\mathbf{x}_i$ $(i = 1, 2, \ldots, N)$ as

$$\mathbf{u}(\mathbf{x}) = \sum_{i=1}^{N} \mathbf{u}_i \phi_i(\mathbf{x}), \tag{5.38}$$

where $\mathbf{u}_i$ are unknown displacement parameters, called *generalized displacements*, and ϕ_i are known functions of position, called *interpolation functions*, with the property that ϕ_i is unity at the ith point (i.e., $\mathbf{x} = \mathbf{x}_i$) and zero at all other points ($\mathbf{x}_j$, $j \neq i$). Then it is possible to represent the strain energy U and potential V due to applied loads in terms of the generalized displacements $\mathbf{u}_i$. The principle of minimum total potential energy can be written as

$$\delta U = -\delta V \qquad \text{or} \qquad \frac{\partial U}{\partial \mathbf{u}_i} \cdot \delta \mathbf{u}_i = -\frac{\partial V}{\partial \mathbf{u}_i} \cdot \delta \mathbf{u}_i$$

where sum on repeated indices is implied. Since

$$\frac{\partial V}{\partial \mathbf{u}_i} = -\mathbf{F}_i,$$

and $\delta \mathbf{u}_i$ are arbitrary, it follows that

$$\left(\frac{\partial U}{\partial \mathbf{u}_i} - \mathbf{F}_i\right) \cdot \delta \mathbf{u}_i = 0 \qquad \text{or} \qquad \frac{\partial U}{\partial \mathbf{u}_i} = \mathbf{F}_i. \tag{5.39}$$

Equation (5.39) is known as the first theorem of Castigliano. For an elastic body, it states that the rate of change of strain energy with respect to the displacement is equal to the load causing the displacement. When applied to a structure with point loads F_i (or moment M_i) moving through displacements u_i (or rotation θ_i), both having the same sense, the theorem states that

$$\frac{\partial U}{\partial u_i} = F_i \qquad \text{or} \qquad \frac{\partial U}{\partial \theta_i} = M_i. \tag{5.40}$$

It is clear from the derivation that Castigliano's Theorem I is a special case of the principle of minimum total potential energy, and hence of the principle of virtual displacements. Consequently, the theorem is equivalent to the unit-dummy-displacement method.

Example 5.5 Let us revisit the truss problem of Example 5.2. The strain energy of the structure can be expressed in terms of the displacements u and v of point O (see Fig. 5.1). We have

$$\begin{aligned} U(u, v) &= \frac{1}{2}\sum_{i=1}^{2}\int_{V_i} E^{(i)}\left(\varepsilon^{(i)}\right)^2 dV \\ &= \frac{EA}{2}\left[h\left(\frac{u}{h}\right)^2 + \sqrt{2}h\left(\frac{u-v}{2h}\right)^2\right]. \end{aligned}$$

Then by Castigliano's Theorem I, we have

$$\begin{aligned} 0 &= \frac{\partial U}{\partial u} = EA\left(\frac{u}{h} + \frac{\sqrt{2}}{2}\frac{u-v}{2h}\right), \\ P &= \frac{\partial U}{\partial v} = EA\left(0 - \frac{\sqrt{2}}{2}\frac{u-v}{2h}\right), \end{aligned}$$

which gives

$$u = \frac{Ph}{EA}, \qquad v = \left(1 + 2\sqrt{2}\right)\frac{Ph}{EA}.$$

Example 5.6 Consider a free–free straight beam of constant bending stiffness EI and subjected to point loads and moments, as shown in Fig. 5.4a. The equation governing the equilibrium of the beam according to the Euler–Bernoulli beam theory [see Examples 4.3, 5.1, and 5.3, and Eq. (5.35)] is

$$EI\frac{d^4 w_0}{dx^4} = 0. \tag{5.41}$$

The solution to this homogeneous fourth-order equation is given by

$$w_0(x) = a_1 + a_2 x + a_3 x^2 + a_4 x^3, \tag{5.42}$$

where a_i $(i = 1, 2, 3, 4)$ are constants of integration. We wish to express these constants in terms of the deflections and rotations at the two ends of an arbitrary beam of length L. Suppose that

$$\begin{aligned} u_1 &\equiv w_0(0) = a_1, \\ u_2 &\equiv \left(-\frac{dw_0}{dx}\right)_{x=0} = -a_2, \\ u_3 &\equiv w_0(L) = a_1 + a_2 L + a_3 L^2 + a_4 L^3, \\ u_4 &\equiv \left(-\frac{dw_0}{dx}\right)_{x=L} = -a_2 - 2a_3 L - 3a_4 L^2. \end{aligned} \tag{5.43}$$

Note from Eq. (5.43) that u_1 and u_3 are the values of the transverse deflection w_0 at $x = 0$ and $x = L$, respectively, and u_2 and u_4 are the rotations $-dw_0/dx$, measured positive counterclockwise, at $x = 0$ and $x = L$, respectively (see Fig. 5.4b). The four equations relate the four constants a_i to the four displacements u_i, called generalized displacements, which serve as the generalized coordinates. These relations can be inverted to solve for u_i in terms of a_i. Then substituting the result into Eq. (5.42) yields

$$w_0(x) = \varphi_1(x)u_1 + \varphi_2(x)u_2 + \varphi_3(x)u_3 + \varphi_4(x)u_4 = \sum_{i=1}^{4} \varphi_i(x)u_i, \tag{5.44}$$

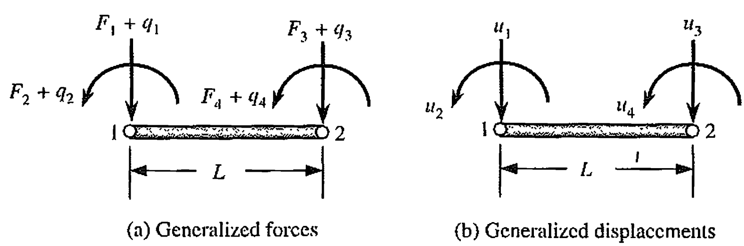

Figure 5.4 (a) Beam with end forces and moments (or generalized forces). (b) Generalized displacements.

where

$$
\begin{aligned}
\varphi_1(x) &= 1 - 3\left(\frac{x}{L}\right)^2 + 2\left(\frac{x}{L}\right)^3, \\
\varphi_2(x) &= -x\left[1 - 2\left(\frac{x}{L}\right) + \left(\frac{x}{L}\right)^2\right], \\
\varphi_3(x) &= \left(\frac{x}{L}\right)^2\left(3 - 2\frac{x}{L}\right), \\
\varphi_4(x) &= x\frac{x}{L}\left(1 - \frac{x}{L}\right).
\end{aligned}
\tag{5.45}
$$

The strain energy of the beam now can be expressed in terms of the generalized coordinates as

$$
\begin{aligned}
U &= \frac{EI}{2}\int_0^L \left(\frac{d^2 w_0}{dx^2}\right)^2 dx \\
&= \frac{EI}{2}\int_0^L \left(\sum_{i=1}^{4} u_i \frac{d^2\varphi_i}{dx^2}\right)\left(\sum_{j=1}^{4} u_j \frac{d^2\varphi_j}{dx^2}\right) dx \\
&= \frac{1}{2}\sum_{i=1}^{4}\sum_{j=1}^{4} K_{ij} u_i u_j,
\end{aligned}
\tag{5.46a}
$$

where

$$
K_{ij} = EI\int_0^L \frac{d^2\varphi_i}{dx^2}\frac{d^2\varphi_j}{dx^2}\,dx. \tag{5.46b}
$$

Note that K_{ij} is symmetric ($K_{ij} = K_{ji}$). The work done by applied forces is given by

$$
\begin{aligned}
V &= -\left[\int_0^L q(x) w_0(x)\,dx + \sum_{i=1}^{4} F_i u_i\right] \\
&= -\sum_{i=1}^{4} (q_i u_i + F_i u_i),
\end{aligned}
\tag{5.47a}
$$

where

$$
q_i = \int_0^L q(x)\varphi_i(x)\,dx, \tag{5.47b}
$$

and F_i are the generalized forces associated with the generalized displacements u_i. Thus, F_1 and F_3 are the transverse forces at $x = 0$ and $x = L$, respectively, and F_2 and F_4 are the bending moments at $x = 0$ and $x = L$, respectively (see Fig. 5.4a). The transverse forces q_1 and q_3 and bending moments q_2 and q_4 together are statically equivalent to the distributed load $q(x)$ on the beam.

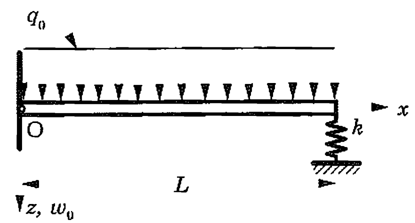

Figure 5.5 A beam fixed at $x = 0$ and supported by a spring at $x = L$.

Using Castigliano's Theorem I, we write

$$F_i + q_i = \frac{\partial U}{\partial u_i} = \sum_{j=1}^{4} K_{ij} u_j, \tag{5.48a}$$

or, in matrix form,

$$\frac{2EI}{L^3}\begin{bmatrix} 6 & -3L & -6 & -3L \\ -3L & 2L^2 & 3L & L^2 \\ -6 & 3L & 6 & 3L \\ -3L & L^2 & 3L & 2L^2 \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{Bmatrix} = \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{Bmatrix} + \begin{Bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \end{Bmatrix}. \tag{5.48b}$$

As a specific example, consider a beam fixed at $x = 0$ and spring supported at $x = L$, and subjected to uniformly distributed load (see Fig. 5.5). We wish to determine the compression in the spring, i.e., determine $w_0(L)$.

For uniformly distributed load $q = q_0$, we have from Eq. (5.47b)

$$\begin{Bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{Bmatrix} = \frac{q_0 L}{12}\begin{Bmatrix} 6 \\ -L \\ 6 \\ L \end{Bmatrix}. \tag{5.49}$$

The geometric boundary conditions require $u_1 = u_2 = 0$. The force boundary conditions require $F_3 = -F_s = -kw_0(L) = -ku_3$ and $F_4 = 0$. Thus we have

$$\frac{2EI}{L^3}\begin{bmatrix} 6 & -3L & -6 & -3L \\ -3L & 2L^2 & 3L & L^2 \\ -6 & 3L & 6 & 3L \\ -3L & L^2 & 3L & 2L^2 \end{bmatrix}\begin{Bmatrix} 0 \\ 0 \\ u_3 \\ u_4 \end{Bmatrix} = \frac{q_0 L}{12}\begin{Bmatrix} 6 \\ -L \\ 6 \\ L \end{Bmatrix} + \begin{Bmatrix} F_1 \\ F_2 \\ -ku_3 \\ 0 \end{Bmatrix}, \tag{5.50}$$

or, setting up the equations for the unknown generalized displacements u_3 and u_4, we obtain

$$\frac{2EI}{L^3}\begin{bmatrix} 6 & 3L \\ 3L & 2L^2 \end{bmatrix}\begin{Bmatrix} u_3 \\ u_4 \end{Bmatrix} = \frac{q_0 L}{12}\begin{Bmatrix} 6 \\ L \end{Bmatrix} + \begin{Bmatrix} -ku_3 \\ 0 \end{Bmatrix},$$

or

$$\begin{bmatrix} \frac{12EI}{L^3}+k & \frac{6EI}{L^2} \\ \frac{6EI}{L^2} & \frac{4EI}{L} \end{bmatrix} \begin{Bmatrix} u_3 \\ u_4 \end{Bmatrix} = \frac{q_0 L}{12} \begin{Bmatrix} 6 \\ L \end{Bmatrix}.$$

Solving for u_3 and u_4 by Cramer's rule, we obtain

$$u_3 = w_0(L) = \frac{q_0 L^4}{8EI} \frac{1}{(1 + (kL^3/3EI))},$$

$$u_4 = -\frac{q_0 L^3}{6EI} \frac{(EI - (kL^3/24))}{(EI + (kL^3/3))}.$$

Note that when $k = 0$, we obtain the deflection u_3 and rotation u_4 at the free end of a cantilever beam under uniformly distributed load of intensity q_0.

5.3 PRINCIPLES OF VIRTUAL FORCES AND COMPLEMENTARY POTENTIAL ENERGY

In Section 5.1 we discussed the principle of virtual displacements. Naturally, the virtual work done by virtual forces in moving through actual displacements should have similar use. Here we formulate the principle of virtual forces. The basic idea is that each particle in the body undergoes actual displacements until the stresses developed satisfy the equations of equilibrium.

Consider single-valued, differentiable variations of a stress field $\delta\overleftrightarrow{\sigma}$ and body forces $\delta\mathbf{f}$ that satisfy the linear equilibrium equations both within the body and on its boundaries:

$$\nabla \cdot \delta\overleftrightarrow{\sigma} + \delta\mathbf{f} = \mathbf{0} \qquad \text{in } V, \tag{5.51}$$

$$\hat{\mathbf{n}} \cdot \delta\overleftrightarrow{\sigma} = \delta\mathbf{t} \qquad \text{on } S_2. \tag{5.52}$$

We shall call such a stress field a *statically admissible field of variation*. These virtual stresses and forces, except for *self-equilibrating*, are completely arbitrary and independent of the true stresses and forces.

The external *complementary virtual work* is defined by

$$\delta W_E^* = -\int_V \mathbf{u} \cdot \delta\mathbf{f}\, dV - \int_{S_1} \hat{\mathbf{u}} \cdot \delta\mathbf{t}\, dS, \tag{5.53a}$$

where $\mathbf{u}$ is the displacement vector, and $\delta\mathbf{f}$ and $\delta\mathbf{t}$ satisfy Eqs. (5.51) and (5.52). The internal complementary virtual work is given by

$$\delta W_I^* = \int_V \overleftrightarrow{\varepsilon} : \delta\overleftrightarrow{\sigma}\, dV, \tag{5.53b}$$

where $\overset{\leftrightarrow}{\varepsilon}$ are actual linear strains and $\delta\overset{\leftrightarrow}{\sigma}$ is the virtual stress field that satisfies Eqs. (5.51) and (5.52).

The principle of complementary virtual work (or virtual forces) states that *the strains and displacements in a deformable body are compatible and consistent with the constraints if and only if the total complementary virtual work is zero*:

$$\delta W_I^* + \delta W_E^* = 0. \tag{5.54}$$

It can be shown that the principle of virtual forces gives the kinematic relations and geometric boundary conditions as the Euler equations. This is illustrated by considering the rectangular component form of δW_I^* and δW_E^* for a deformable solid:

$$\begin{aligned} 0 &= \int_V \varepsilon_{ij}\delta\sigma_{ij}\, dV - \int_V u_i \delta f_i\, dV - \int_{S_1} \hat{u}_i \delta t_i\, dS \\ &= \int_V \varepsilon_{ij}\delta\sigma_{ij}\, dV - \int_V u_i(-\delta\sigma_{ij,j})\, dV - \int_{S_1} \hat{u}_i n_j \delta\sigma_{ij}\, dS \\ &= -\int_V \left[\frac{1}{2}(u_{i,j} + u_{j,i}) - \varepsilon_{ij}\right]\delta\sigma_{ij}\, dV + \int_{S_1} (u_i - \hat{u}_i) n_j \delta\sigma_{ij}\, dS, \end{aligned} \tag{5.55}$$

where sum on repeated indices is assumed. Because $\delta\sigma_{ij}$ is arbitrary, we obtain the strain–displacement equations and the displacement boundary conditions as the Euler equations

$$\varepsilon_{ij} - \frac{1}{2}(u_{i,j} + u_{j,i}) = 0 \qquad \text{in } V \tag{5.56a}$$

$$u_i - \hat{u}_i = 0 \qquad \text{on } S_1. \tag{5.56b}$$

The unit-dummy-load method is a special case of the complementary virtual work, and it can be used to determine displacements and forces in structures. The basic idea can be described by analogy with the unit-dummy-displacement method. If u_0 is the true displacement at point O in an elastic structure, we can prescribe a virtual force δR_0 at the point. The application of virtual force induces a system of virtual stresses $\delta\sigma_{ij}$ that satisfy the equilibrium equations. Then from the principle of virtual forces (5.54), we have ($\delta W_E = -u_0\delta R_0$):

$$u_0\delta R_0 = \int_V \varepsilon_{ij}\delta\sigma_{ij}^0\, dV. \tag{5.57}$$

Once again, one can take $\delta R_0 = 1$ and calculate corresponding virtual internal stresses $\delta\sigma_{ij}^0$. Equation (5.57) represents the unit-dummy-load method.

For the Euler–Bernoulli beams, Eq. (5.57) takes the form

$$\begin{aligned} &u_0\delta P_0 + w_0\delta F_0 + \theta_0\delta M_0 + \phi_0\delta T_0 \\ &= \int_0^L \left(\frac{N}{EA}\delta N + \frac{M}{EI}\delta M + f_s\frac{V}{GA}\delta V + \frac{T}{GJ}\delta T\right) dx, \end{aligned} \tag{5.58}$$

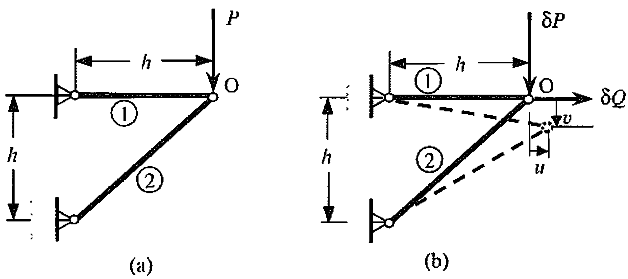

Figure 5.6 (a) Original truss. (b) Truss with virtual forces.

where $(\delta P_0, \ \delta F_0, \ \delta M_0, \ \delta T_0)$ are the generalized virtual forces at point O, and $(u_0, w_0, \theta_0, \phi_0)$ are the corresponding generalized displacements.

Example 5.7 Consider the frame structure shown in Fig. 5.6a. We wish to determine the vertical displacement v and horizontal displacement u of point O using the unit-dummy-load method. Assume virtual (or dummy) forces of δP downward and δQ horizontal (see Fig. 5.6b). Then we have

$$u \cdot \delta Q + v \cdot \delta P = \int_V \varepsilon_{ij} \delta\sigma_{ij} \, dV = \sum_{i=1}^{2} \int_0^{h_i} A^{(i)} \varepsilon_{11}^{(i)} \delta\sigma_{11}^{(i)} \, dx, \tag{5.59}$$

where $A^{(1)} = A^{(2)} = A$ and $h_1 = h, h_2 = \sqrt{2}h$. The actual stresses are computed in terms of the actual forces in each member due to the applied load P. For static equilibrium of the structure, the member forces are found to be

$$F^{(1)} = P, \qquad F^{(2)} = -\sqrt{2}P, \qquad \sigma_{11}^{(1)} = \frac{P}{A}, \qquad \sigma_{11}^{(2)} = -\frac{\sqrt{2}P}{A}. \tag{5.60a}$$

Hence

$$\varepsilon_{11}^{(1)} = \frac{P}{EA}, \qquad \varepsilon_{11}^{(2)} = -\frac{\sqrt{2}P}{EA}. \tag{5.60b}$$

The virtual stresses are computed from virtual forces, which in turn are computed from the virtual loads δP and δQ on the structure. We obtain

$$\delta F^{(1)} = \delta Q + \delta P, \qquad \delta F^{(2)} = -\sqrt{2}\,\delta P, \tag{5.61a}$$

and

$$\delta\sigma_{11}^{(1)} = \frac{\delta Q + \delta P}{A}, \qquad \delta\sigma_{11}^{(2)} = -\frac{\sqrt{2}\,\delta P}{A}. \tag{5.61b}$$

Substituting Eqs. (5.60b) and (5.61b) in (5.59), we obtain

$$\delta Q \cdot u + \delta P \cdot v = Ah\frac{\delta Q + \delta P}{A}\frac{P}{EA} + \sqrt{2}Ah\frac{\sqrt{2}\,\delta P}{A}\frac{\sqrt{2}P}{EA},$$

or, collecting the coefficients of δQ and δP separately, we obtain

$$u = \frac{Ph}{EA}, \qquad v = \frac{Ph}{EA} + 2\sqrt{2}\frac{Ph}{EA} = \frac{Ph}{AE}\left(1 + 2\sqrt{2}\right). \tag{5.62}$$

Example 5.8 Consider the overhang simply supported beam shown in Fig. 5.7a. We wish to determine the rotation (or slope) at the left end of the beam using the unit-dummy-load method.

Since no moment is applied at the left end where we wish to determine the rotation, we assume a virtual moment of δM_0 at the left end (clockwise). The virtual moment should be in self-equilibrium. A convenient set of self-equilibrating forces consist of a force of $\delta M_0/20$ downward at the left end and another force of $\delta M_0/20$ upward at the other support (see Fig. 5.7b). Using the unit-dummy-load method, we obtain

$$\theta_0 \cdot \delta M_0 = \frac{1}{EI}\int_0^L M(x)\delta M^0(x)\,dx, \tag{5.63}$$

where $M(x)$ is the actual bending moment and $\delta M^0(x)$ is the virtual bending moment due to the virtual forces at any location x along the beam. We have (first we must compute the reaction forces at the supports using static equilibrium of forces):

$$M(x) = \begin{cases} 3750x - 200x^2, & 0 \le x \le 20, \\ 3750x - 200x^2 + 6250(x - 20), & 20 \le x \le 25, \end{cases} \tag{5.64a}$$

$$\delta M^0(x) = \begin{cases} \delta M_0 - \left(\frac{1}{20}\delta M_0\right)x, & 0 \le x \le 20, \\ 0, & 20 \le x \le 25. \end{cases} \tag{5.64b}$$

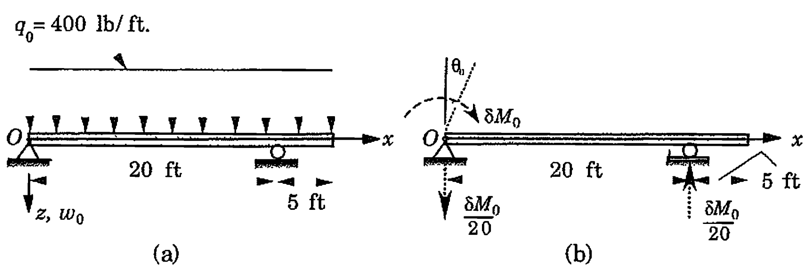

Figure 5.7 (a) Given beam. (b) Virtual force system.

Using (5.64a,b) in (5.63) and evaluating the integral (the integral over the interval [20 to 25] is zero), we obtain

$$\theta_0 = \frac{1}{EI}\int_0^{20}\left(3750x - 200x^2 - 187.5x^2 + 10x^3\right) dx = -\frac{7\times 10^5}{6EI}. \qquad (5.65)$$

Example 5.9 We wish to determine the horizontal and vertical displacements of point A of the frame structure shown in Fig. 5.8a. We use a dummy vertical load of $\delta Q = 1$ kN at point A to determine the vertical deflection. The self-equilibrating virtual force system is shown in Fig. 5.8b.

We have

$$\delta Q \cdot v = \int \frac{M}{EI}\delta M\, dx,$$

$$1 \cdot v = \int_0^a \frac{0}{EI}(x)\, dx + \int_0^a \frac{F_0 x}{EI}(a+x)\, dx + \int_0^{2.5a} \frac{F_0 a}{EI}(2a)\, dx$$

$$= \frac{5F_0a^3}{6EI} + \frac{5F_0a^3}{EI} = \frac{35F_0a^3}{6EI} = 0.0583 \text{ (m)}.$$

Similarly, we use a dummy horizontal load of $\delta P = 1$ kN at point A to determine the horizontal deflection and find

$$\delta P \cdot u = \int \frac{M}{EI}\delta M\, dx,$$

$$1 \cdot u = \int_0^a \frac{0}{EI}(0)\, dx + \int_0^a \frac{F_0 x}{EI}(0)\, dx + \int_0^{2.5a} \frac{F_0 a}{EI}(-x)\, dx$$

$$= -\frac{3.125F_0a^3}{EI} = -0.0313 \text{ (m)}.$$

The negative sign for u indicates that the displacement is in the opposite direction to the assumed dummy load. Thus, point A moves downward and to the left.

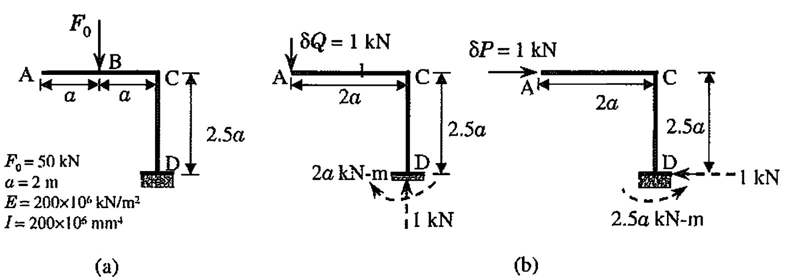

Figure 5.8 (a) Given frame structure. (b) Virtual force system.

5.4 PRINCIPLE OF COMPLEMENTARY POTENTIAL ENERGY AND CASTIGLIANO'S THEOREM II

Noting that the complementary potential energy due to virtual loads, and the complementary strain energy, are given by

$$\delta V^* = \delta W_E^*, \qquad \delta U^* = \delta W_I^*, \tag{5.66}$$

we can arrive at the *principle of maximum complementary potential energy*:

$$\delta \Pi^* \equiv \delta(U^* + V^*) = 0, \tag{5.67}$$

where Π^* denotes the total complementary potential energy.

Analogous to the unit-dummy-load method, we can derive Castigliano's Theorem II from the principle of maximum complementary energy. We have

$$\delta \Pi^* \equiv \delta U^* + \delta V^* = 0 \;\rightarrow\; \delta U^* = -\delta V^*.$$

If U^* and V^* can be expressed in terms of point loads F_i, then we have

$$\delta U^* = \frac{\partial U^*}{\partial F_i}\delta F_i, \qquad \delta V^* = \frac{\partial V^*}{\partial F_i}\delta F_i = -u_i \delta F_i,$$

and

$$\left(\frac{\partial U^*}{\partial F_i} - u_i\right)\delta F_i = 0 \qquad \text{or} \qquad \frac{\partial U^*}{\partial F_i} = u_i. \tag{5.68}$$

Equation (5.67) represents Castigliano's Theorem II. Equation (5.67) is valid for structures that are linearly elastic as well as nonlinearly elastic. When the material of the structure is linearly elastic, we have $U_0 = U_0^*$ and $U = U^*$ in value. However, U is always expressed in terms of displacements while U^* is in terms of forces, and Castigliano's Theorem I is based on U while Theorem II is based on U^*.

Castigliano's Theorem II, which is essentially the same as the unit-dummy-load method, is used to determine deflections and slopes under applied point forces and moments. However, when a structure does not have a load (or moment) at the point at which displacement (or slope) is required, the determination of the displacement (or slope) at that point requires the use of a fictitious (or dummy) load. For example, consider a beam that is subjected to uniformly distributed load q_0, and point loads $F_1, F_2, \ldots,$ etc. Suppose that we wish to determine the vertical deflection w_0 at a point at which there is no point load. We introduce a fictitious vertical load R at the point, and then write the complementary strain energy in terms of q_0, R, and $F_1, F_2, \ldots,$ etc. Then we use Castigliano's Theorem II to determine the desired displacement at the point:

$$w_0 = \left.\frac{\partial U^*}{\partial R}\right|_{R=0}.$$

Example 5.10 Consider the frame structure of Example 5.7 (see Fig. 5.6a). We wish to determine the vertical displacement v and horizontal displacement u of point O using Castigliano's Theorem II.

First, assume that there is a hypothetical load of Q applied in the horizontal direction at point O. This is necessary because, unless U^* is a function of Q, we cannot use Castigliano's Theorem II to compute u. The complementary strain energy of the structure now becomes

$$\begin{aligned} U^* &= \frac{1}{2}\sum_{i=1}^{2}\int_{V_i}\frac{1}{E^{(i)}}\left(\sigma^{(i)}\right)^2 dV \\ &= \frac{A}{2E}\left[h\left(\frac{P+Q}{A}\right)^2 + \sqrt{2}h\left(-\frac{\sqrt{2}P}{A}\right)^2\right] \\ &= \frac{h}{2EA}\left(P^2 + Q^2 + 2PQ + 2\sqrt{2}P^2\right). \end{aligned}$$

Then, by Castigliano's Theorem II, we have

$$u = \left(\frac{\partial U^*}{\partial Q}\right)_{Q=0} = \frac{Ph}{EA},$$

$$v = \left(\frac{\partial U^*}{\partial P}\right)_{Q=0} = \frac{Ph}{EA}\left(1 + 2\sqrt{2}\right).$$

Example 5.11 Consider the pin-connected structure consisting of two bars and supporting a load P (see Fig. 5.9). We assume that the bars have the same cross-sectional area A, and are made of nonlinear elastic material whose stress–strain behavior is given by

$$\sigma = \begin{cases} E\sqrt{\varepsilon}, & \varepsilon \geq 0, \\ -E\sqrt{-\varepsilon}, & \varepsilon < 0, \end{cases} \qquad \text{or} \qquad \varepsilon = \begin{cases} \dfrac{\sigma^2}{E^2}, & \sigma \geq 0, \\ -\dfrac{\sigma^2}{E^2}, & \sigma < 0. \end{cases}$$

We wish to determine the vertical deflection using Castigliano's Theorem II.

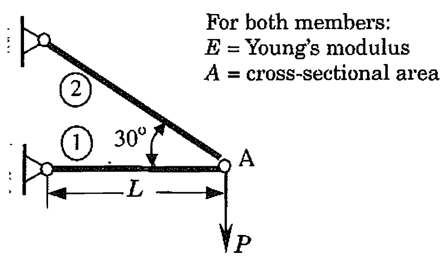

Figure 5.9 A two-member truss.

The forces in the horizontal and inclined members are given by

$$F_1 = -\sqrt{3}P, \quad F_2 = 2P \rightarrow \sigma^{(1)} = -\frac{\sqrt{3}P}{A}, \quad \sigma^{(2)} = \frac{2P}{A}.$$

The complementary strain energy of the structure is

$$\begin{aligned} U^* &= \sum_{i=1}^{2} \int_{V_i} \left(\int_0^{\sigma^{(i)}} \varepsilon^{(i)} d\sigma^{(i)} \right) dV \\ &= \left[-AL\frac{1}{3E^2}\left(\sigma^{(1)}\right)^3 + \frac{2}{\sqrt{3}}AL\frac{1}{3E^2}\left(\sigma^{(2)}\right)^3 \right] \\ &= \left(\frac{\sqrt{3}P^3L}{E^2A^2} + \frac{16\sqrt{3}P^3L}{9E^2A^2} \right) \\ &= \frac{25\sqrt{3}}{9}\frac{P^3L}{E^2A^2}. \end{aligned}$$

The displacement v under the load P is given by

$$v = \frac{\partial U^*}{\partial P} = \frac{25}{\sqrt{3}}\frac{P^2L}{E^2A^2}.$$

Example 5.12 Consider the indeterminate beam structure shown in Fig. 5.10. We wish to use Castigliano's Theorem II to determine the reaction force in the elastic support at the center of the beam. We assume linear elastic behavior of the beam as well as the support.

Suppose R is the reaction in the center support. Since the center member is replaced with its reaction, we must consider the energy of the beam only. Due to symmetry, we can consider only half the beam, which receives half the reaction $R/2$. For the linear elastic case, neglecting the energy due to transverse shear forces, the complementary strain energy is given by

$$U^* = \int_0^{L/2} \frac{M^2}{2EI} dx, \tag{a}$$

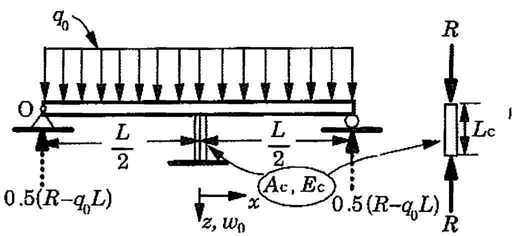

Figure 5.10 The indeterminate beam used in Example 5.12.

where

$$M(x) = \frac{1}{2}\left[-(q_0L - R)\,x + q_0x^2\right], \qquad 0 \le x \le \frac{L}{2}. \tag{b}$$

Using Castigliano's Theorem II, we obtain

$$\begin{aligned}
-w_0(0) &= \frac{\partial U^*}{\partial (R/2)} = 2\int_0^{L/2} \frac{M}{EI}\frac{\partial M}{\partial R}\,dx \\
&= \frac{1}{2EI}\int_0^{L/2} \left[-(q_0L - R)x + q_0x^2\right]x\,dx \\
&= \frac{1}{2EI}\left(-\frac{q_0L^4}{24} + \frac{RL^3}{24} + \frac{q_0L^4}{64}\right) \\
&= \frac{1}{2EI}\left(-\frac{5q_0L^4}{192} + \frac{RL^3}{24}\right).
\end{aligned} \tag{c}$$

The minus sign indicates that $w_0(0)$ is taken (by the sign convention adopted) opposite in sense to that of the reaction R. From the uniaxial deformation of the central support, it is clear that

$$w_0(0) = \frac{RL_c}{E_cA_c}. \tag{5.69}$$

Hence, we have

$$-\frac{RL_c}{E_cA_c} = \frac{1}{EI}\left(\frac{RL^3}{48} - \frac{5q_0L^4}{384}\right), \tag{d}$$

from which we obtain

$$R = \frac{5q_0L^4}{384EI}\left(\frac{L_c}{A_cE_c} + \frac{L^3}{48EI}\right)^{-1}$$

The center deflection is then given by

$$w_0(0) = \frac{5q_0L^4}{384EI}\left(1 + \frac{k_cL^3}{48EI}\right)^{-1}, \tag{5.70}$$

where $k_c = (E_cA_c)/L_c$. Note that if the central support is rigid, i.e., $E_c = \infty$, Eq. (a) gives the reaction $R = 5q_0L/8$ and deflection $w_0(0) = 0$ at the midspan of a three-point supported beam. If the central support is not present, i.e., $E_c = 0$ (hence $k_c = 0$), then $R = 0$ and the deflection becomes that of a simply supported beam under uniform load:

$$w_0(0) = \frac{5q_0L^4}{384EI}. \tag{e}$$

Finally, if the energy due to transverse shear force is included, the total complementary energy is

$$U^* = \int_0^{L/2} \frac{M^2}{2EI}\,dx + f_s \int_0^{L/2} \frac{V^2}{2GA}\,dx, \tag{f}$$

where the factor f_s is a geometric factor defined by Eq. (4.43). For rectangular cross sections it is equal to $f_s = 6/5$. The shear force for the problem at hand is

$$V(x) = \frac{1}{2}\left[-(q_0 L - R) + 2q_0 x\right], \qquad 0 \le x \le \frac{L}{2}. \tag{g}$$

Then Eq. (d) for the case with shear deformation becomes

$$-\frac{R}{k_c} = \frac{1}{EI}\left(\frac{RL^3}{48} - \frac{5q_0L^4}{384}\right) + \frac{1}{K_sGA}\left(\frac{RL}{4} - \frac{q_0L^2}{8}\right), \tag{h}$$

or

$$R = \frac{5q_0L^4}{384EI}\left(\frac{1}{k_c} + \frac{L^3}{48EI}\right)^{-1}.$$

The deflection in Eq. (5.69) becomes

$$w_0(0) = \left(\frac{5q_0L^4}{384EI} + \frac{q_0L^2}{8K_sGA}\right)\left(1 + \frac{k_cL^3}{48EI} + \frac{k_cL}{4K_sGA}\right)^{-1}, \tag{5.71}$$

where $K_s = 1/f_s$ is known as the shear correction factor. Clearly, shear deformation has the effect of increasing the deflection.

Example 5.13 Castigliano's Theorem II can be used to determine the reactions of an indeterminate beam more easily than using the statics and kinematics of the problem. As an example, consider the beam shown in Fig. 5.11. We wish to determine the reactions at the right support.

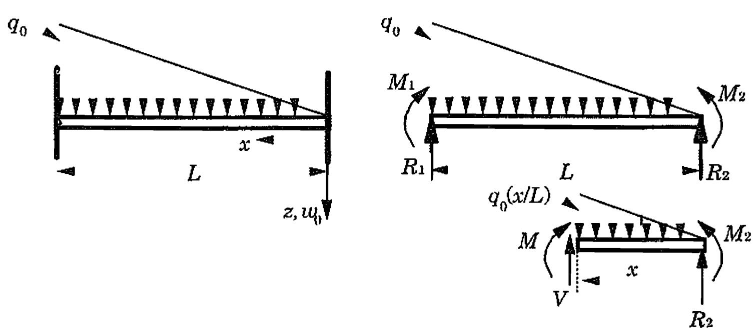

Figure 5.11 The indeterminate beam used in Example 5.13.

First we write the expression for the bending moment,

$$M(x) = M_2 + R_2 x - \left(q_0 \frac{x}{L}\right)\frac{x}{2}\frac{x}{3}, \qquad \frac{\partial M}{\partial R_2} = x, \qquad \frac{\partial M}{\partial M_2} = 1, \qquad 0 < x < L.$$

The complementary strain energy is given by

$$U^* = \int_0^L \frac{M^2}{2EI}\, dx.$$

Using Castigliano's Theorem II, we obtain

$$w_0(L) = \frac{\partial U^*}{\partial R_2} = \frac{1}{EI}\int_0^L M \frac{\partial M}{\partial R_2}\, dx,$$

$$-\left.\frac{dw_0}{dx}\right|_{x=L} = \frac{\partial U^*}{\partial M_2} = \frac{1}{EI}\int_0^L M \frac{\partial M}{\partial M_2}\, dx.$$

Since the deflection and rotation are zero at $x = L$, we obtain

$$0 = \frac{1}{EI}\int_0^L \left[M_2 + R_2 x - \left(\frac{q_0}{6L}\right)\frac{x^3}{6}\right] x\, dx = \frac{1}{EI}\left(\frac{1}{2}M_2 L^2 + \frac{1}{3}R_2 L^3 - \frac{1}{30}q_0 L^4\right),$$

$$0 = \frac{1}{EI}\int_0^L \left[M_2 + R_2 x - \left(\frac{q_0}{6L}\right)\frac{x^3}{6}\right] dx = \frac{1}{EI}\left(M_2 L + \frac{1}{2}R_2 L^2 - \frac{1}{24}q_0 L^3\right).$$

Solving the equations, we arrive at

$$R_2 = \frac{3}{20}q_0 L, \qquad M_2 = -\frac{1}{30}q_0 L^2.$$

Example 5.14 Consider the structure shown in Fig. 5.12. We wish to determine the transverse deflections of points A and B, assuming linear elastic behavior with E, G, I, and J as the material and geometric parameters.

The structure has bending and torsional deformations. First we note that the portion AB of the structure experiences only bending while the part BC experiences

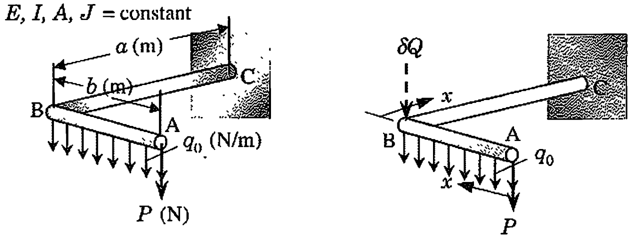

Figure 5.12 The frame structure used in Example 5.14.

bending as well as torsion. Therefore, the strain energy of the structure involves computing the bending energies of the two parts and the torsional energy of part BC. The complementary strain energy (neglecting the energy due to shear) is given by

$$U^* = \int_0^L \left(\frac{M^2}{2EI} + \frac{T^2}{2GJ} \right) dx. \tag{a}$$

For the structure at hand, we have ($U^* = U^*_{AB} + U^*_{BC}$):

$$\begin{aligned} U^* &= \frac{1}{2EI} \int_0^b M^2\,dx + \frac{1}{2EI} \int_0^a M^2\,dx + \frac{1}{2GJ} \int_0^a T^2\,dx \\ &= \frac{1}{2EI} \int_0^b \left(Px + \frac{q_0 x^2}{2} \right)^2 dx + \frac{1}{2EI} \int_0^a (P + q_0 b)^2 x^2\,dx \\ &\quad + \frac{1}{2GJ} \int_0^a \left(Pb + \frac{q_0 b^2}{2} \right)^2 dx \\ &= \frac{1}{2EI} \left(\frac{P^2 b^3}{3} + \frac{q_0^2 b^5}{20} + \frac{P q_0 b^4}{4} \right) + \frac{1}{2EI} (P + q_0 b)^2 \frac{a^3}{3} \\ &\quad + \frac{1}{2GJ} \left(Pb + \frac{q_0 b^2}{2} \right)^2 a. \end{aligned} \tag{b}$$

The deflection at point A, by Castigliano's Theorem II, is given by

$$w_0^A = \frac{\partial U^*}{\partial P} = \frac{1}{EI} \left(\frac{Pb^3}{3} + \frac{q_0 b^4}{8} \right) + \frac{1}{EI} (P + q_0 b) \frac{a^3}{3} + \frac{1}{GJ} \left(Pb + \frac{q_0 b^2}{2} \right) ab. \tag{c}$$

Clearly, the unit dummy-load-method would give exactly the same expression as above, except that the both sides of Eq. (c) would be multiplied by δP.

In order to determine the vertical deflection at point B, we introduce a dummy vertical load δQ there (which is in equilibrium by the force δQ and bending moment $a\delta Q$ at the fixed end). The portion BC of the bar is in a state of pure bending due to the dummy load δQ at B. The actual bending moment at any distance x from point B toward C is $M(x) = -(P + q_0 b)x$. The virtual moment is $\delta M^B(x) = -\delta Q \cdot x$. Note that the torque $T = (Pb + q_0 b^2/2)$ does not figure into the calculation because δT due to δQ is zero. Using the unit-dummy-load method, we write

$$\begin{aligned} w_0^B \delta Q &= \int_0^a \frac{1}{EI} M \delta M^B\,dx = \frac{(P + q_0 b)}{EI} \left(\int_0^a x^2\,dx \right) \delta Q \\ &= \frac{(P + q_0 b) a^3}{3EI} \delta Q, \end{aligned} \tag{d}$$

or

$$w_0^B = \frac{(P + q_0 b)a^3}{3EI}. \tag{e}$$

The answer can be easily verified by noting that the deflection at the free end of a cantilever beam of length L and end load F_0 is $F_0 L^3/EI$.

Example 5.15 Consider the frame structure shown in Fig. 5.13. We wish to determine the reaction forces at support A using Castigliano's Theorem II. The material obeys Hooke's law, and EA and EI are constant throughout.

The expressions for the bending moment and axial force in each member can be expressed in terms of the reactions at point A. The sense of the local coordinate along the member length is indicated in the figures.

Member AB:

$$M_1(x) = F_A x - M_A, \quad \frac{\partial M_1}{\partial F_A} = x, \quad \frac{\partial M_1}{\partial M_A} = -1, \quad \frac{\partial M_1}{\partial N_A} = 0,$$

$$N_1(x) = -N_A, \quad \frac{\partial N_1}{\partial F_A} = 0, \quad \frac{\partial N_1}{\partial M_A} = 0, \quad \frac{\partial N_1}{\partial N_A} = -1.$$

Member BC:

$$M_2(x) = F_A a - M_A + N_A x, \quad \frac{\partial M_2}{\partial F_A} = a, \quad \frac{\partial M_2}{\partial M_A} = -1, \quad \frac{\partial M_2}{\partial N_A} = x,$$

$$N_2(x) = F_A - P_0, \quad \frac{\partial N_2}{\partial F_A} = 1, \quad \frac{\partial N_2}{\partial M_A} = 0, \quad \frac{\partial N_2}{\partial N_A} = 0.$$

Member CD:

$$M_3(x) = F_A(a - x) + P_0 x - M_A - M_0 + N_A a, \quad \frac{\partial M_3}{\partial F_A} = a - x,$$

$$\frac{\partial M_3}{\partial M_A} = -1, \quad \frac{\partial M_3}{\partial N_A} = a, \quad N_3(x) = N_A, \quad \frac{\partial N_3}{\partial F_A} = 0,$$

$$\frac{\partial N_3}{\partial M_A} = 0, \quad \frac{\partial N_3}{\partial N_A} = 1.$$

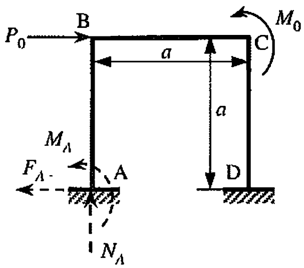

Figure 5.13 The frame structure used in Example 5.15.

The complementary strain energy of the structure (neglecting the energy due to shear) is given by

$$U^* = \sum_{i=1}^{3} \int_0^a \left(\frac{M_i^2}{2EI} + \frac{N_i^2}{2EA} \right) dx = U^*(F_A, N_A, M_A, P_0, M_0).$$

Since the support is rigid (i.e., all generalized displacements are zero), we have

$$\frac{\partial U^*}{\partial N_A} = 0, \qquad \frac{\partial U^*}{\partial F_A} = 0, \qquad \frac{\partial U^*}{\partial M_A} = 0.$$

Carrying out the integration, we obtain

$$0 = \frac{1}{EI}\left(F_A a^3 - \frac{3M_A a^2}{2} + \frac{4N_A a^3}{3} + \frac{P_0 a^3}{2} - M_0 a^2 \right) + \frac{2N_A a}{EA},$$

$$0 = \frac{1}{EI}\left(\frac{5F_A a^3}{3} - 2M_A a^2 + N_A a^3 + \frac{P_0 a^3}{6} - \frac{M_0 a^2}{2} \right) + \frac{1}{EA}\left(F_A a - P_0 a\right),$$

$$0 = \frac{1}{EI}\left(-2F_A a^2 + 3M_A a - \frac{3N_A a^2}{2} - \frac{P_0 a^2}{2} + M_0 a \right),$$

or, in matrix form, we have

$$\begin{bmatrix} \frac{4a^3}{3EI} + \frac{2a}{EA} & \frac{a^3}{EI} & -\frac{3a^2}{2EI} \\ \frac{a^3}{EI} & \frac{5a^3}{3EI} + \frac{a}{EA} & -\frac{2a^2}{EI} \\ -\frac{3a^2}{2EI} & -\frac{2a^2}{EI} & \frac{3a}{EI} \end{bmatrix} \begin{Bmatrix} N_A \\ F_A \\ M_A \end{Bmatrix} = \begin{Bmatrix} -\frac{P_0 a^3}{2EI} + \frac{M_0 a^2}{EI} \\ -\frac{P_0 a^3}{6EI} + \frac{P_0 a}{EA} + \frac{M_0 a^2}{2EI} \\ \frac{P_0 a^2}{2EI} - \frac{M_0 a}{EI} \end{Bmatrix}.$$

The coefficient matrix is known as the *flexibility matrix.*

5.5 BETTI'S AND MAXWELL'S RECIPROCITY THEOREMS

The principle of superposition is said to hold for a solid body if the displacements obtained under a given set of forces is equal to the sum of the individual displacements that would be obtained by applying the single forces separately. Consider, for example, the indeterminate beam shown in Fig. 5.14. The beam problem is equivalent to the two beam problems shown there. At point A the beam experiences the deflections w_0^q and w_0^s, due respectively to the distributed load q_0 and spring force F_s. Within the restrictions of the linear Euler–Bernoulli beam theory, the deflections are linear functions of the loads. Therefore, the principle of superposition is valid:

$$w_0^A = w_0^q + w_0^s = \frac{q_0 L^4}{8EI} - \frac{F_s L^3}{3EI}. \tag{5.72}$$

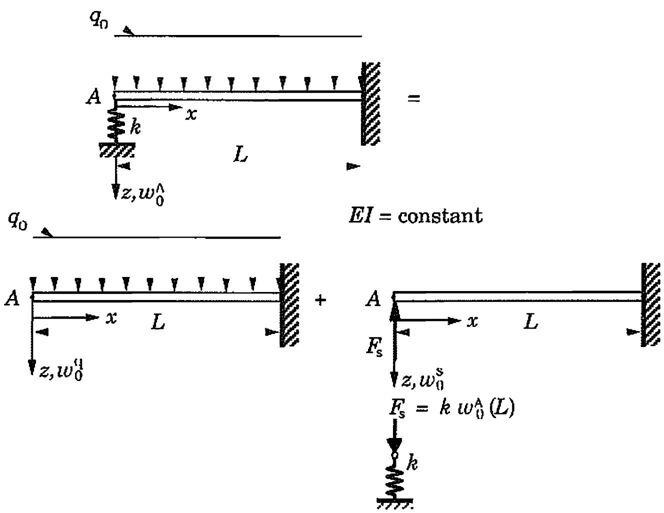

Figure 5.14 Representation of an indeterminate beam as a combination of two determinate beams.

Because the spring force F_s is equal to kw_0^A, we can calculate w_0^A from Eq. (5.72):

$$w_0^A = \frac{q_0 L^4}{8EI(1 + (kL^3/3EI))}. \tag{5.73}$$

The procedure can be used to represent a complicated linear problem as a linear combination of simple linear problems.

The principle of superposition is not valid for strain and potential energies because they are quadratic functions of displacements and/or forces. In other words, when a linear elastic body is subjected to more than one external force, the total work due to external forces is not equal to the sum of the works that are obtained by applying the single forces separately. However, there exist theorems that relate the work done by two different forces applied in different orders. The present section is devoted to the discussion of two reciprocity theorems for work done on structural systems.

Consider a linear elastic solid which is in equilibrium under the action of two external forces, $\mathbf{F}_1$ and $\mathbf{F}_2$ (see Fig. 5.15). Since the order of application of the forces is arbitrary, we suppose that force $\mathbf{F}_1$ is applied first. Let W_1 be the work produced by $\mathbf{F}_1$. Then we apply force $\mathbf{F}_2$, which produces work W_2. This work is the same as that produced by force $\mathbf{F}_2$, if it alone were acting on the body. When force $\mathbf{F}_2$ is applied, force $\mathbf{F}_1$ (which is already acting on the body) does additional work because its point of application is displaced due to the deformation caused by force $\mathbf{F}_2$. Let us denote this work by W_{12}. Thus the total work done by the application of forces $\mathbf{F}_1$ and $\mathbf{F}_2$, $\mathbf{F}_1$ first and $\mathbf{F}_2$ next, is

$$W = W_1 + W_2 + W_{12}. \tag{5.74a}$$

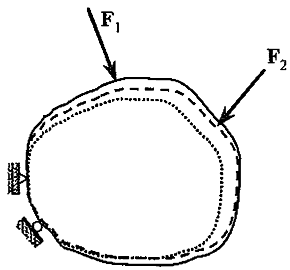

Figure 5.15 Configurations of an elastic body due to the application of loads $\mathbf{F}_1$ and $\mathbf{F}_2$. — Undeformed configuration. - - - Deformed configuration after the application of load $\mathbf{F}_1$. ..... Deformed configuration after the application of load $\mathbf{F}_2$.

Work W_{12}, which can be positive or negative, is zero if and only if the displacement of the point of application of force $\mathbf{F}_1$ produced by force $\mathbf{F}_2$ is zero or perpendicular to the direction of $\mathbf{F}_1$.

Now suppose that we change the order of application. Then the total work done is equal to

$$\overline{W} = W_1 + W_2 + W_{21}, \tag{5.74b}$$

where W_{21} is the work done by force F_2 due to the application of force F_1. The work done in both cases should be the same because at the end the elastic body is loaded by the same pair of external forces. Thus we have

$$W = \overline{W} \qquad \text{or} \qquad W_{12} = W_{21}. \tag{5.75}$$

Equation (5.75) is a mathematical statement of Betti's (1823–1892) reciprocity theorem. Applied to a three-dimensional elastic body, Eq. (5.75) takes the form

$$\int_V f_i \bar{u}_i \, dV + \int_{S_2} t_i \bar{u}_i \, dS = \int_V \bar{f}_i u_i \, dV + \int_{S_2} \bar{t}_i u_i \, dS, \tag{5.76}$$

where $\bar{u}_i$ are the displacements produced by body forces $\bar{f}_i$ and surface forces $\bar{t}_i$, and u_i are the displacements produced by body forces f_i and surface forces t_i. The left-hand side of Eq. (5.76), for example, denotes the work done by forces f_i and t_i in moving through the displacements $\bar{u}_i$ produced by forces $\bar{f}_i$ and $\bar{t}_i$. We now state Betti's reciprocity theorem: *if a linear elastic body is subjected to two different sets of forces, the work done by the first system of forces in moving through the displacements produced by the second system of forces is equal to the work done by the second system of forces in moving through the displacements produced by the first system of forces.*

Example 5.16 Consider a cantilever beam of length L subjected to a concentrated load F_0 at the free end and to a uniformly distributed load of intensity q_0 (see Fig. 5.16).

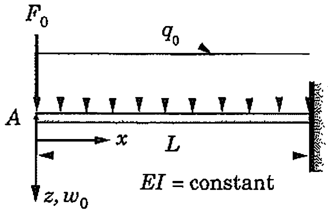

Figure 5.16 Configuration of a cantilever beam.

The deflection equation due to the concentrated load alone is

$$w_0^F(x) = \frac{F_0}{6EI}\left(x^3 - 3L^2x + 2L^3\right),$$

and the deflection equation due to the distributed load is

$$w_0^q(x) = \frac{q_0}{24EI}\left(x^4 - 4L^3x + 3L^4\right).$$

The work done by the load F_0 in moving through the displacement due to the application of the uniformly distributed load q_0 is

$$W_{12} = F_0 w_0^q(0) = \frac{F_0 q_0 L^4}{8EI}.$$

The work done by the uniformly distributed load q_0 in moving through the displacement field due to the application of point load F_0 is

$$W_{21} = \int_0^L \frac{F_0}{6EI}(x^3 - 3L^2x + 2L^3)q_0\, dx = \frac{F_0 q_0 L^4}{8EI},$$

which is in agreement with W_{12}.

An important special case of Betti's reciprocity theorem is given by Maxwell's (1831–1879) reciprocity theorem. Maxwell's theorem was given in 1864, whereas Betti's theorem was given in 1872. Therefore, it may be considered that Betti generalized the work of Maxwell. We derive Maxwell's reciprocity theorem from Betti's reciprocity theorem.

Consider a linear elastic solid subjected to force $\mathbf{F}^1$ of unit magnitude acting at point A, and force $\mathbf{F}^2$ of unit magnitude acting at a different point B of the body. Let $\mathbf{u}_{AB}$ be the displacement of point A in the direction of force $\mathbf{F}^1$ produced by unit force $\mathbf{F}^2$, and $\mathbf{u}_{BA}$ be the displacement of point B in the direction of force $\mathbf{F}^2$ produced by unit force $\mathbf{F}^1$ (see Fig. 5.17). From Betti's theorem it follows that

$$\mathbf{F}^1 \cdot \mathbf{u}_{AB} = \mathbf{F}^2 \cdot \mathbf{u}_{BA} \qquad \text{or} \qquad u_{AB} = u_{BA}. \tag{5.77}$$

Equation (5.77) is a statement of Maxwell's theorem. If $\hat{\mathbf{e}}_1$ and $\hat{\mathbf{e}}_2$ denote the unit vectors along forces $\mathbf{F}^1$ and $\mathbf{F}^2$, respectively, Maxwell's theorem states that

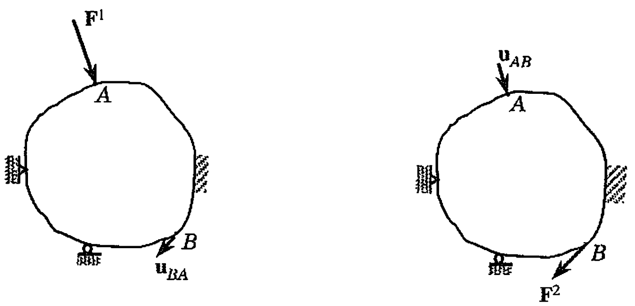

Figure 5.17 Configurations of the body discussed in the Maxwell theorem.

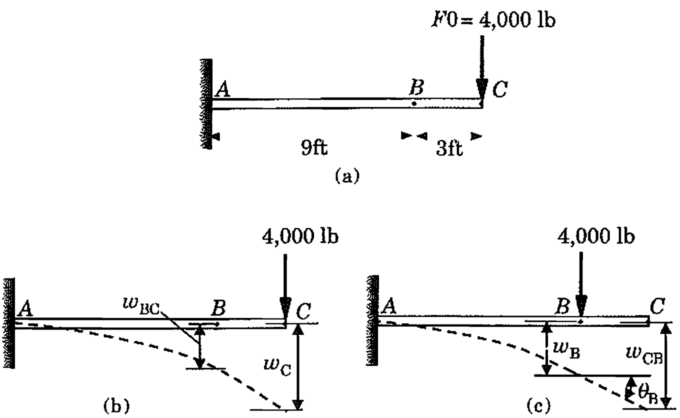

Figure 5.18 The cantilever beam of Example 5.17.

the displacement of point A in the $\hat{e}_1$-direction produced by a unit force acting at point B in the $\hat{e}_2$-direction is equal to the displacement of point B in the $\hat{e}_2$ direction produced by a unit force acting at point A in the $\hat{e}_1$-direction.

We close this section with the following two examples that illustrate the usefulness of Maxwell's theorem.

Example 5.17 Consider a cantilever beam ($E = 24 \times 10^6$ psi, $I = 120$ in.^{4}) of length 12 ft subjected to a point load of 4000 lb at the free end. We wish to find the deflection at a point 3 ft from the free end (see Fig. 5.18).

By Maxwell's theorem, the displacement w_{BC} at point B ($x = 3$ ft) produced by the 4000 lb load at point C ($x = 0$) is equal to the deflection w_{CB} at point C produced by applying the 4000 lb load at point B. Let w_B and θ_B denote the deflection and slope, respectively, at point B owing to load $F = 4000$ lb applied at point B. Then the deflection at point B ($x = 3$ ft) caused by load $F_0 = 4000$ lb at point C ($x = 0$)

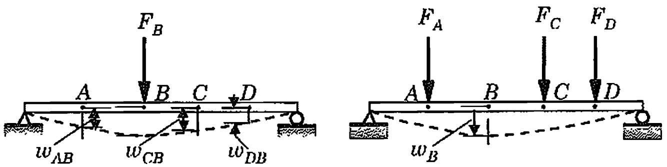

Figure 5.19 The simply supported beam of Example 5.18.

is ($w_B = FL^3/3EI$ and $\theta_B = FL^2/2EI$):

$$\begin{aligned} w_{BC} &= w_{CB} = w_B + (3 \times 12)\theta_B \\ &= \frac{4000(9 \times 12)^3}{3EI} + \frac{(3 \times 12)4000(9 \times 12)^2}{2EI} \\ &= \frac{243 \times 6000 \times (12)^3}{24 \times 10^6 \times 120} = 0.8748 \text{ in.} \end{aligned}$$

Example 5.18 Consider a simply supported beam subjected to a point load F_B at point B. The load produces deflections w_{AB}, w_{CB}, and w_{DB} at points A, C, and D, respectively (see Fig. 5.19a). We wish to find the deflection w_B at point B produced by loads F_A, F_C, and F_D (see Fig. 5.19b).

A unit load acting at point B produces displacements w_{AB}/F_B, w_{CB}/F_B, and w_{DB}/F_B at points A, C, and D, respectively. But, by Maxwell's theorem, w_{AB}/F_B, w_{CB}/F_B, and w_{DB}/F_B are the displacements of point B caused by a unit load acting at points A, C, and D, respectively. Therefore, the displacement at point B owing to forces F_A, F_C, and F_D is given by

$$\begin{aligned} w_B &= F_A\left(\frac{w_{AB}}{F_B}\right) + F_C\left(\frac{w_{CB}}{F_B}\right) + F_D\left(\frac{w_{DB}}{F_B}\right) \\ &= \frac{F_A w_{AB} + F_C w_{CB} + F_D w_{DB}}{F_B}. \end{aligned}$$

EXERCISES

5.1 Use the principle of virtual displacements to determine the governing equations and natural boundary conditions associated with the problem in Exercise 4.4.

5.2 Use the principle of virtual displacements to determine the governing equations and natural boundary conditions associated with the problem in Exercise 4.5.

5.3 Use the principle of virtual displacements to derive the equations governing the Euler–Bernoulli beam theory with geometric nonlinearity. Use a distributed transverse load of $q = q(x)$ and assume that the displacement field is given by

$$u(x, y, z) = u_0(x) - z\frac{dw_0}{dx} - y\frac{dw_0}{dx}, \qquad v = 0, \qquad w(x, y, z) = w_0(x)$$

and the only nonzero strain is (nonlinear)

$$\varepsilon_{xx}(x, y, z) = \frac{du_0}{dx} + \frac{1}{2}\left(\frac{dw_0}{dx}\right)^2 - z\frac{d^2w_0}{dx^2} - y\frac{d^2w_0}{dx^2}.$$

Here u_0 and w_0 are the axial and transverse displacements of a point $(x, y, 0)$ in the beam. Your answer should be expressed in terms of the area-integrated quantities [e.g., stress resultants of Eq. (5.9)].

5.4 *The Timoshenko beam theory.* The Euler–Bernoulli beam theory is based on the assumption that a straight line transverse to the axis of the beam before deformation remains (i) straight, (ii) inextensible, and (iii) normal to the midplane after deformation. In the Timoshenko beam theory, the first two assumptions are kept but the normality condition is relaxed by assuming that the rotation is independent of the slope (dw_0/dx) of the beam. Using these assumptions, the displacement field of the beam can be expressed as

$$\begin{aligned} u(x, z) &= u_0(x) + z\phi(x) \\ v &= 0 \\ w &= w_0(x), \end{aligned} \tag{a}$$

where (u, v, w) are the displacements of a point along the (x, y, z) coordinates, (u_0, w_0) are the displacements of a point on the midplane of an undeformed beam, and ϕ is the rotation (about the y-axis) of a transverse normal line. Derive the equations of equilibrium of the Timoshenko beam theory using the principle of virtual displacements for the case of infinitesimal strains. Also derive the natural boundary conditions of the theory.

5.5 Use the unit-dummy-displacement method to determine the displacement in the direction of the applied load P in the structure in Exercise 4.10.

5.6 Use Castigliano's Theorem I to determine the axial forces in the wires of the structure shown in Fig. E5.6. Assume small deformation.

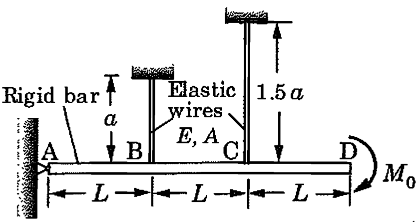

Figure E5.6

5.7 Use Castigliano's Theorem I to determine the unknown generalized displacements and forces of the beam shown in Fig. E5.7. Use the results of Example 5.7.

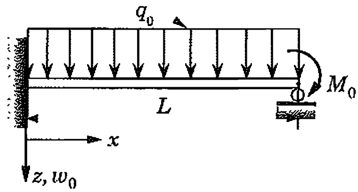

Figure E5.7

5.8 Use the unit-dummy-load method to determine the horizontal and vertical displacements at point D of the structure in Exercise 4.9.

5.9 Use the unit-dummy-load method to determine the center deflection of a simply supported beam under uniformly distributed transverse load, q_0.

5.10 Use the unit-dummy-load (moment) method to determine the rotation at $x = L$ in Exercise 5.7.

5.11 Use the unit-dummy-load method to determine a relationship between the tip deflection w_e and deflection w_c at the center of the beam shown in Fig. E5.11.

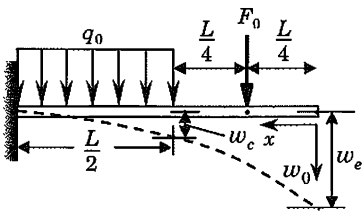

Figure E5.11

5.12 Use the unit-dummy-load method to determine the deflection at point B of the beam shown in Fig. E5.12.

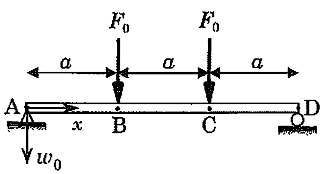

Figure E5.12

5.13 Use the unit-dummy-load method to determine the force in the cable of the cable-supported cantilever beam shown in Fig. E5.13.

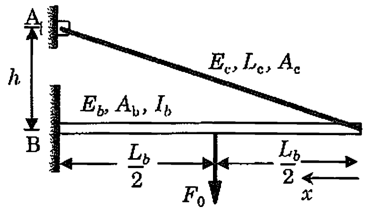

Figure E5.13

5.14 Use the unit-dummy-load method to determine the rotation of joint B of the frame shown in Fig. E5.14. Consider energy due to bending only.

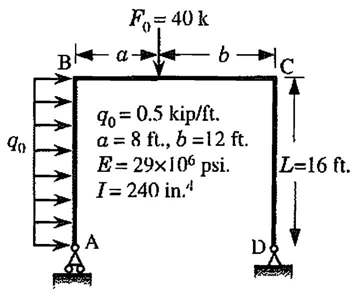

Figure E5.14

5.15 Use the unit-dummy-load method to determine the vertical deflection of point A of the structure in Fig. E5.15. Consider energies due to both bending and torsion.

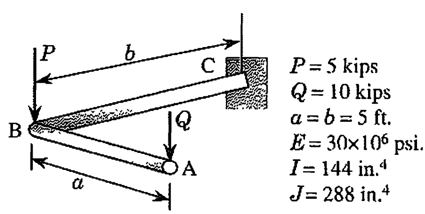

Figure E5.15

5.16 Consider the frame structure shown in Fig. 4.9 of Example 4.4. Each member of the structure has the same geometric and material properties. Assume linear elastic behavior. Determine the horizontal and vertical displacements at the point of load application. Use the unit-dummy-load method.

5.17 Consider the pin-connected structure shown in Fig. E4.9. Suppose that the material of all members obeys the following stress–strain relation:

$$\sigma = \begin{cases} E\sqrt{\varepsilon}, & \varepsilon \geq 0, \\ -E\sqrt{-\varepsilon}, & \varepsilon \leq 0, \end{cases}$$

where E is a constant. All members have the same cross-sectional area, A. Determine the horizontal and vertical displacements at the point of load application using the unit-dummy-load method.

5.18 Solve the problem in Exercise 5.6 using Castigliano's Theorem II.

5.19 Use Castigliano's Theorem II to determine the compressive force and displacements in the linear elastic spring (spring constant, k) supporting the free end of a cantilever beam under a triangular distributed load (see Fig. E5.19).

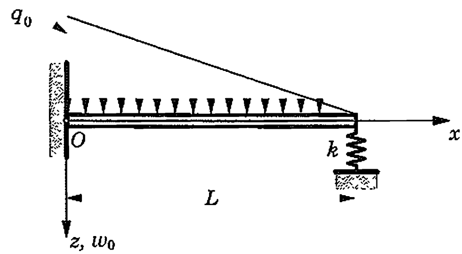

Figure E5.19

5.20 The thin curved beam shown in Fig. E5.20 has a radius of curvature R, modulus E, and moment of inertia I about the axis of bending. Use Castigliano's Theorem II to determine the vertical and horizontal deflections at the free end. Include only energy due to bending.

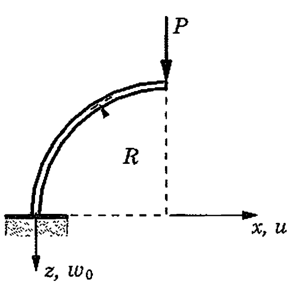

Figure E5.20

5.21 The thin curved beam shown in Fig. E5.21 has a radius of curvature R, modulus E, and moment of inertia I about the axis of bending. Use Castigliano's Theorem II to determine the vertical and horizontal deflections at the free end. Include only energy due to bending.

5.22 The thin curved beam shown in Fig. E5.22 has a radius of curvature R, modulus E, and moment of inertia I about the axis of bending. The radial distributed load (i.e., pressure) is constant, p per unit arc. Use Castigliano's Theorem II to determine the horizontal and vertical deflections. Include only energy due to bending.

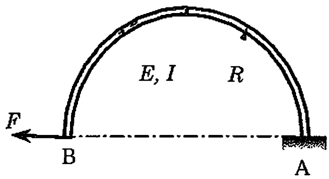

Figure E5.21

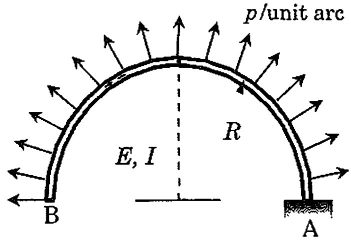

Figure E5.22

5.23 Determine the horizontal displacement of the curved beam problem of Exercise 5.21 by accounting for the energy due to extension, bending, as well as transverse shear.

5.24 Use Castigliano's Theorem II to determine the rotation at the free end of a spring-supported cantilever beam under uniformly distributed load (see Fig. 5.14).

5.25 Determine the transverse deflection of the free end of the cantilever beam shown in Fig. 5.16 using Castigliano's Theorem II. Solve the problem (a) neglecting the energy due to shear force $V(x)$, and (b) including the shear energy [see Eq. (4.42); write the answer in terms of f_s].

5.26 Determine the deflection at point C of the beam shown in Fig. E5.26 using Castigliano's Theorem II.

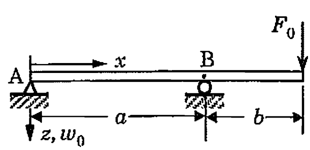

Figure E5.26

5.27 Determine the slope at point A of the beam shown in Fig. E5.26 using Castigliano's Theorem II.

5.28 Use Castigliano's Theorem II to determine the vertical and horizontal deflections at point C of the statically determinate structure shown in Fig. E5.28.

Assume that all connections are pin connections and that all members have the same area of cross section (A) and modulus (E).

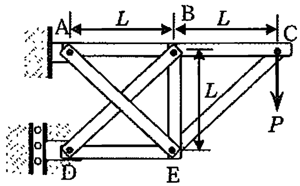

Figure E5.28

5.29 Suppose that the structure of Exercise 5.28 is made indeterminate by replacing the roller at point D with a connection identical to that at point A (see Fig. E5.29). Determine the reactions at points C and D.

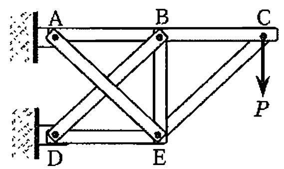

Figure E5.29

5.30 Prove Betti's theorem for a three-dimensional elastic solid.

5.31 Consider a simply supported beam of length L subjected to a concentrated load F_0 at the midspan and a bending moment M_0 at the left end. Verify that Betti's theorem holds.

5.32 Verify Maxwell's reciprocal relationship for the cantilever problem with two different loadings shown in Fig. E5.32. You must determine the required deflections and slopes to verify Maxwell's reciprocal relationship.

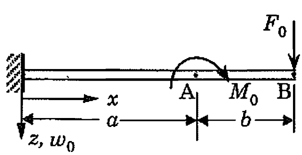

Figure E5.32

5.33 Determine the deflection at the midspan of a cantilever beam subjected to uniformly distributed load q_0 throughout the span and a point load F_0 at the free end (see Fig. E5.33). Use Maxwell's theorem and superposition.

5.34 A load $P = 4000$ lb. acting at a point A of a beam produces 0.25 in. at point B and 0.75 in. at point C of the beam. Find the deflection of point A produced by loads of 4500 lb and 2000 lb acting at points B and C, respectively.

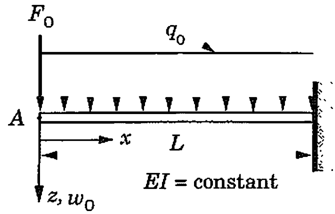

Figure E5.33

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6

DYNAMICAL SYSTEMS: HAMILTON'S PRINCIPLE

6.1 INTRODUCTION

The principles of virtual work and their special cases discussed in Chapter 5 were limited to static equilibrium of solids. Hamilton's principle is a generalization of the principle of virtual displacements to dynamics of systems of particles, rigid bodies, or deformable solids. The principle assumes that the system under consideration is characterized by two energy functions, a *kinetic energy* K and a *potential energy* Π. For *discrete* systems (i.e., systems with a finite number of degrees of freedom), these energies can be described in terms of a finite number of generalized coordinates and their derivatives with respect to time t. For *continuous* systems (i.e., systems that cannot be described by a finite number of generalized coordinates), the energies can be expressed in terms of the dependent variables (which are functions of position) of the problem. Hamilton's principle reduces to the principle of virtual displacements for systems that are in static equilibrium.

The present chapter is devoted to the application of Hamilton's principle to dynamics of discrete systems as well as continuous systems. We begin the discussion with Hamilton's principle for particles.

6.2 HAMILTON'S PRINCIPLE FOR PARTICLES AND RIGID BODIES

Consider a single particle of mass m moving under the influence of a force $\mathbf{F} = \mathbf{F}(\mathbf{r})$. The path $\mathbf{r}(t)$ followed by the particle is related to the force $\mathbf{F}$ and mass m by Newton's second law of motion:

$$\mathbf{F}(\mathbf{r}) = m\frac{d^2\mathbf{r}}{dt^2}. \tag{6.1}$$

A path that differs from the actual path is expressed as $\mathbf{r}+\delta\mathbf{r}$, where $\delta\mathbf{r}$ is the variation of the path for any *fixed* time t. We suppose that the actual path $\mathbf{r}$ and the *varied* path differ except at two distinct times t_1 and t_2, i.e., $\delta\mathbf{r}(t_1) = \delta\mathbf{r}(t_2) = \mathbf{0}$. Taking the scalar product of Eq. (6.1) with the variation $\delta\mathbf{r}$, and integrating with respect to time between t_1 and t_2, we obtain

$$\int_{t_1}^{t_2} \left[m\frac{d^2\mathbf{r}}{dt^2} - \mathbf{F}(\mathbf{r}) \right] \cdot \delta\mathbf{r}\, dt = 0. \tag{6.2}$$

Integration by parts of the first term in Eq. (6.2) yields

$$-\int_{t_1}^{t_2} \left(m\frac{d\mathbf{r}}{dt} \cdot \frac{d\delta\mathbf{r}}{dt} + \mathbf{F}(\mathbf{r}) \cdot \delta\mathbf{r} \right) dt + \left(m\frac{d\mathbf{r}}{dt} \cdot \delta\mathbf{r} \right) \Big|_{t_1}^{t_2} = 0. \tag{6.3}$$

The last term in Eq. (6.3) vanishes because $\delta\mathbf{r}(t_1) = \delta\mathbf{r}(t_2) = \mathbf{0}$. Also, note that

$$m\frac{d\mathbf{r}}{dt} \cdot \frac{d\delta\mathbf{r}}{dt} = \delta \left[\frac{m}{2}\frac{d\mathbf{r}}{dt} \cdot \frac{d\mathbf{r}}{dt} \right] \equiv \delta K, \tag{6.4}$$

where K is the kinetic energy of the particle:

$$K = \frac{1}{2}m\frac{d\mathbf{r}}{dt} \cdot \frac{d\mathbf{r}}{dt} = \frac{1}{2}m\mathbf{v} \cdot \mathbf{v}. \tag{6.5}$$

Equation (6.3) now takes the form

$$\int_{t_1}^{t_2} (\delta K + \mathbf{F} \cdot \delta\mathbf{r})\, dt = 0, \tag{6.6}$$

which is known as the *general form of Hamilton's principle* for a single particle.

Suppose that the force $\mathbf{F}$ is conservative (that is, the sum of the potential and kinetic energies is conserved) such that it can be replaced by the gradient of a potential:

$$\mathbf{F} = -\text{grad } V, \tag{6.7}$$

where $V = V(\mathbf{r})$ is the potential energy due to the loads on the particle. Then Eq. (6.6) can be expressed in the form

$$\delta \int_{t_1}^{t_2} (K - V)\, dt = 0, \tag{6.8}$$

because

$$\text{grad } V \cdot \delta\mathbf{r} = \frac{\partial V}{\partial x_1}\delta x_1 + \frac{\partial V}{\partial x_2}\delta x_2 + \frac{\partial V}{\partial x_3}\delta x_3 = \delta V(x_1, x_2, x_3).$$

The difference between the kinetic and potential energies is called the *Lagrangian* function:

$$L \equiv K - V. \tag{6.9}$$

Note that for particles and rigid bodies considered here, the internal energy W_I is assumed to be zero (and $W_E = V$).

Equation (6.8) represents Hamilton's principle for the conservative motion of a particle. It states that *the motion of a particle acted on by conservative forces between two arbitrary instants of time t_1 and t_2 is such that the line integral over the Lagrangian function is an extremum for the path motion.* Stated in other words, of all possible paths that the particle could travel from its position at time t_1 to its position at time t_2, its actual path will be one for which the integral

$$I \equiv \int_{t_1}^{t_2} L\,dt \tag{6.10}$$

is an extremum (i.e., a minimum, maximum, or an inflection).

If the path $\mathbf{r}$ can be expressed in terms of the generalized coordinates $q_i (i = 1, 2, 3)$, the Lagrangian function can be written in terms of q_i and their time derivatives:

$$L = L(q_1, q_2, q_3, \dot{q}_1, \dot{q}_2, \dot{q}_3). \tag{6.11}$$

Then from Section 4.4 the condition for the extremum of I results in the equation (note that $\delta q_i = 0$ at t_1 and t_2):

$$\begin{aligned}\delta I &= \delta \int_{t_1}^{t_2} L(q_1, q_2, q_3, \dot{q}_1, \dot{q}_2, \dot{q}_3)\,dt = 0\\ &= \int_{t_1}^{t_2} \sum_{i=1}^{3} \left[\frac{\partial L}{\partial q_i} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right)\right] \delta q_i\,dt.\end{aligned} \tag{6.12}$$

When all q_i are linearly independent (i.e., no constraints among q_i), the variations δq_i are independent for all t, except $\delta q_i = 0$ at t_1 and t_2. Therefore, the coefficients of δq_1, δq_2, and δq_3 vanish separately:

$$\frac{\partial L}{\partial q_i} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) = 0, \qquad i = 1, 2, 3. \tag{6.13}$$

These equations are called the *Lagrange equations of motion.* In Section 4.4.6 these equations were also called the Euler equations. In the discussions to follow we shall refer to these equations as the Euler–Lagrange equations.

When the forces are not conservative, we must deal with the general form of Hamilton's principle in Eq. (6.6). Recognizing that $\mathbf{F} \cdot \delta\mathbf{r}$ represents the virtual work $-\delta W_E$ due to forces $\mathbf{F}$, we can write Eq. (6.6) in the form

$$\int_{t_1}^{t_2} \delta K\,dt - \int_{t_1}^{t_2} \delta W_E\,dt = 0. \tag{6.14}$$

In this case, there exists no functional I that must be an extremum. If the virtual work can be expressed in terms of the generalized coordinates q_i by

$$\delta W_E = -(Q_1\delta q_1 + Q_2\delta q_2 + Q_3\delta q_3), \tag{6.15}$$

where Q_i are the *generalized forces*, then we can write Eq. (6.14) as

$$\int_{t_1}^{t_2} \sum_{i=1}^{3} \left[\frac{\partial K}{\partial q_i} - \frac{d}{dt}\left(\frac{\partial K}{\partial \dot{q}_i}\right) + Q_i\right] \delta q_i \, dt = 0, \tag{6.16}$$

and the Euler–Lagrange equations for the nonconservative forces are given by

$$\frac{\partial K}{\partial q_i} - \frac{d}{dt}\left(\frac{\partial K}{\partial \dot{q}_i}\right) + Q_i = 0, \qquad i = 1, 2, 3. \tag{6.17}$$

Example 6.1 Consider the spring–mass system shown in Fig. 6.1. First, consider the case in which the mass is sliding on a smooth, frictionless surface (see Fig. 6.1a). The kinetic and potential energies (measured from the unstretched position of the spring) are

$$K = \frac{m}{2}(\dot{x})^2, \qquad V = \frac{k}{2}x^2 - fx.$$

Hence the equation of motion is ($L = K - V$):

$$\frac{\partial L}{\partial x} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = 0,$$

which yields

$$m\ddot{x} + kx = f(t).$$

Next, suppose that the surface on which the mass m slides is dry and offers resistance to motion (see Fig. 6.1b). For motion to begin, there must be a force acting on the body that overcomes the resistance to motion. The friction force is parallel to the surface, in the direction opposite to the motion, and proportional to the force normal to the surface. The force normal to the surface in this case is the weight $W = mg$, and the constant of proportionality is known as the static coefficient of friction μ_s.

Figure 6.1 A spring–mass system, with the mass sliding on a surface.

The value of μ_s varies from 0 (when the surface is frictionless) to 1, depending on the surface finish and material. Once motion ensues, the value of the friction coefficient drops to $\mu_k < \mu_s$, known as the kinematic friction coefficient. The friction force remains constant in magnitude during the motion as long as the inertia force and the force in the spring are sufficiently large to overcome the static friction.

Thus, in this case, the system involves a force that is nonconservative. Denoting the magnitude of the friction force by $F_d = mg\mu_k$, we can write the equation of motion as

$$-\frac{d}{dt}\left(\frac{\partial K}{\partial \dot{x}}\right) + Q = 0, \qquad Q = -kx - \text{sign}(\dot{x})F_d + f(t),$$

where sign$(\dot{x})$ denotes the sign of $\dot{x}$ (i.e., sign$(\dot{x})$ is $+1$ when $\dot{x}$ is positive and -1 if $\dot{x}$ is negative). The above equation of motion can be separated into two equations as

$$m\ddot{x} + kx = -F_d + f(t) \quad \text{when } \dot{x} > 0, \qquad m\ddot{x} + kx = F_d + f(t) \quad \text{when } \dot{x} < 0.$$

Example 6.2 Consider the motion of a pendulum in a plane. Ideally speaking, the pendulum is imagined to be a mass m (called the bob) attached at the end of a rigid massless rod of length l that pivots about a fixed point O, as shown in Fig. 6.2. Hence, $q_1 = l$ is fixed, $\dot{q}_1 = 0$, and $q_2 = \theta$ is the only independent generalized coordinate. The coordinate θ is measured from the vertical position. The force **F** acting on the bob is the component of the gravitational force:

$$\mathbf{F} = mg(-\sin\theta\hat{\mathbf{e}}_\theta + \cos\theta\hat{\mathbf{e}}_r).$$

The component along $\hat{\mathbf{e}}_r$ does no work (because l is a constant). The first component is derivable from the potential ($\nabla V = -\mathbf{F}$):

$$V = -[-mgl(1-\cos\theta)] = mgl(1-\cos\theta). \tag{6.18a}$$

The expression signifies the potential energy of the pendulum bob at any instant of time with respect to the static equilibrium position $\theta = 0$. The gradient operator in

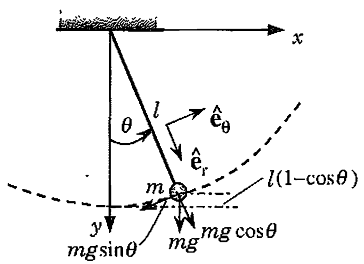

Figure 6.2 Configuration of the pendulum.

the present case is given by ($r = l$):

$$\nabla = \hat{\mathbf{e}}_r \frac{\partial}{\partial r} + \frac{\hat{\mathbf{e}}_\theta}{r} \frac{\partial}{\partial \theta}.$$

Thus the kinetic and potential energies are given by

$$K = \frac{m}{2}(l\dot{\theta})^2, \qquad V = mgl(1 - \cos\theta), \tag{6.18b}$$

$$\delta K = ml^2\dot{\theta}\delta\dot{\theta}, \qquad \delta V = mgl \sin\theta\delta\theta. \tag{6.18c}$$

Therefore, the Lagrangian function L is a function of θ and $\dot{\theta}$. The Euler–Lagrange equation is given by

$$\frac{\partial L}{\partial \theta} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = 0,$$

which yields

$$-mgl \sin\theta - \frac{d}{dt}(ml^2\dot{\theta}) = 0 \qquad \text{or} \qquad \ddot{\theta} + \frac{g}{l}\sin\theta = 0. \tag{6.19}$$

Equation (6.19) represents a second-order nonlinear differential equation governing θ. For small angular motions, Eq. (6.19) can be linearized by replacing $\sin\theta \approx \theta$:

$$\ddot{\theta} + \frac{g}{l}\theta = 0. \tag{6.20}$$

Now suppose that the mass experiences a resistance force $\mathbf{F}^*$ proportional to its speed (i.e., the pendulum is suspended in a medium that offers resistance to motion). According to Stoke's law,

$$\mathbf{F}^* = -6\pi\mu a l\dot{\theta}\hat{\mathbf{e}}_\theta, \tag{6.21}$$

where μ is the viscosity of the surrounding medium, a is the radius of the pendulum bob, and $\hat{\mathbf{e}}_\theta$ is the unit vector tangential to the circular path. The resistance of the massless rod supporting the bob is neglected. The force $\mathbf{F}^*$ is not derivable from a potential function. Thus, we have one part of the force (i.e., gravitational force) conservative and the other (i.e., viscous force) nonconservative. Hence, we use Hamilton's principle expressed by Eq. (6.14) or Eq. (6.16) with

$$\delta W_E = \delta V - \mathbf{F}^* \cdot (l\delta\theta\hat{\mathbf{e}}_\theta) = \left(mgl \sin\theta + 6\pi\mu a l^2\dot{\theta}\right)\delta\theta \equiv -Q\delta\theta.$$

Then the equation of motion is given by [$K = K(\dot{\theta})$]:

$$-\frac{d}{dt}\left(\frac{\partial K}{\partial \dot{\theta}}\right) + Q = 0 \qquad \text{or} \qquad \ddot{\theta} + \frac{g}{l}\sin\theta + \frac{6\pi a\mu}{m}\dot{\theta} = 0. \tag{6.22}$$

The coefficient $c = 6\pi a\mu/m$ is called the *damping* coefficient.

Hamilton's principle in Eqs. (6.8) and (6.14) can be easily extended to a system of N particles, hence to rigid bodies. We have

$$\delta \int_{t_1}^{t_2} L(q_i, \dot{q}_i)\, dt = 0 \qquad \text{for conservative forces,} \tag{6.23}$$

$$\int_{t_1}^{t_2} \delta K\, dt - \int_{t_1}^{t_2} \delta W_E\, dt = 0 \qquad \text{for nonconservative forces,} \tag{6.24}$$

where

$$K = \sum_{i=1}^{N} \frac{m_i}{2} \frac{d\mathbf{r}_i}{dt} \cdot \frac{d\mathbf{r}_i}{dt}, \qquad \delta W_E = -\sum_{i=1}^{N} \mathbf{F}_i \cdot \delta \mathbf{r}_i, \tag{6.25}$$

and m_i and $\mathbf{r}_i$ denote the mass and path of the ith particle. Of course, Eqs. (6.23) and (6.24) apply to a rigid body, because a rigid body can be viewed as a set of particles with fixed positions $\mathbf{r}_i$ relative to the mass center.

6.3 HAMILTON'S PRINCIPLE FOR A CONTINUUM

Following essentially the same procedure as that used for discrete systems, we can develop Hamilton's principle for the dynamics of deformable bodies. The main difference between rigid bodies and deformable bodies is the presence of internal energy W_I for deformable bodies. Newton's second law of motion applied to deformable bodies expresses the global statement of the principle of conservation of linear momentum. However, it should be noted that Newton's second law of motion for continuous media is not sufficient to determine its motion $\mathbf{u} = \mathbf{u}(\mathbf{x}, t)$; the kinematic conditions and constitutive equations derived in Chapter 3 are needed to completely determine the motion.

Newton's second law of motion for a continuous body can be written in general terms as

$$\mathbf{F} - m\mathbf{a} = \mathbf{0}, \tag{6.26}$$

where m is the mass, $\mathbf{a}$ the acceleration vector, and $\mathbf{F}$ is the resultant of *all* forces acting on the body. The actual path $\mathbf{u} = \mathbf{u}(\mathbf{x}, t)$ followed by a material particle in position $\mathbf{x}$ in the body is varied, consistent with kinematic (essential) boundary conditions, to $\mathbf{u} + \delta\mathbf{u}$, where $\delta\mathbf{u}$ is the admissible variation (or virtual displacement) of the path. We suppose that the varied path differs from the actual path except at initial and final times, t_1 and t_2, respectively. Thus, an admissible variation $\delta\mathbf{u}$ satisfies the conditions

$$\delta\mathbf{u} = \mathbf{0} \qquad \text{on } S_1 \text{ for all } t, \tag{6.27a}$$

$$\delta\mathbf{u}(\mathbf{x}, t_1) = \delta\mathbf{u}(\mathbf{x}, t_2) = \mathbf{0} \qquad \text{for all } \mathbf{x}, \tag{6.27b}$$

where S_1 denotes the portion of the boundary of the body where the displacement vector $\mathbf{u}$ is specified. Note that the scalar product of Eq. (6.26) with $\delta\mathbf{u}$ gives work done at point $\mathbf{x}$, because $\mathbf{F}$, $\mathbf{a}$, and $\mathbf{u}$ are vector functions of position (whereas the work is a scalar). Integration of the product over the volume (and surface) of the body gives the total work done by all points.

The *work done on the body* at time t by the resultant force in moving through the virtual displacement $\delta\mathbf{u}$ is given by

$$\int_V \mathbf{f}\cdot\delta\mathbf{u}\,dV + \int_{S_2}\hat{\mathbf{t}}\cdot\delta\mathbf{u}\,dS - \int_V \overleftrightarrow{\sigma} : \delta\overleftrightarrow{\varepsilon}\,dV, \tag{6.28}$$

where $\mathbf{f}$ is the body force vector, $\hat{\mathbf{t}}$ the specified surface traction vector, and $\overleftrightarrow{\sigma}$ and $\overleftrightarrow{\varepsilon}$ are the stress and strain tensors. The last term in Eq. (6.28) represents the *virtual work* of internal forces *stored in the body*. The strains $\delta\overleftrightarrow{\varepsilon}$ are assumed to be compatible in the sense that the strain–displacement relations (3.20) or (3.23) are satisfied. The work done by the inertia force $m\mathbf{a}$ in moving through the virtual displacement $\delta\mathbf{u}$ is given by

$$\int_V \rho\frac{\partial^2\mathbf{u}}{\partial t^2}\cdot\delta\mathbf{u}\,dV, \tag{6.29}$$

where ρ is the mass density (which can be a function of position) of the medium. We have, analogous to Eq. (6.2) for discrete systems, the result

$$\int_{t_1}^{t_2}\left\{\int_V \rho\frac{\partial^2\mathbf{u}}{\partial t^2}\cdot\delta\mathbf{u}\,dV - \left[\int_V(\mathbf{f}\cdot\delta\mathbf{u} - \overleftrightarrow{\sigma} : \delta\overleftrightarrow{\varepsilon})dV + \int_{S_2}\hat{\mathbf{t}}\cdot\delta\mathbf{u}\,dS\right]\right\}dt = 0$$

or

$$-\int_{t_1}^{t_2}\left[\int_V \rho\frac{\partial\mathbf{u}}{\partial t}\cdot\frac{\partial\delta\mathbf{u}}{\partial t}\,dV + \int_V(\mathbf{f}\cdot\delta\mathbf{u} - \overleftrightarrow{\sigma} : \delta\overleftrightarrow{\varepsilon})dV + \int_{S_2}\hat{\mathbf{t}}\cdot\delta\mathbf{u}\,dS\right]dt = 0. \tag{6.30}$$

In arriving at the expression in Eq. (6.30), integration by parts is used on the first term; the integrated terms vanish because of the initial and final conditions in Eq. (6.27b). Equation (6.30) is known as the general form of Hamilton's principle for a continuous medium (conservative or not, and elastic or not).

For an ideal elastic body, we recall from the previous discussions that the forces $\mathbf{f}$ and $\mathbf{t}$ are conservative:

$$\delta V = -\left(\int_V \mathbf{f}\cdot\delta\mathbf{u}\,dV + \int_{S_2}\hat{\mathbf{t}}\cdot\delta\mathbf{u}\,dS\right), \tag{6.31a}$$

and that there exists a strain energy density function $U_0 = U_0(\varepsilon_{ij})$ such that

$$\sigma_{ij} = \frac{\partial U_0}{\partial\varepsilon_{ij}}. \tag{6.31b}$$

Substituting Eqs. (6.31a,b) into Eq. (6.30), we obtain

$$\delta \int_{t_1}^{t_2} [K - (V + U)]\, dt = 0, \tag{6.32}$$

where K and U are the kinetic and strain energies:

$$K = \int_V \frac{\rho}{2} \frac{\partial \mathbf{u}}{\partial t} \cdot \frac{\partial \mathbf{u}}{\partial t}\, dV, \qquad U = \int_V U_0\, dV. \tag{6.33}$$

Equation (6.32) represents Hamilton's principle for an elastic body (linear or nonlinear). Recall that the sum of the strain energy and potential energy of external forces, $U + V$, is called the total potential energy, Π, of the body. For bodies involving no motion (i.e., forces are applied sufficiently slowly such that the motion is independent of time, and the inertia forces are negligible), Hamilton's principle (6.32) reduces to the principle of virtual displacements. Equation (6.32) may be viewed as the dynamics version of the principle of virtual displacements.

The Euler–Lagrange equations associated with the Lagrangian, $L = K - \Pi$, can be obtained from Eq. (6.32):

$$\begin{aligned} 0 &= \delta \int_{t_1}^{t_2} L(\mathbf{u}, \nabla\mathbf{u}, \dot{\mathbf{u}})\, dt \\ &= \int_{t_1}^{t_2} \left[\int_V \left(\rho \frac{\partial^2 \mathbf{u}}{\partial t^2} - \operatorname{div} \overset{\leftrightarrow}{\sigma} - \mathbf{f} \right) \cdot \delta\mathbf{u}\, dV + \int_{S_2} (\mathbf{t} - \hat{\mathbf{t}}) \cdot \delta\mathbf{u}\, dS \right] dt, \end{aligned} \tag{6.34}$$

where integration by parts, gradient theorems, and Eqs. (6.27a,b) were used in arriving at Eq. (6.34) from Eq. (6.32). Because $\delta\mathbf{u}$ is arbitrary for t, $t_1 < t < t_2$, and for $\mathbf{x}$ in V and also on S_2, it follows that

$$\begin{aligned} \rho \frac{\partial^2 \mathbf{u}}{\partial t^2} - \operatorname{div} \overset{\leftrightarrow}{\sigma} - \mathbf{f} = \mathbf{0} &\qquad \text{in } V, \\ \mathbf{t} - \hat{\mathbf{t}} = \mathbf{0} &\qquad \text{on } S_2. \end{aligned} \tag{6.35}$$

Equations (6.35) are the Euler–Lagrange equations for an elastic body.

Using the results of this section and the previous sections, one can derive statements of Hamilton's principle corresponding to the principle of virtual forces (or complementary potential energy) and stationary principles. These are left as exercises for the reader.

Example 6.3 Consider the axial motion of an elastic bar of length L, area of cross section A, modulus of elasticity E, mass density ρ, and subjected to distributed force f per unit length and an end load P. We wish to determine the equations of motion for the bar. The kinetic and total potential energies of the system are

$$K = \int_V \frac{\rho}{2}\left(\frac{\partial u}{\partial t}\right)^2 dV = \int_0^L \frac{\rho A}{2}\left(\frac{\partial u}{\partial t}\right)^2 dx, \tag{6.36a}$$

$$\begin{aligned}\Pi &= \int_V \frac{1}{2}\sigma_{ij}\varepsilon_{ij}\, dV - \int_0^L f u\, dx - Pu(L) \\ &= \int_0^L \frac{A}{2}\sigma_{xx}\varepsilon_{xx}\, dx - \int_0^L f u\, dx - Pu(L),\end{aligned} \tag{6.36b}$$

where u, σ_{xx}, and ε_{xx} are assumed to be functions of x and t only, and

$$\begin{aligned} u(0,t) &= 0 \qquad &&\text{(bar is fixed at } x = 0\text{)},\\ \varepsilon_{xx} &= \frac{\partial u}{\partial x} \qquad &&\text{(strain–displacement relation)}.\end{aligned} \tag{6.37}$$

Substituting for K and Π from Eqs. (6.36a,b) into Eq. (6.32), we obtain

$$\begin{aligned}0 &= \int_{t_1}^{t_2}\left\{\int_0^L \left[A\rho\frac{\partial u}{\partial t}\frac{\partial \delta u}{\partial t} - A\sigma_{xx}\delta\left(\frac{\partial u}{\partial x}\right) + f\delta u\right] dx + P\delta u(L)\right\} dt \\ &= \int_0^L \left[\int_{t_1}^{t_2} -\frac{\partial}{\partial t}\left(\rho A\frac{\partial u}{\partial t}\right)\delta u\, dt + \rho A\frac{\partial u}{\partial t}\delta u\Big|_{t_1}^{t_2}\right] dx \\ &\quad + \int_{t_1}^{t_2}\left\{\int_0^L \left[\frac{\partial}{\partial x}(A\sigma_{xx}) + f\right]\delta u\, dx - (A\sigma_{xx}\delta u)\Big|_0^L + P\delta u(L)\right\} dt \\ &= -\int_{t_1}^{t_2}\left\{\int_0^L \left[\frac{\partial}{\partial t}\left(\rho A\frac{\partial u}{\partial t}\right) - \frac{\partial}{\partial x}(A\sigma_{xx}) - f\right]\delta u\, dx \right.\\ &\qquad \left. - (A\sigma_{xx} - P)\big|_{x=L}\delta u(L)\right\} dt,\end{aligned} \tag{6.38}$$

where $\delta u(0,t) = 0$ and $\delta u(x,t_1) = \delta u(x,t_2) = 0$ are used to simplify the expression. The Euler–Lagrange equations are obtained by setting the coefficients of δu in $(0, L)$ and at $x = L$ to zero separately:

$$\frac{\partial}{\partial t}\left(\rho A\frac{\partial u}{\partial t}\right) - \frac{\partial}{\partial x}(A\sigma_{xx}) - f = 0, \qquad 0 < x < L, \tag{6.39a}$$

$$(A\sigma_{xx})\big|_{x=L} - P = 0. \tag{6.39b}$$

for all t, $t_1 < t < t_2$. For linear elastic materials, we have $\sigma_{xx} = E\varepsilon_{xx} = E(\partial u/\partial x)$, and Eqs. (6.39a,b) become

$$\frac{\partial}{\partial t}\left(\rho A\frac{\partial u}{\partial t}\right) - \frac{\partial}{\partial x}\left(EA\frac{\partial u}{\partial x}\right) - f = 0, \qquad 0 < x < L, \tag{6.40a}$$

$$\left(AE\frac{\partial u}{\partial x}\right)\bigg|_{x=L} - P = 0. \tag{6.40b}$$

Now suppose that the bar also experiences a nonconservative (viscous damping) force proportional to the velocity:

$$F^* = -\mu\frac{\partial u}{\partial t}, \tag{6.41}$$

where μ is the damping coefficient (a constant). Then the Euler–Lagrange equations from Eq. (6.30) are given by

$$\frac{\partial}{\partial t}\left(\rho A\frac{\partial u}{\partial t}\right) - \frac{\partial}{\partial x}\left(EA\frac{\partial}{\partial x}\right) - f + \mu\frac{\partial u}{\partial t} = 0, \qquad 0 < x < L, \tag{6.42a}$$

$$\left(AE\frac{\partial u}{\partial x}\right)_{x=L} - P = 0. \tag{6.42b}$$

The next example illustrates the use of Hamilton's principle to derive the equations of motion associated with an assumed displacement field of a refined theory of straight beams. The displacement field is often arrived at by placing certain kinematic hypotheses on the system (see Reddy [7]).

Example 6.4 *(Third-order beam theory)* Consider the displacement field

$$\begin{aligned} u(x,z,t) &= u_0(x,t) + z\phi(x,t) - c_1 z^3\left(\phi + \frac{\partial w_0}{\partial x}\right), \\ w(x,z,t) &= w_0(x,t), \end{aligned} \tag{6.43}$$

where $c_1 = 4/(3h^2)$, u_0 is the axial displacement, w_0 the transverse displacement, and ϕ the rotation of a point on the centroidal axis x of the beam. The displacement field is arrived at by (a) relaxing the Euler–Bernoulli hypotheses to let the straight lines normal to the beam axis before deformation become (cubic) curves with arbitrary slope at $z = 0$, and (b) requiring the transverse shear stress to vanish at the top and bottom of the beam. Thus, the only restriction from the Euler–Bernoulli beam theory that is kept is $w(x,z,t) = w_0(x,t)$ (i.e., transverse deflection is independent of the thickness coordinate z). The displacement field (6.43) accommodates quadratic variation of transverse shear strain ε_{xz} and shear stress σ_{xz} through the beam height, as can be seen from the strains computed next.

Now suppose that the beam is subjected to a distributed transverse load of $q(x,t)$ along the length of the beam. Since we are primarily interested in deriving the equations of motion and the nature of the boundary conditions of the beam that experiences a displacement field of the form in Eq. (6.43), we will not consider specific geometric or force boundary conditions here. The procedure to obtain the equations of motion and boundary conditions involves the following steps: (i) compute the strains, (ii) compute the virtual energies required in Hamilton's principle, and (iii) use Hamilton's principle and derive the Euler–Lagrange equations of motion and identify the primary and secondary variables of the theory (which in turn help identify the nature of the boundary conditions).

Although one can use the general nonlinear strain–displacement relations, here we restrict the development to small strains and displacements. The linear strains associated with the displacement field are

$$\begin{aligned}\varepsilon_{xx} &= \varepsilon_{xx}^{(0)} + z\varepsilon_{xx}^{(1)} + z^3\varepsilon_{xx}^{(3)},\\ \gamma_{xz} &= \gamma_{xz}^{(0)} + z^2\gamma_{xz}^{(2)},\end{aligned} \tag{6.44a}$$

where

$$\begin{aligned}\varepsilon_{xx}^{(0)} &= \frac{\partial u_0}{\partial x}, \qquad \varepsilon_{xx}^{(1)} = \frac{\partial \phi}{\partial x}, \qquad \varepsilon_{xx}^{(3)} = -c_1\left(\frac{\partial \phi}{\partial x} + \frac{\partial^2 w_0}{\partial x^2}\right),\\ \gamma_{xz}^{(0)} &= \phi + \frac{\partial w_0}{\partial x}, \qquad \gamma_{xz}^{(2)} = -c_2\left(\phi + \frac{\partial w_0}{\partial x}\right),\end{aligned} \tag{6.44b}$$

and $c_2 = 4/h^2$. Note that $\gamma_{xz} = 2\varepsilon_{xz}$ is a quadratic function of z. Hence, $\sigma_{xz} = G\gamma_{xz}$ is also quadratic in z.

From the dynamic version of the principle of virtual displacements (i.e., Hamilton's principle for deformable bodies) we have

$$\begin{aligned}0 &= \int_0^T\int_0^L\int_A \left[\sigma_{xx}\left(\delta\varepsilon_{xx}^{(0)} + z\delta\varepsilon_{xx}^{(1)} + z^3\delta\varepsilon_{xx}^{(3)}\right) + \sigma_{xz}\left(\delta\gamma_{xz}^{(0)} + z^2\delta\gamma_{xz}^{(2)}\right)\right] dA\,dx\,dt\\ &\quad - \int_0^T\int_0^L\int_A \rho\left\{\left[\dot{u}_0 + z\dot{\phi} - c_1 z^3\left(\dot{\phi} + \frac{\partial \dot{w}_0}{\partial x}\right)\right]\right.\\ &\qquad \left.\times\left[\delta\dot{u}_0 + z\delta\dot{\phi} - c_1 z^3\left(\delta\dot{\phi} + \frac{\partial \delta\dot{w}_0}{\partial x}\right)\right] + \dot{w}_0\delta\dot{w}_0\right\} dA\,dx\,dt\\ &\quad - \int_0^T\int_0^L q\delta w_0\,dx\,dt\\ &= \int_0^T\int_0^L \left(N_{xx}\delta\varepsilon_{xx}^{(0)} + M_{xx}\delta\varepsilon_{xx}^{(1)} + P_{xx}\delta\varepsilon_{xx}^{(3)} + Q_x\delta\gamma_{xz}^{(0)} + R_x\delta\gamma_{xz}^{(2)}\right) dx\,dt\\ &\quad - \int_0^T\int_0^L \left\{I_0\dot{u}_0\delta\dot{u}_0 + \left[I_2\dot{\phi} - c_1 I_4\left(\dot{\phi} + \frac{\partial \dot{w}_0}{\partial x}\right)\right]\delta\dot{\phi} + q\delta w_0\right\} dx\,dt\\ &\quad - \int_0^T\int_0^L \left\{-c_1\left[I_4\dot{\phi} - c_1 I_6\left(\dot{\phi} + \frac{\partial \dot{w}_0}{\partial x}\right)\right]\left(\delta\dot{\phi} + \frac{\partial \delta\dot{w}_0}{\partial x}\right) + I_0\dot{w}_0\delta\dot{w}_0\right\} dx\,dt\end{aligned}$$

$$
\begin{aligned}
&= \int_0^T \int_0^L \left\{ \left(-\frac{\partial N_{xx}}{\partial x} + I_0 \frac{\partial^2 u_0}{\partial t^2} \right) \delta u_0 \right. \\
&\quad + \left(-\frac{\partial \overline{M}_{xx}}{\partial x} + \overline{Q}_x + K_2 \frac{\partial^2 \phi}{\partial t^2} - c_1 J_4 \frac{\partial^3 w_0}{\partial x \partial t^2} \right) \delta \phi \\
&\quad + \left[-c_1 \frac{\partial^2 P_{xx}}{\partial x^2} - \frac{\partial \overline{Q}_x}{\partial x} - q + c_1 \left(J_4 \frac{\partial^3 \phi}{\partial x \partial t^2} - c_1 I_6 \frac{\partial^4 w_0}{\partial x^2 \partial t^2} \right) \right. \\
&\quad \left. \left. + I_0 \frac{\partial^2 w_0}{\partial t^2} \right] \delta w_0 \right\} dx dt \\
&+ \int_0^T \left\{ N_{xx} \delta u_0 + \overline{M}_{xx} \delta \phi - c_1 P_{xx} \frac{\partial \delta w_0}{\partial x} \right. \\
&\quad \left. + \left[\overline{Q}_x + c_1 \left(\frac{\partial P_{xx}}{\partial x} - J_4 \frac{\partial^2 \phi}{\partial t^2} + c_1 I_6 \frac{\partial^3 w_0}{\partial x \partial t^2} \right) \right] \delta w_0 \right\}_0^L dt,
\end{aligned}
\tag{6.45}
$$

where all the terms involving $[\ \cdot\ ]_0^T$ vanish on account of the assumption that all variations and their derivatives are zero at $t = 0$ and $t = T$, and the new variables introduced in arriving at the last expression are defined as follows:

$$
\begin{Bmatrix} N_{xx} \\ M_{xx} \\ P_{xx} \end{Bmatrix} = \int_{-h/2}^{h/2} \begin{Bmatrix} 1 \\ z \\ z^3 \end{Bmatrix} \sigma_{xx}\, dz, \qquad \begin{Bmatrix} Q_x \\ R_x \end{Bmatrix} = \int_{-h/2}^{h/2} \begin{Bmatrix} 1 \\ z^2 \end{Bmatrix} \sigma_{xz}\, dz, \tag{6.46a}
$$

$$
\overline{M}_{xx} = M_{xx} - c_1 P_{xx}, \quad \overline{Q}_x = Q_x - c_2 R_x, \quad c_1 = \frac{4}{3h^2}, \quad c_2 = \frac{4}{h^2}, \tag{6.46b}
$$

$$
J_4 = I_4 - c_1 I_6, \qquad K_2 = I_2 - 2c_1 I_4 + c_1^2 I_6, \qquad I_i = \int_{-h/2}^{h/2} \rho(z)^i\, dz. \tag{6.46c}
$$

Note that I_i are zero for odd values of i (i.e., $I_1 = I_3 = I_5 = 0$).

Thus, the Euler–Lagrange equations are

$$
\delta u_0\colon \quad \frac{\partial N_{xx}}{\partial x} = I_0 \frac{\partial^2 u_0}{\partial t^2}, \tag{6.47}
$$

$$
\begin{aligned}
\delta w_0\colon \quad & \frac{\partial \overline{Q}_x}{\partial x} + c_1 \frac{\partial^2 P_{xx}}{\partial x^2} + q \\
&= I_0 \frac{\partial^2 w_0}{\partial t^2} + c_1 \left(J_4 \frac{\partial^3 \phi}{\partial x \partial t^2} - c_1 I_6 \frac{\partial^4 w_0}{\partial x^2 \partial t^2} \right),
\end{aligned}
\tag{6.48a}
$$

$$
\delta \phi : \quad \frac{\partial \overline{M}_{xx}}{\partial x} - \overline{Q}_x = K_2 \frac{\partial^2 \phi}{\partial t^2} - c_1 J_4 \frac{\partial^3 w_0}{\partial x \partial t^2}. \tag{6.48b}
$$

The last two lines of Eq. (6.45) include the boundary terms, and they show that the primary variables of the theory are (those with the variational symbol) u_0, w_0, ϕ, and $\partial w_0/\partial x$. The corresponding secondary variables are the coefficients of δu_0, δw_0, $\delta\phi$, and $\partial\delta w_0/\partial x$:

$$N_{xx}, \qquad \overline{Q}_x + c_1\left(\frac{\partial P_{xx}}{\partial x} - J_4\frac{\partial^2\phi}{\partial t^2} + c_1 I_6\frac{\partial^3 w_0}{\partial x\partial t^2}\right), \qquad \overline{M}_{xx}, \qquad -c_1 P_{xx}. \tag{6.49}$$

When $c_1 = 0$ in Eq. (6.43), it corresponds to the displacement field of the Timoshenko beam theory. Thus, the equations of motion of the Timoshenko beam theory can be obtained directly from Eqs. (6.47)–(6.48a,b) by setting $c_1 = c_2 = 0$:

$$\frac{\partial N_{xx}}{\partial x} = I_0\frac{\partial^2 u_0}{\partial t^2}, \tag{6.50}$$

$$\frac{\partial Q_x}{\partial x} + q = I_0\frac{\partial^2 w_0}{\partial t^2}, \tag{6.51a}$$

$$\frac{\partial M_{xx}}{\partial x} - Q_x = I_2\frac{\partial^2 \phi}{\partial t^2}. \tag{6.51b}$$

The primary and secondary variables of the Timoshenko beam theory are: (u_0, w_0, ϕ) and (N_{xx}, Q_x, M_{xx}).

6.4 HAMILTON'S PRINCIPLE FOR CONSTRAINED SYSTEMS

The kinematic relations that restrict a motion are called *constraints*. When the relations are between generalized coordinates, and possible position and time, in the form

$$G(t, \mathbf{q}) = 0 \qquad \text{for a discrete system,} \tag{6.52a}$$

$$G(\mathbf{x}, t, \mathbf{u}, \text{grad } \mathbf{u}) = 0 \qquad \text{for a continuous system,} \tag{6.52b}$$

they are called *holonomic constraints*. When the constraints are of inequality type or relate q_i to $\dot{q}_i$, they are called *nonholonomic constraints*. We consider only holonomic constraints here.

As discussed in Section 4.4.8, constraints can be included into the Lagrangian function either by means of the Lagrange multipliers or by the penalty function method. To illustrate the ideas for a dynamical system, we consider the motion of a single particle whose motion is constrained by holonomic constraints:

$$G_1(t, q_1, q_2, q_3) = 0, \qquad G_2(t, q_1, q_2, q_3) = 0. \tag{6.53}$$

The variation of these equations yields

$$\frac{\partial G_1}{\partial q_1}\delta q_1 + \frac{\partial G_1}{\partial q_2}\delta q_2 + \frac{\partial G_1}{\partial q_3}\delta q_3 = 0, \tag{6.54a}$$

$$\frac{\partial G_2}{\partial q_1}\delta q_1 + \frac{\partial G_2}{\partial q_2}\delta q_2 + \frac{\partial G_2}{\partial q_3}\delta q_3 = 0. \tag{6.54b}$$

Since there are two constraints and three degrees of freedom, only one of the variations δq_1, δq_2, and δq_3 is independent and the other two are related to the independent ones by Eq. (6.53). We multiply Eq. (6.54a) by the Lagrange multiplier λ_1 and Eq. (6.54b) by λ_2, integrate each equation over t_1 to t_2, and add the results to Eq. (6.16):

$$\int_{t_1}^{t_2} \sum_{i=1}^{3} \left[\frac{\partial K}{\partial q_i} - \frac{d}{dt}\left(\frac{\partial K}{\partial \dot{q}_i}\right) + Q_i + \lambda_1 \frac{\partial G_1}{\partial q_i} + \lambda_2 \frac{\partial G_2}{\partial q_i} \right] \delta q_i \, dt = 0. \tag{6.55}$$

Since the variations δq_1, δq_2, and δq_3 are not all independent, we cannot use the usual argument to set the coefficients of δq_i ($i = 1, 2, 3$) to zero. However, the Lagrange multipliers λ_1 and λ_2 are arbitrary. Therefore, we choose λ_1 and λ_2 such that the coefficients of two of the variations out of δq_1, δq_2, and δq_3 vanish. Since the remaining variation is linearly independent, its coefficient should be zero. Thus we obtain

$$\frac{\partial K}{\partial q_i} - \frac{d}{dt}\left(\frac{\partial K}{\partial \dot{q}_i}\right) + Q_i + \lambda_1 \frac{\partial G_1}{\partial q_i} + \lambda_2 \frac{\partial G_2}{\partial q_i} = 0 \tag{6.56}$$

for $i = 1, 2, 3$. Equations (6.56) together with (6.53) provide five equations for the five unknowns $(q_1, q_2, q_3, \lambda_1, \lambda_2)$.

In the penalty function method, Hamilton's principle in Eq. (6.14) is modified to read

$$\int_{t_1}^{t_2} \delta K_p \, dt - \int_{t_1}^{t_2} \delta W_E \, dt + \delta \int_{t_1}^{t_2} \left(\frac{\gamma_1}{2} G_1^2 + \frac{\gamma_2}{2} G_2^2 \right) dt = 0, \tag{6.57}$$

where γ_1 and γ_2 are the penalty parameters, and K_p is the kinetic energy of the constrained system. Since the constraint conditions (6.53) are now included, in the least-squares sense, we suppose that the variations δq_1, δq_2, and δq_3 are independent of each other. Performing the variation indicated in Eq. (6.57), we obtain (after the usual argument) the Euler–Lagrange equations

$$\frac{\partial K_p}{\partial q_i} - \frac{d}{dt}\left(\frac{\partial K_p}{\partial \dot{q}_i}\right) + Q_i + \gamma_1 G_1 \frac{\partial G_1}{\partial q_i} + \gamma_2 G_2 \frac{\partial G_2}{\partial q_i} = 0. \tag{6.58}$$

A comparison of Eq. (6.58) with Eq. (6.56) ($K_p = K$) shows that the Lagrange multipliers λ_1 and λ_2 can be computed from

$$\lambda_1 = \gamma_1 G_1, \qquad \lambda_2 = \gamma_2 G_2, \tag{6.59}$$

where G_1 and G_2 are given by Eq. (6.53) for the generalized coordinates q_i computed from Eq. (6.58); G_1 and G_2 thus computed are not identically zero because q_i computed from Eq. (6.58) are different from the true values, the error being inversely proportional to the penalty parameters γ_1 and γ_2.

Example 6.5 To illustrate the ideas presented in the preceding paragraphs, we reconsider the damped motion of a pendulum of Example 6.2. The generalized coordinates are

$$q_1 = r, \qquad q_2 = \theta, \tag{6.60a}$$

and the constraint is

$$G_1(q_1) \equiv q_1 - l = 0. \tag{6.60b}$$

The work done by external forces in moving through virtual displacement $\delta\mathbf{q}$ is given by

$$\begin{aligned} \delta W_E &= -\mathbf{F}\cdot\delta\mathbf{q} = -\left[mg(-\sin\theta\hat{\mathbf{e}}_\theta + \cos\theta\hat{\mathbf{e}}_r) - 6\pi\mu a r\dot{\theta}\hat{\mathbf{e}}_\theta\right]\cdot\left(\delta q_1\hat{\mathbf{e}}_r + r\delta q_2\hat{\mathbf{e}}_\theta\right) \\ &= -\left[mg\cos\theta\delta q_1 - r(mg\sin\theta + 6\pi\mu a r\dot{\theta})\delta q_2\right] \equiv -\left(Q_1\delta q_1 + Q_2\delta q_2\right). \end{aligned} \tag{6.61}$$

The kinetic energy is given by

$$K = \frac{m}{2}\left[(\dot{r})^2 + (l\dot{\theta})^2\right] = \frac{m}{2}\left(\dot{q}_1^2 + l^2\dot{q}_2^2\right) = \frac{1}{2}m(l\dot{\theta})^2. \tag{6.62}$$

For the Lagrange multiplier method, the Euler–Lagrange equations are obtained by substituting Eq. (6.62) and Q_i from (6.61) into Eq. (6.56):

$$-m\ddot{r} + mg\cos\theta + \lambda_1 = 0, \tag{6.63a}$$

$$-ml^2\ddot{\theta} - r(mg\sin\theta + 6\pi\mu a r\dot{\theta}) = 0. \tag{6.63b}$$

In view of the constraint condition (6.60b), we have $\ddot{r} = 0$, and Eqs. (6.63a,b) become

$$\lambda_1 = -mg\cos\theta, \qquad \ddot{\theta} + \frac{6\pi\mu a}{m}\dot{\theta} + \frac{g}{l}\sin\theta = 0. \tag{6.63c}$$

The Lagrange multiplier λ_1 can be interpreted as the force exerted on the pendulum bob by the massless rod. It is the force necessary to oppose gravity and maintain the motion of the bob in a circular arc.

In the penalty function method, the Euler–Lagrange equations are obtained by substituting Eqs. (6.61) and (6.62) into Eq. (6.58):

$$mg\cos\theta + \gamma_1(r - l) = 0, \tag{6.64a}$$

$$-ml^2\ddot{\theta} - r(mg\sin\theta + 6\pi\mu a r\dot{\theta}) = 0. \tag{6.64b}$$

A comparison of the first equation with the first equation in (6.63) shows that the approximate Lagrange multiplier is given by

$$\lambda_1(\gamma_1) = \gamma_1(r - l) = -mg\cos\theta.$$

The error in the constraint is given by

$$r - l = -\frac{mg\cos\theta}{\gamma_1},$$

which goes to zero as γ_1 goes to infinity.

The ideas discussed above for particles and rigid bodies can be readily extended to deformable bodies by accounting for the internal energy due to deformation. In particular, Eqs. (6.56) and (6.58) take the following forms:

Lagrange multiplier method

$$\frac{\partial L_c}{\partial q_i} - \frac{\partial}{\partial t}\left(\frac{\partial L_c}{\partial \dot{q}_i}\right) - \frac{\partial}{\partial x}\left(\frac{\partial L_c}{\partial q_{i,x}}\right) + Q_i + \lambda_1\frac{\partial G_1}{\partial q_i} + \lambda_2\frac{\partial G_2}{\partial q_i} = 0. \tag{6.65}$$

Penalty method

$$\frac{\partial L_c}{\partial q_i} - \frac{\partial}{\partial t}\left(\frac{\partial L_c}{\partial \dot{q}_i}\right) - \frac{\partial}{\partial x}\left(\frac{\partial L_c}{\partial q_{i,x}}\right) + Q_i + \gamma_1 G_1\frac{\partial G_1}{\partial q_i} + \gamma_2 G_2\frac{\partial G_2}{\partial q_i} = 0. \tag{6.66}$$

where $q_{i,x} = (\partial q_i/\partial x)$, and $L_c = K_c - (U_c + V_c)$. The quantities K_c, U_c, and V_c are the kinetic energy, strain energy, and the potential energy due to conservative loads, respectively, of the constrained system. The next example illustrates the use of Eqs. (6.65) and (6.66).

Example 6.6 The Lagrange function associated with the dynamics of an Euler–Bernoulli beam is given by $L = K - (U + V)$, where

$$\begin{aligned} K &= \int_0^L \int_A \left[\frac{\rho}{2}\left(\frac{\partial u_0}{\partial t} - z\frac{\partial^2 w_0}{\partial x \partial t}\right)^2 + \frac{\rho}{2}\left(\frac{\partial w_0}{\partial t}\right)^2\right] dA\,dx \\ &= \int_0^L \left[\frac{\rho A}{2}\left(\frac{\partial u_0}{\partial t}\right)^2 + \frac{\rho I}{2}\left(\frac{\partial^2 w_0}{\partial x \partial t}\right)^2 + \frac{\rho A}{2}\left(\frac{\partial w_0}{\partial t}\right)^2\right] dx, \end{aligned} \tag{6.67a}$$

$$\begin{aligned} U &= \int_0^L \int_A \frac{E}{2}\left(\frac{\partial u_0}{\partial x} - z\frac{\partial^2 w_0}{\partial x^2}\right)^2 dA\,dx \\ &= \int_0^L \left[\frac{EA}{2}\left(\frac{\partial u_0}{\partial x}\right)^2 + \frac{EI}{2}\left(\frac{\partial^2 w_0}{\partial x^2}\right)^2\right] dx, \end{aligned} \tag{6.67b}$$

$$V = -\int_0^L [f(x,t)u_0 + q(x,t)w_0]\,dx. \tag{6.67c}$$

Here, u_0 denotes the axial displacement and w_0 the transverse displacement, which are functions of x and t, and f and q are the axial and transverse distributed loads. In arriving at the expressions for K and U, we have used the fact that the x-axis coincides with the geometric centroidal axis, $\int_A z\,dA = 0$.

Note that there is no constraint among the two dependent variables $q_1 = u_0$ and $q_2 = w_0$, and $L = L(u_0, w_0, u_0', w_0'', \dot{u}_0, \dot{w}_0, \dot{w}_0')$. Hence, the Euler–Lagrange equations of the Euler–Bernoulli beam theory are given by

$$\delta u_0:\quad \frac{\partial L}{\partial u_0} - \frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \dot{u}_0}\right) - \frac{\partial}{\partial x}\left(\frac{\partial L}{\partial u_0'}\right) = 0,$$

$$\delta w_0:\quad \frac{\partial L}{\partial w_0} - \frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \dot{w}_0}\right) + \frac{\partial^2}{\partial x^2}\left(\frac{\partial L}{\partial w_0''}\right) + \frac{\partial^2}{\partial x \partial t}\left(\frac{\partial L}{\partial \dot{w}_0'}\right) = 0.$$

Note that

$$\frac{\partial L}{\partial u_0} = f, \qquad \frac{\partial L}{\partial \dot{u}_0} = \rho A \dot{u}_0, \qquad \frac{\partial L}{\partial u_0'} = -EAu_0',$$

$$\frac{\partial L}{\partial w_0} = q, \qquad \frac{\partial L}{\partial \dot{w}_0} = \rho A \dot{w}_0, \qquad \frac{\partial L}{\partial \dot{w}_0'} = \rho I \dot{w}_0', \qquad \frac{\partial L}{\partial w_0''} = -EIw_0''.$$

Hence we have

$$\delta u_0:\quad \frac{\partial}{\partial t}\left(\rho A \frac{\partial u_0}{\partial t}\right) - \frac{\partial}{\partial x}\left(EA\frac{\partial u_0}{\partial x}\right) = f, \tag{6.68a}$$

$$\delta w_0:\quad \frac{\partial}{\partial t}\left(\rho A \frac{\partial w_0}{\partial t}\right) + \frac{\partial^2}{\partial x^2}\left(EI\frac{\partial^2 w_0}{\partial x^2}\right) - \frac{\partial^2}{\partial x \partial t}\left(\rho I \frac{\partial^2 w_0}{\partial x \partial t}\right) = q. \tag{6.68b}$$

Now suppose that we introduce the function $\phi(x, t)$ such that

$$\phi + \frac{\partial w_0}{\partial x} = 0. \tag{6.69}$$

Now the problem can be looked at as one of minimizing L subject to the constraint in Eq. (6.69). We wish to determine the equations of motion of the beam using (a) the Lagrange multiplier method and (b) the penalty method.

First we write out the expressions K, U, and V for the constrained problem. We have ($V = V_c$):

$$K_c = \int_0^L \left[\frac{\rho A}{2}\left(\frac{\partial u_0}{\partial t}\right)^2 + \frac{\rho I}{2}\left(\frac{\partial \phi}{\partial t}\right)^2 + \frac{\rho A}{2}\left(\frac{\partial w_0}{\partial t}\right)^2\right] dx, \tag{6.70a}$$

$$U_c = \int_0^L \left[\frac{EA}{2}\left(\frac{\partial u_0}{\partial x}\right)^2 + \frac{EI}{2}\left(\frac{\partial \phi}{\partial x}\right)^2\right] dx, \tag{6.70b}$$

and $L_c = K_c - (U_c + V_c)$.

(a) *Lagrange multiplier method:* We have

$$L_l = L_c + \int_0^L \lambda \left(\phi + \frac{\partial w_0}{\partial x}\right) dx. \tag{6.71a}$$

Hence, the Euler–Lagrange equations are

$$\begin{aligned}
\delta u_0: &\quad \frac{\partial L_l}{\partial u_0} - \frac{\partial}{\partial x}\left(\frac{\partial L_l}{\partial u_0'}\right) - \frac{\partial}{\partial t}\left(\frac{\partial L_l}{\partial \dot{u}_0}\right) = 0,\\
\delta w_0: &\quad \frac{\partial L_l}{\partial w_0} - \frac{\partial}{\partial x}\left(\frac{\partial L_l}{\partial w_0'}\right) - \frac{\partial}{\partial t}\left(\frac{\partial L_l}{\partial \dot{w}_0}\right) = 0,\\
\delta \phi: &\quad \frac{\partial L_l}{\partial \phi} - \frac{\partial}{\partial x}\left(\frac{\partial L_l}{\partial \phi_0'}\right) - \frac{\partial}{\partial t}\left(\frac{\partial L_l}{\partial \dot{\phi}_0}\right) = 0,\\
\delta \lambda: &\quad \frac{\partial L_l}{\partial \lambda} = 0.
\end{aligned}$$

We have

$$\begin{aligned}
\delta u_0: &\quad f + \frac{\partial}{\partial x}\left(EA\frac{\partial u_0}{\partial x}\right) - \frac{\partial}{\partial t}\left(\rho A\frac{\partial u_0}{\partial t}\right) = 0,\\
\delta w_0: &\quad q - \frac{\partial \lambda}{\partial x} - \frac{\partial}{\partial t}\left(\rho A\frac{\partial w_0}{\partial t}\right) = 0,\\
\delta \phi: &\quad \lambda + \frac{\partial}{\partial x}\left(EI\frac{\partial \phi}{\partial x}\right) - \frac{\partial}{\partial t}\left(\rho I\frac{\partial \phi}{\partial t}\right) = 0,\\
\delta \lambda: &\quad \phi + \frac{\partial w_0}{\partial x} = 0.
\end{aligned}$$

(b) *Penalty method:* In this case we have

$$L_p = L_c + \frac{1}{2}\int_0^L \gamma \left(\phi + \frac{\partial w_0}{\partial x}\right)^2 dx. \tag{6.71b}$$

Note that

$$L_p = L_p(u_0, w_0, \phi, u_0', w_0', \phi', \dot{u}_0, \dot{w}_0, \dot{\phi}).$$

Hence, the Euler–Lagrange equations are

$$\delta u_0: \quad f + \frac{\partial}{\partial x}\left(EA\frac{\partial u_0}{\partial x}\right) - \frac{\partial}{\partial t}\left(\rho A\frac{\partial u_0}{\partial t}\right) = 0,$$

$$\delta w_0: \quad q - \frac{\partial}{\partial x}\left[\gamma\left(\phi + \frac{\partial w_0}{\partial x}\right)\right] - \frac{\partial}{\partial t}\left(\rho A \frac{\partial w_0}{\partial t}\right) = 0,$$

$$\delta \phi: \quad \gamma\left(\phi + \frac{\partial w_0}{\partial x}\right) + \frac{\partial}{\partial x}\left(EI\frac{\partial \phi}{\partial x}\right) - \frac{\partial}{\partial t}\left(\rho I \frac{\partial \phi}{\partial t}\right) = 0.$$

6.5 RAYLEIGH'S METHOD

For natural vibration of conservative systems, the Lagrangian function consists of only the strain energy and kinetic energy, $L = K - U$. For an ideal mechanical system (i.e., without thermal and other dissipative effects) the kinetic energy is the maximum when the strain energy is zero, and vice versa. Thus, in the absence of other energies, the principle of conservation of energy requires that

$$K_{max} = U_{max}. \tag{6.72}$$

Equation (6.72) merely implies that the maximum kinetic energy has the same value as the maximum strain energy. However, at any given instant, the two energies are *not* equal to each other but their sum is constant. For example, for the spring–mass system shown in Fig. 6.3, we have

$$K_{max} = \tfrac{1}{2}m(\dot{x}_{max})^2, \qquad U_{max} = \tfrac{1}{2}k(x_{max})^2. \tag{6.73}$$

If the system is in a state of natural vibration, then

$$x = A\sin\omega t, \qquad x_{max} = A,$$

where A is the amplitude and ω is the frequency of natural vibration. Therefore,

$$\dot{x} = A\omega\cos\omega t, \qquad \dot{x}_{max} = A\omega.$$

Then

$$\tfrac{1}{2}mA^2\omega^2 = \tfrac{1}{2}kA^2 \;\rightarrow\; \omega^2 = \frac{k}{m}.$$

Thus, given the stiffness of the spring and mass attached to it, we can calculate the natural frequency.

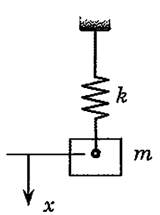

Figure 6.3 A spring–mass system.

Extending the discussion to natural vibration of a continuous system, say to an elastic bar, we have

$$K = \frac{1}{2}\int_0^L \rho A\left(\frac{\partial u}{\partial t}\right)^2 dx, \qquad U = \frac{1}{2}\int_0^L EA\left(\frac{\partial u}{\partial x}\right)^2 dx, \tag{6.74}$$

where ρ is the mass density (measured per unit volume), E the Young's modulus, and A is the area of cross section of the bar. For natural vibration, we have

$$u(x,t) = u_n(x)\sin(\omega_n t + \theta_n), \qquad \dot{u}(x,t) = u_n(x)\omega\cos(\omega_n t + \theta_n).$$

Hence,

$$K = \frac{1}{2}\int_0^L \rho A u_n^2 \omega_n^2 \cos^2(\omega_n t + \theta_n)\, dx,$$
$$U = \frac{1}{2}\int_0^L EA\left(\frac{du_n}{dx}\right)^2 \sin^2(\omega_n t + \theta_n)\, dx.$$

Equating the maximum values of K and U ($K_{max} = U_{max}$), we obtain

$$\omega_n^2 \int_0^L \rho A u_n^2\, dx = \int_0^L EA\left(\frac{du_n}{dx}\right)^2 dx \quad \text{or} \quad \omega_n^2 = \frac{\int_0^L EA(du_n/dx)^2\, dx}{\int_0^L \rho A u_n^2\, dx}. \tag{6.75}$$

The right-hand side of Eq. (6.75) is known as the *Rayleigh quotient* (see [9]) applied to a bar. With an appropriate choice of the eigenfunctions $u_n(x)$, the Rayleigh quotient allows us to compute the eigenvalue ω_n^2. A suitable candidate for $u_n(x)$ is the function that is sufficiently differentiable as required in Eq. (6.75) and satisfies the geometric boundary conditions of the problem. As we shall see in Chapter 7, the Rayleigh quotient is a special case of the Ritz approximation. The Rayleigh quotient is good for the estimation of fundamental frequencies.

Example 6.7 We wish to use the Rayleigh quotient to estimate the fundamental frequency of vibration of a cantilever beam. For this case, the Rayleigh quotient takes the form

$$\omega_n^2 = \frac{\int_0^L EI(d^2 w_n/dx^2)^2\, dx}{\int_0^L \rho A w_n^2\, dx}, \tag{6.76}$$

where $w_n(x)$ is the eigenmode associated with the transverse motion of the beam.

A suitable candidate for the eigenfunction w_1 is provided by the deflection of the beam under its own weight. For a cantilever beam, the deflection is given by

$$w_0(x) = \rho A\left(x^4 - 4x^3 L + 6x^2 L^2\right).$$

Therefore, we choose $w_n(x) = x^4 - 4x^3L + 6x^2L^2$, and compute the Rayleigh quotient

$$\omega_1^2 = \frac{\int_0^L EI\left[12(x-L)^2\right]^2 dx}{\int_0^L \rho A(x^4 - 4x^3L + 6x^2L^2)^2\, dx}$$

$$= \frac{EI}{\rho AL^4}\frac{144 \times 63}{728} \rightarrow \omega_1 = \frac{3.53}{L^2}\sqrt{\frac{EI}{\rho A}}.$$

The exact fundamental frequencies can be obtained by solving the transcendental equation

$$\cos\lambda\cosh\lambda + 1 = 0, \qquad \lambda^2 = \omega L^2\sqrt{\frac{\rho A}{EI}}.$$

The first root of this equation is $\lambda_1 = 1.875$, giving $\omega_1 = (3.516/L^2)\sqrt{EI/\rho A}$. Thus, the solution estimated by the Rayleigh quotient is very accurate.

A higher-order frequency can be obtained only if we can find an appropriate candidate for the corresponding eigenfunction. For example, in the case of a simply supported beam, the eigenfunction associated with the nth mode is $w_n(x) = \sin(n\pi x/L)$, which can be used in the Rayleigh quotient to determine $\omega_n = (n\pi/L)^2$. This turns out to be the exact value of the frequency because of the selection of the exact eigenfunction for the simply supported beam.

EXERCISES

6.1–6.3 Derive the Lagrangian function for the linear spring, mass, and linear dashpot systems shown in Figs. E6.1–E6.3. Assume that motion starts from rest and that the contact surfaces are free of friction. The constitutive equations for the spring and dashpot are given by equations of the form Force $= k \times$ displacement, for springs ($k =$ spring constant), and Force $= \eta \times$ velocity, for dashpots ($\eta =$ dashpot constant).

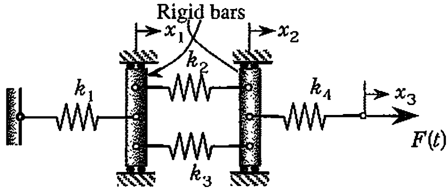

Figure E6.1

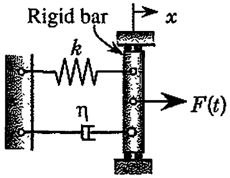

Figure E6.2

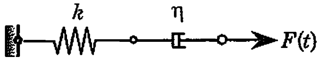

Figure E6.3

6.4 Derive the equations of motion for the double pendulum of Exercise 4.1.

6.5 Derive the equations of motion for the rigid-body assemblage of Exercise 4.2.

6.6 Consider a pendulum of mass m_1 with a flexible suspension, as shown in Fig. E6.6. The hinge of the pendulum is in a block of mass m_2, which can move up and down between the frictionless guides. The block is connected by a linear spring (of spring constant k) to an immovable support. The coordinate x is measured from the position of the block in which the system remains stationary. Derive the Euler–Lagrange equations of motion for the system.

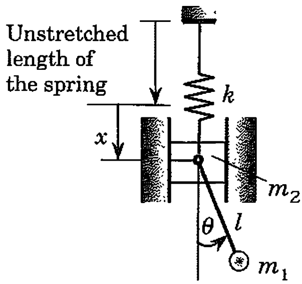

Figure E6.6

6.7 Consider a block of mass m_2 sliding on another block of mass m_1, which in turn slides on a horizontal surface, as shown in Fig. E6.7. Using u_1 and u_2 as coordinates, obtain the equations of motion. Assume that all surfaces are frictionless.

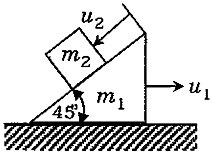

Figure E6.7

6.8 Figure E6.8 represents a double pendulum that is suspended from a block which moves horizontally with a prescribed motion, $x = f(t)$. Derive the Euler–Lagrange equations of motion of the pendulum when it oscillates in the (x, y)-plane under the action of gravity and prescribed motion.

6.9 Two masses m_1 and m_2 are attached to the ends of an inextensible cord that is suspended over a frictionless stationary pulley, as shown in Fig. E6.9. Find the equations of motion of the system.

6.10 Repeat Exercise 6.9 for the case in which a monkey of mass m_3 is climbing up the cord above mass m_1 with a speed v_0 relative to mass m_1 (see Fig. E6.10).

6.11 Determine the motion of all masses in the suspended double-pulley problem represented in Fig. E6.11.

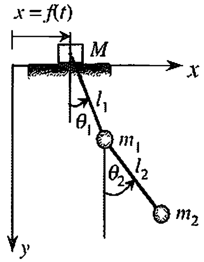

Figure E6.8

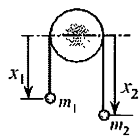

Figure E6.9

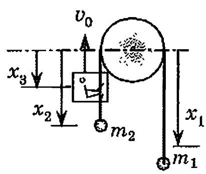

Figure E6.10

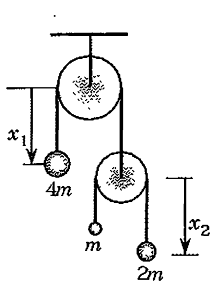

Figure E6.11

6.12 Derive the equations of motion of the system shown in Fig. E6.12. Assume that the mass moment of inertia of the link about its mass center is $J = m\Omega^2$, where Ω is the radius of gyration.

6.13 Consider a cantilever beam supporting a lumped mass m at its end (J is the mass moment of inertia), as shown in Fig. E6.13. Derive the equations of motion and natural boundary conditions for the problem (see Example 6.6).

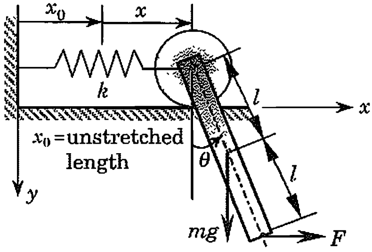

Figure E6.12

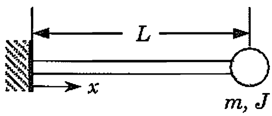

Figure E6.13

6.14 Derive the linear equations of motion of the Timoshenko beam theory, starting with the displacement field

$$u(x,z,t) = u_0(x,t) + z\phi(x,t), \qquad v = 0, \qquad w(x,z,t) = w_0(x,t).$$

Assume that the beam is subjected to distributed axial load $f(x,t)$ and transverse load $q(x,t)$, and that the x-axis coincides with the geometric centroidal axis. Compare your results with Eqs. (6.50)–(6.51a,b).

6.15 Express the equations of motion of the Timoshenko beam theory, Eqs. (6.51a,b), in terms of the generalized displacements (w_0, ϕ). Assume linear elastic behavior. In particular, show that

$$-\frac{\partial}{\partial x}\left[K_s GA\left(\frac{\partial w_0}{\partial x} + \phi\right)\right] + I_0\frac{\partial^2 w_0}{\partial t^2} = q,$$

$$-\frac{\partial}{\partial x}\left(EI\frac{\partial \phi}{\partial x}\right) + K_s GA\left(\frac{\partial w_0}{\partial x} + \phi\right) + I_2\frac{\partial^2 \phi}{\partial t^2} = 0.$$

6.16 Rewrite the equations of motion of the Timoshenko beam theory from Exercise 6.15 solely in terms of the transverse deflection w_0.

6.17 Consider the Euler–Bernoulli beam theory, whose displacement field is given by

$$u(x,z,t) = u_0(x,t) - z\frac{\partial w_0}{\partial x}, \qquad v = 0, \qquad w(x,z,t) = w_0(x,t).$$

Assume that the beam has two types of viscous (velocity-dependent) damping: (1) viscous resistance to transverse displacement of the beam, and (2) a viscous resistance to straining of the beam material. If the resistance to transverse velocity is denoted by $c(x)$, the corresponding damping force is given by $q_D(x,t) = c(x)\dot{w}_0$. If the resistance to strain velocity is c_s, the damping stress is $\sigma_{xx}^D = c_s\dot{\varepsilon}_{xx}$. Derive the equations of motion of the beam with both types of damping.

6.18 Derive the equations of motion of the Euler–Bernoulli beam theory (see Example 6.6) when the following nonlinear strain ε_{xx} is used:

$$\varepsilon_{xx} = \frac{\partial u_0}{\partial x} + \frac{1}{2}\left(\frac{\partial w_0}{\partial x}\right)^2 - z\frac{\partial^2 w_0}{\partial x^2}.$$

6.19 Suppose that the displacements (u_1, u_2, u_3) along the three coordinate axes $x_1 = x, x_2 = y$, and $x_3 = z$ of a point (x, y, z) in the beam can be expressed as

$$\begin{aligned} u_1(x,z,t) &= u(x,t) + z\left[c_1\frac{\partial w}{\partial x} + c_2\phi(x,t)\right], \\ u_2(x,z,t) &= 0, \\ u_3(x,z,t) &= w(x,t), \end{aligned} \tag{a}$$

where (c_1, c_2) are constants, (u_1, u_2, u_3) denote the total displacements along the $x = x_1$, $y = x_2$, and $z = x_3$ coordinates, respectively, of a point (x, y, z) in the beam, $u(x,t)$ is the displacement of a point on the midplane, $w(x,t)$ is the transverse deflection of a point on the midplane, and $\phi(x,t)$ is the rotation of a transverse normal about the y-axis. Assume that the beam is subjected to distributed transverse load $q(x,t)$ and axial load $f(x,t)$, both measured per unit length.

(1) Compute the *linear* strains, using the strain–displacement relations.

(2) Use the dynamics version of the principle of virtual displacements and the following definitions of stress resultants to (i) derive the governing equations of motion (i.e., the Euler–Lagrange equations) and (ii) identify the primary and secondary variables of the theory:

$$N_x = \int_A \sigma_{xx}\,dA, \qquad Q_x = \int_A \sigma_{xz}\,dA, \qquad M_x = \int_A z\cdot\sigma_{xx}\,dA \tag{b}$$

Here N_x denotes the axial force, Q_x is the transverse (shear) force, and M_x is the bending moment. Your answer should be in terms of the stress resultants defined in Eq. (b).

6.20 Derive the equations of motion of the Timoshenko beam theory when the strains ε_{xx} and ε_{xz} are given by

$$\varepsilon_{xx} = \frac{\partial u_0}{\partial x} + \frac{1}{2}\left(\frac{\partial w_0}{\partial x}\right)^2 + z\frac{\partial \phi}{\partial x}, \qquad 2\varepsilon_{xz} = \phi + \frac{\partial w_0}{\partial x}.$$

6.21 Consider a rectangular membrane of dimensions a by b, and mass density ρ, and subjected to distributed transverse load $f(x, t)$. The membrane is fixed at all points of its boundary. If the strain energy stored in the membrane is given by

$$U = \int_0^a \int_0^b \frac{T_0}{2} \left[\left(\frac{\partial u}{\partial x} \right)^2 + \left(\frac{\partial u}{\partial y} \right)^2 \right] dxdy,$$

where T_0 is the tension in the membrane and u is the transverse displacement, determine the equation of motion.

6.22 Use the Rayleigh quotient to estimate the fundamental natural frequency of a simply supported beam. Use the static deflection of the beam under uniform load as the eigenfunction.

6.23 Use the Rayleigh quotient to estimate the fundamental natural frequency of a fixed–hinged beam. Use

$$w_1(x) = x^2 \left(2x^2 - 5xL + 3L^2 \right).$$

6.24 Use the Rayleigh quotient to estimate the fundamental natural frequency of a fixed–fixed beam. Use

$$w_1(x) = x^2 (L - x)^2.$$

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7

DIRECT VARIATIONAL METHODS

7.1 INTRODUCTION

In Chapters 5 and 6 we saw how energy principles can be used to obtain governing equations, associated boundary conditions, and, in certain simple cases, solutions for displacements and forces at selective points of a structure. However, the energy methods considered in Chapters 5 and 6 cannot be used, in general, to determine continuous solutions to complex problems.

The present chapter deals with approximate methods that employ the variational statements (i.e., either variational principles or weak formulations) to determine continuous solutions of problems of mechanics. Recall that the energy principles contain, in a single statement, the governing equation(s) and the natural boundary condition(s) of the problem. The energy principles involved setting the first variation of an appropriate functional with respect to the dependent variables to zero. The procedures of the calculus of variations were then used to obtain the governing (Euler–Lagrange) equations of the problem. In contrast, the methods described in this chapter seek a solution in terms of adjustable parameters that are determined by substituting the assumed solution into the functional and finding its extremum or stationary value with respect to the parameters. Such solution methods are called *direct methods*, because the approximate solutions are obtained directly by using the same variational principle that was used to derive the governing equations.

The assumed solutions in the variational methods are in the form of a finite linear combination of *undetermined parameters* with appropriately chosen functions. This amounts to representing a continuous function by a finite linear combination of functions. Since the solution of a continuum problem in general cannot be represented by a finite set of functions, error is introduced into the solution. Therefore, the solution obtained is an *approximation* of the true solution for the equations describing a physical problem. As the number of linearly independent terms in the assumed solution is increased, the error in the approximation will be reduced, and the assumed solution converges to the desired solution.

The equations governing a physical problem themselves are approximate. The approximations are introduced via several sources, including the geometry, the representation of specified loads and displacements, and the material behavior. In the present study, our primary interest is to determine accurate approximate solutions to appropriate analytical descriptions of physical problems.

The variational methods of approximation described here include the classical methods of Ritz, Galerkin, and Petrov–Galerkin (weighted residuals). Examples of applications of these methods are drawn from the problems of bars, beams, torsion, and membranes [1–23]. Applications of these methods to circular and rectangular plates are considered in Chapter 8. We begin with some mathematical preliminaries.

7.2 CONCEPTS FROM FUNCTIONAL ANALYSIS

7.2.1 General Introduction

Before we discuss the variational methods of approximation, it is useful to equip ourselves with certain mathematical concepts. These include the vector spaces, norm, inner product, linear independence, orthogonality, and linear and bilinear forms of functions. Since the objective of the present study is to learn about variational methods, we limit our discussion here only to concepts that are pertinent in the context. The reader already familiar with these concepts may browse through the section to gain familiarity with the notation. Others who do not wish to burden themselves with the formalism of functional analysis may skip this section; it would not prevent them from gaining an understanding of the main ideas of variational methods.

A set X is any well-defined collection of things, which are called *members* or *elements* of X. In the present study we are concerned with collections of numbers, sequences, functions, and functions of functions. Examples of sets are provided below:

1. The set $\Re$ of all real numbers.
2. The set $C[0, L]$ of all real-valued continuous functions $f(x)$ defined on the closed interval $0 \leq x \leq L$.
3. The collection of all closed intervals, $I_i = [x_i, x_{i+1}]$, on the real line.

The following notation, very standard in mathematics, is adopted here:

$$
\begin{aligned}
&\subset \text{ means "a subset of"} \\
&\not\subset \text{ means "not a subset of"} \\
&\in \text{ means "an element of"} \\
&\notin \text{ means "not a element of"}
\end{aligned}
\tag{7.1}
$$

$\forall$ means "for all"

$\exists$ means "there exists"

$\ni$ means "such that".

One way of defining a set S is to specify two pieces of information: (1) assume that each element of S is an element of a universal set (i.e., a well-known set), say X, and (2) list the properties that elements of the universal set must satisfy in order to be in S. For example, let X be the set of all sequences of complex numbers $x = \{x_1, x_2, x_3, \ldots\}$ and S be all elements of X possessing the property

$$\sum_{n=1}^{\infty} |x_n| < \infty.$$

We shall use the following notation

$$S = \left\{ x \in X \colon \sum_{n=1}^{\infty} |x_n| < \infty \right\}, \tag{7.2}$$

which is read "S is the set of all elements of X such that (the colon stands for 'such that') $\sum_{n=1}^{\infty} |x_n| < \infty$."

A set $A \subset \Re$ is said to be *bounded from above* if there exists a real number μ such that $a \leq \mu$ for all $a \in A$. The real number μ is said to be an *upper bound* of the set A. Similarly, a set A is said to be *bounded from below* if there exists a real number γ such that $a \geq \gamma$ for all $a \in A$. The real number γ is said to be a *lower bound* of the set A. If a set A is bounded from above and from below, we say that A is bounded. An upper (lower) bound $M\,(m)$ for A is said to be the maximum (minimum) of $A \subset \Re$ if $M \in A$ $(m \in A)$. It should be noted that even a bounded set need not have a maximum or a minimum. Every nonempty set of real numbers bounded from above has a "least upper bound," and every nonempty set of real numbers bounded from below has a "greatest lower bound." The least upper bound of a set A is denoted by sup A ("supremum of A"), and the greatest lower bound of A is denoted by inf A ("infimum of A").

7.2.2 Linear Vector Spaces

As we have seen in Chapter 2, the term *vector* is used often to imply a *physical* vector that has "magnitude and direction" and obeys certain rules of vector addition and scalar multiplication. These ideas can be extended to functions, which are also called vectors, provided that the rules of vector addition and scalar multiplication are defined. While the definition of a vector "from a linear vector space" does not require the vector to have a magnitude, in nearly all cases of practical interest the vector is endowed with a magnitude, called the *norm*. In such cases the vector is said to belong to a normed vector space. We begin with a formal definition of an abstract vector space.

A collection of vectors, $u, v, w, \ldots$ is called a *real linear vector space* V over the real number field $\Re$ if the following rules of vector addition and scalar multiplication of a vector are satisfied by the elements of the vector space.

Vector Addition To every pair of vectors u and v there corresponds a unique vector $u + v \in V$, called the *sum* of u and v, with the following properties:

(1a) $u + v = v + u$ (commutative);

(1b) $(u + v) + w = u + (v + w)$ (associative);

(1c) there exists a unique vector, Θ, independent of u such that $u + \Theta = u$ for every $u \in V$ (existence of an identity element);

(1d) to every u there exists a unique vector, $-u$ (that depends on u), such that $u + (-u) = \Theta$ for every $u \in V$ (existence of the additive inverse element). (7.3)

Scalar Multiplication To every vector u and every real number $\alpha \in \Re$ there corresponds a unique vector $\alpha u \in V$, called the *product* of u and α, such that the following properties hold:

(2a) $\alpha(\beta u) = (\alpha\beta)u$ (associative);

(2b) $(\alpha + \beta)u = \alpha u + \beta u$ (distributive w.r.t. the scalar addition);

(2c) $\alpha(u + v) = \alpha u + \alpha v$ (distributive w.r.t. the vector addition);

(2d) $1 \cdot u = u \cdot 1$. (7.4)

Note that in order to prove that a set of vectors qualifies as a vector space, one must define the identity and inverse elements and prove the "closure property" $u + v \in V$ and $\alpha u \in V$ for all $u, v \in V$ and $\alpha \in \Re$.

A subset S of a vector space V is called a *subspace* of V, denoted $S \subset V$, if S itself is a vector space with respect to vector addition and scalar multiplication defined over V.

Example 7.1

1. The set of ordered n-tuples $(x_1, x_2, x_3, \ldots, x_n)$ of real numbers $x_1, x_2, \ldots, x_n$ is called the *Cartesian space*, denoted $\Re^n$. A typical element of $\Re^n$ is denoted $\mathbf{x} = (x_1, x_2, x_3, \ldots, x_n)$. The Cartesian space is a linear vector space with respect to the usual rules of addition and scalar multiplication:

   Vector addition: $\mathbf{x} + \mathbf{y} = (x_1 + y_1, x_2 + y_2, \ldots, x_n + y_n) \quad \forall \mathbf{x}, \mathbf{y} \in \Re^n$.

   Scalar multiplication: $\alpha\mathbf{x} = (\alpha x_1, \alpha x_2, \ldots, \alpha x_n) \quad \forall \mathbf{x} \in \Re^n$ and $\alpha \in \Re$.

   The identity element is $\mathbf{0} = (0, 0, 0, \ldots)$ (n zeros) and the inverse element is the negative of the vector.

2. Let $\mathcal{P}$ be the set of *all* polynomials in x with real coefficients. A typical element of $\mathcal{P}$ is of the form

$$p(x) = a_0 + a_1 x + a_2 x^2 + \cdots,$$

where $a_0, a_1, \ldots$ are real numbers. Then $\mathcal{P}$ is a linear vector space with respect to the usual rules of addition and scalar multiplication. Also, the set $\mathcal{P}_n$ of polynomials of degree *less than or equal to* degree n is also a linear vector space, as can be verified (by the closure property). Moreover, $\mathcal{P}_n \subset \mathcal{P}$. However, the set of polynomials of degree *equal to* n is not a vector space as the closure property is violated. For example, consider the set of all cubic polynomials. The sum of $p_1(x) = 1 - 2x + 3x^2 + 6x^3$ and $p_2 = -3 + 5x + 2x^2 - 6x^3$ is not a cubic polynomial.

3. Let $C^n[a, b]$, where $n \geq 0$ is an integer, denote the set of all real-valued functions $u(x)$ defined on the interval $a \leq x \leq b$ such that u is continuous, and the derivatives $d^k u/dx^k$ of order $k \leq n$ exist and are continuous on $[a, b]$. It can be shown that $C^n[a, b]$ is a linear vector space with respect to the usual rules of vector addition and scalar multiplication.

4. The set

$$S_0 = \left\{u : u(x) \in C^2[0, L], -\frac{d}{dx}\left(a(x)\frac{du}{dx}\right) + c(x)u = 0, \qquad 0 < x < L\right\}$$

is a vector space with respect to the usual addition and scalar multiplication. However, the set

$$S = \left\{u : u(x) \in C^2[0, L], -\frac{d}{dx}\left(a(x)\frac{du}{dx}\right) + c(x)u = f(x), \qquad 0 < x < L\right\}$$

is not a linear vector space (why?).

5. Consider the transverse motion of a cable of length L, fixed at its ends (see Fig. 7.1). Let $C[0, L]$ denote the set of all real-valued, continuous functions $u(x, t)$ defined on the closed interval $0 \leq x \leq L$ for any time t. The transverse deflection $u(\cdot, t)$ (i.e., configuration) of the cable at any time t can be viewed as an element of $C[0, L]$. However, not every element of $C[0, L]$ is a possible

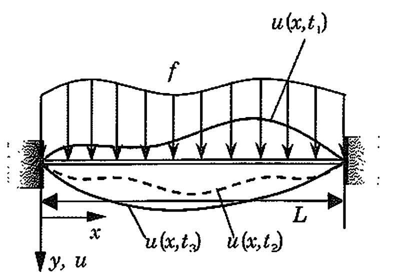

Figure 7.1 Transverse motion of a cable fixed at both ends.

configuration of the cable because all possible configurations must pass through the points $x = 0$ and $x = L$; i.e., the boundary conditions $u(0, t) = 0$ and $u(L, t) = 0$ must be satisfied. Let S be the subset of $C[0, L]$ made up of all real-valued, continuous functions $u(x, t)$ such that $u(0, t) = 0$ and $u(L, t) = 0$:

$$S = \{u\colon u(x) \in C[0, L], \qquad u(0, t) = 0, u(L, t) = 0\}.$$

Then S is a subspace of $C[0, L]$, and all possible configurations (i.e., deflections) are contained in this space.

Let U and V be each a linear vector space. An *ordered pair* is a pair of elements $u \in U$ and $v \in V$ where one of the elements is designated as the first member of the pair and the other is designated as the second. We denote ordered pairs by (u, v) with the obvious order. Then $U \times V$ is called a *product space* W with elements $w = (u, v)$, $u \in U$ and $v \in V$:

$$W = \{w\colon w = (u, v),\ u \in U,\ v \in V\},$$

which is also a linear vector space with respect to the following definitions of vector addition and scalar multiplication of a vector in the product space $U \times V$:

$$(u_1, v_1) + (u_2, v_2) = (u_1 + u_2, v_1 + v_2), \tag{7.5a}$$

$$\alpha(u, v) = (\alpha u, \alpha v), \qquad \alpha \in \Re \tag{7.5b}$$

for $(u_1, v_1), (u_2, v_2) \in W = U \times V$ with $u_1, u_2 \in U$ and $v_1, v_2 \in V$.

Consider an open-bounded domain $\Omega \subset \Re^3$. Note that Ω is a set of points $\mathbf{x} = (x_1, x_2, x_3)$. A real-valued function $u(\mathbf{x})$ is said to be *square-integrable* in the domain Ω if the integrals (in the Lebesgue sense)

$$\int_\Omega u(\mathbf{x})\, d\mathbf{x}, \qquad \int_\Omega |u(\mathbf{x})|^2\, d\mathbf{x} \tag{7.6}$$

exist and are finite. The space of square-integrable functions u defined over a domain Ω is called the L_2 space:

$$L_2(\Omega) = \left\{u(\mathbf{x})\colon \int_\Omega |u(\mathbf{x})|^2\, d\mathbf{x}\right\}. \tag{7.7}$$

There is a corresponding space $L_\infty(\Omega)$, which consists of all real-valued functions $u(\mathbf{x})$ defined in the domain Ω such that there exists an N with the property that

$$|u(\mathbf{x})| \leq N.$$

Linear Independence Recall the concepts of coplanar and collinear vectors in Euclidean space from Chapter 2. These concepts can be generalized to function spaces. An expression of the form

$$\alpha_1 u_1 + \alpha_2 u_2 + \cdots = \sum_{i=1}^{n} \alpha_i u_i \tag{7.8}$$

for all functions $u_i(x)$ and scalars $\alpha_i \in \Re$ (real number field) is called a *linear combination* of u_i. The equation $\sum_i^n \alpha_i u_i = 0$ is called a *linear relation* among the functions u_i. A set of n functions, $u_1, u_2, \ldots, u_n$, is said to be *linearly dependent* if a set of n numbers, $\alpha_1, \alpha_2, \ldots, \alpha_n$, not all of which are zero, can be found such that the following linear relation holds:

$$\sum_{i=1}^{n} \alpha_i u_i = 0. \tag{7.9}$$

If there does not exist at least one nonzero number among α_i such that the above relation is satisfied, the vectors are said to be *linearly independent.*

Example 7.2

1. Consider the following set of polynomials, $\{p_i\}$, with

$$p_1(x) = 1 + x, \qquad p_2(x) = 1 + x^2, \qquad p_3(x) = 1 + x + x^3.$$

Consider the linear relation

$$\alpha_1 p_1 + \alpha_2 p_2 + \alpha_3 p_3 = 0$$

for $\alpha_i \in \Re$. Since the above relation must hold for all x, it follows that the coefficients of powers of x must be zero separately. Collecting the coefficients of various powers of x and setting them to zero, we obtain

$$\alpha_1 + \alpha_2 + \alpha_3 = 0, \qquad \alpha_1 + \alpha_3 = 0, \qquad \alpha_2 = 0, \qquad \alpha_3 = 0.$$

The solution to these equations is trivial (i.e., all $\alpha_i = 0$); hence, the set $\{p_1, p_2, p_3\}$ is linearly independent.

2. If p_3 is replaced by $p_4 = 2 + x + x^2$, we see that the linear relation

$$\alpha_1 p_1 + \alpha_2 p_2 + \alpha_4 p_4 = 0$$

requires that

$$\alpha_1 + \alpha_2 + 2\alpha_4 = 0, \qquad \alpha_1 + \alpha_4 = 0, \qquad \alpha_2 + \alpha_4 = 0.$$

An infinite number of solutions to the above set of equations exists. For example,

$$\alpha_4 = 1, \qquad \alpha_1 = \alpha_2 = -1$$

is a solution. Hence, the set $\{p_1, p_2, p_4\}$ is linearly dependent. Indeed, p_4 can be expressed as a linear combination of p_1 and p_2:

$$p_4 = \alpha_1 p_1 + \alpha_2 p_2, \qquad \alpha_1 = \alpha_2 = 1.$$

7.2.3 Normed and Inner Product Spaces

Norm The concepts of distance between two points and length of a physical vector can be generalized to abstract vectors, i.e., vectors that are functions. Let V be a linear vector space over the real number field $\Re$. We shall use the notation $\|\cdot\|$ to denote the norm of real-valued functions $u(\mathbf{x})$, $\mathbf{x} \in \Omega \subset \Re^3$. Then, associated with every vector $u \in V$, there exists a real number $\|\cdot\| \in \Re$, called the *norm*, that satisfies certain rules, as discussed below. Thus the norm is the operation, $\|\cdot\| : V \to \Re$.

(1) Nonnegative:

(a) $\|u\| \geq 0$ for all u.

(b) $\|u\| = 0$ only if $u = 0$. (7.10)

(2) Homogeneous: $\|\alpha u\| = |\alpha| \; \|u\|$.

(3) Triangle inequality: $\|u + v\| \leq \|u\| + \|v\|$.

If $\|u\|$ satisfies (1a), (2), and (3), it is called a *seminorm*, and is denoted by $|u|$.

A linear vector space endowed with a norm is called a *normed vector space*. A linear subspace S of a normed vector space V is a linear subspace equipped with the norm of V.

A norm $\|\cdot\|$ can be used to define a notion of distance between vectors, called *natural metric*:

$$d(u, v) \equiv \|u - v\| \qquad \text{for } u, v \in V. \tag{7.11}$$

Examples of norms will be given shortly.

For $1 \leq p \leq \infty$, we define the *Lebesgue spaces* [see Eq. (7.7)]:

$$L_p(\Omega) = \{u : \|u\|_p < \infty\}, \tag{7.12}$$

where

$$\|u\|_{L_p(\Omega)} \equiv \|u\|_p = \left[\int_\Omega |u(\mathbf{x})|^p \, d\mathbf{x}\right]^{1/p} < \infty. \tag{7.13}$$

For $p = \infty$ we set

$$\|u\|_{L_\infty(\Omega)} \equiv \|u\|_\infty = \sup\{|u(\mathbf{x})| : \mathbf{x} \in \Omega\}. \tag{7.14}$$

This is called the "sup-norm."

Two norms $\|\cdot\|_1$ and $\|\cdot\|_2$ on a normed vector space V are said to be *equivalent* if there exist positive numbers c_1 and c_2, independent of $u \in V$, such that the following double inequality holds:

$$c_1\|u\|_1 \leq \|u\|_2 \leq c_2\|u\|_1. \tag{7.15}$$

A normed space V is called *complete* if every Cauchy sequence $\{u_j\}$ of elements of V has a limit $u \in V$. For a normed vector space, a Cauchy sequence is one such that

$$\|u_j - u_k\| \to 0 \qquad \text{as } j, k \to \infty,$$

and completeness means that

$$\|u - u_j\| \to 0 \qquad \text{as } j \to \infty.$$

A normed vector space which is complete in its natural metric is called a *Banach space*. A linear subspace of a Banach space is itself a Banach space if and only if the subspace is complete.

Example 7.3

1. The n-dimensional Euclidean space $\Re^n$ is a Banach space with respect to the *Euclidean norm*:

$$\|\mathbf{x}\| \equiv \sqrt{\sum_{i=1}^{n} x_i^2}. \tag{7.16}$$

2. The space $C[0, 1]$ of real-valued continuous functions $f(x)$ defined on the closed interval $[0, 1]$ with the sup-norm (7.14) is a Banach space. It is a linear vector space with respect to the vector addition and scalar multiplication defined as

$$(f+g)(x) = f(x) + g(x), \qquad (\alpha f)(x) = \alpha f(x), \qquad \alpha \in \Re.$$

Further, it is complete with respect to the sup-norm in (7.14):

$$\|f\|_\infty \equiv \max |f(x)|.$$

3. *Sobolev space*, $W^{m,p}(\Omega)$. Let $C^m(\Omega)$ denote the set of all real-valued functions with m continuous derivatives defined in $\Omega \in \Re^3$, and let $C^\infty(\Omega)$ denote the set of infinitely differentiable continuous functions. We define on $C^m(\Omega)$ the norm, called the *Sobolev norm*,

$$\|u\|_{m,p} = \left[\int_\Omega \sum_{|\alpha| \leq m} |D^\alpha u(\mathbf{x})|^p \, d\mathbf{x}\right]^{1/p}, \tag{7.17}$$

for $1 \leq p \leq \infty$ and for all $u \in C^m(\Omega)$. In Eq. (7.17), α denotes an n-tuple of integers:

$$\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n), \qquad |\alpha| = \sum_i^n \alpha_i, \quad \alpha_i \geq 0,$$
$$D^\alpha = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \cdots \partial x_n^{\alpha_n}}. \tag{7.18}$$

For $m = 1$, $n = 2$, and $1 \leq p < \infty$, we have $[\alpha = (\alpha_1, \alpha_2), \alpha_1, \alpha_2 = 0, 1]$, and

$$\|u\|_{1,p} = \left\{ \int_{\Omega \subset \Re^2} \left[|u|^p + \left| \frac{\partial u}{\partial x} \right|^p + \left| \frac{\partial u}{\partial y} \right|^p \right] dxdy \right\}^{1/p} \tag{7.19}$$

The space $C^m(\Omega)$ is not complete with respect to the Sobolev norm $\|\cdot\|_{m,p}$. The completion of $C^m(\Omega)$ with respect to the norm $\|\cdot\|_{m,p}$ is called the *Sobolev space of order* (m, p), denoted by $W^{m,p}(\Omega)$. The completion of $C(\Omega)$ is the $L_2(\Omega)$ space. Hence the Sobolev space is a Banach space. Of course, the Lebesque space $L_p(\Omega)$ is a special case of the Sobolev space $W^{m,p}$ for $m = 0$, and $L_2(\Omega)$ is a special case of $L_p(\Omega)$ for $p = 2$, with the norms defined in (7.13).

If U and V are each normed vector spaces, we can define a norm on the product space $U \times V$ in one of the following ways:

$$\begin{aligned} &(1) \quad \|(u, v)\| = \|u\|_U + \|v\|_V. \\ &(2) \quad \|(u, v)\| = \left(\|u\|_U^p + \|v\|_V^p\right)^{1/p}, \qquad p \geq 1. \\ &(3) \quad \|(u, v)\| = \max\,(\|u\|_U, \|v\|_V). \end{aligned} \tag{7.20}$$

Then $U \times V$ is a normed vector space with respect to any one of the above norms.

Inner Product Analogous to the scalar product of physical vectors, the *inner product* of a pair of vectors u and v from an abstract vector space V is defined to be a real number, denoted $(u, v)_V$ [i.e., $(\cdot, \cdot)_V : V \times V \to \Re$], which satisfies the following rules for every $u_1, u_2, u, v \in V$ and $\alpha \in \Re$:

$$\begin{aligned} &(1) \quad \text{Symmetry: } (u, v)_V = (v, u)_V. \\ &(2a) \quad \text{Homogeneous: } (\alpha u, v)_V = \alpha(u, v)_V. \\ &(2b) \quad \text{Additive: } (u_1 + u_2, v)_V = (u_1, v)_V + (u_2, v)_V. \\ &(3) \quad \text{Positive-definite: } (u, u)_V > 0 \text{ for all } u \neq 0. \end{aligned} \tag{7.21}$$

One can define a number of inner products and associated *natural norms* for pairs of functions that, along with their derivatives, are square-integrable. In particular, the Sobolev space $W^{m,2}(\Omega) \equiv H^m(\Omega)$, which is also known as the *Hilbert space* of order m, is endowed with the inner product

$$(u, v)_m = \int_\Omega \sum_{|\alpha| \le m} D^\alpha u(\mathbf{x}) D^\alpha v(\mathbf{x})\, d\mathbf{x}, \tag{7.22}$$

for all $u, v \in H^m(\Omega)$. Note that for $m = 0$, we have $H^0(\Omega) = L_2(\Omega)$. Some special cases of Eq. (7.22) are given by

$$(u, v)_0 = \int_\Omega uv\, dxdy, \qquad \|u\|_0 = \sqrt{(u, u)_0}, \tag{7.23a}$$

$$(u, v)_1 = \int_\Omega \left(uv + \frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\frac{\partial v}{\partial y} \right) dxdy, \qquad \|u\|_1 = \sqrt{(u, u)_1}, \tag{7.23b}$$

$$(u, v)_2 = \int_\Omega \left(uv + \frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\frac{\partial v}{\partial y} + \frac{\partial^2 u}{\partial x \partial y}\frac{\partial^2 v}{\partial x \partial y} \right.$$
$$\left. + \frac{\partial^2 u}{\partial x^2}\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\frac{\partial^2 v}{\partial y^2} \right) dxdy, \qquad \|u\|_2 = \sqrt{(u, u)_2}. \tag{7.23c}$$

A linear vector space on which an inner product can be defined is called an *inner product space*. A linear subspace S of an inner product space V is a subspace with the inner product of V. Note that the square root of the inner product of a vector with itself satisfies the axioms of a norm. Consequently, one can associate with every inner product in vector space V a norm

$$\|u\|_V = \sqrt{(u, u)_V}. \tag{7.24}$$

The norm thus obtained is called the *norm induced by the inner product.* Since we can associate with each inner product a norm, every inner product space is also a normed vector space. It should be obvious to the reader that the converse does not hold in general.

Orthogonality Two vectors $u, v \in V$ are said to be *orthogonal* if

$$(u, v)_V = 0, \tag{7.25}$$

where $(\cdot, \cdot)_V$ denotes an inner product in V. Note that the concept of orthogonality is a generalization of the familiar notion of perpendicularity of one vector to another in Euclidean space. A set of mutually orthogonal vectors is called an orthogonal set.

A sequence of functions $\{\phi_i\}$ in $L_2(\Omega)$ is called *orthonormal* if

$$(\phi_i, \phi_j)_V = \delta_{ij} = \begin{cases} 1, & \text{if } i = j, \\ 0, & \text{if } i \ne j. \end{cases} \tag{7.26}$$

Here δ_{ij} denotes the Kronecker delta. It can be shown that every orthonormal system is linearly independent.

If two vectors u and v of an inner product space V are orthogonal, then the Pythagorean theorem holds even in function spaces:

$$\|u+v\|_V^2 = (u+v, u+v)_V = (u,u)_V + 2(u,v)_V + (v,v)_V = \|u\|_V^2 + \|v\|_V^2.$$

A set of functions $\{\phi_i\}$ is said to be *complete* in $L_2(\Omega)$ if every piecewise continuous function f can be approximated in Ω by the sum $\sum_{j=1}^{n} c_j\phi_j$ in such a way that

$$\mathcal{E}_n \equiv \int_\Omega \left(f - \sum_{j=1}^{n} c_j\phi_j \right)^2 d\mathbf{x} \tag{7.27}$$

can be made as small as we wish (by increasing n). This property of the coordinate functions is the key to the proof of the convergence of the Ritz and Galerkin approximations.

A complete (in its natural metric) inner product space is called a *Hilbert space*. We mention without proof the fact that every inner product space (hence a normed space) has a completion.

The following lemma, referred to as the *fundamental lemma of variational calculus*, plays an important role in variational theory.

Lemma 7.1 Let V be an inner product space. If $(u, v)_V = 0$ for all $v \in V$, then $u = 0$.

Proof: Since $(u, v)_V = 0$ for all v, it must also hold for $v = u$. Then $(u, u)_V = 0$ implies that $u = 0$.

7.2.4 Transformations, and Linear and Bilinear Forms

A transformation T from a linear vector space U into another linear vector space V (both vector spaces are defined on the same field of scalars) is a correspondence that assigns to each element $u \in U$ a unique element $v = Tu \in V$. We use the terms *transformation*, *mapping*, and *operator* interchangeably, and the transformation is expressed as $T : U \to V$.

A transformation $T : U \to V$, where U and V vector spaces that have the same scalar field, is said to be *linear* if

1. $T(\alpha u) = \alpha T(u)$, for all $u \in U$, $\alpha \in \Re$ (homogeneous);
2. $T(u_1 + u_2) = T(u_1) + T(u_2)$, for all $u_1, u_2 \in U$, (additive). (7.28)

Otherwise it is said to be a *nonlinear transformation*.

Transformations that map vectors (functions) into real numbers are of special interest in the present study. Such transformations are called *functionals*. A linear transformation $l : V \to \Re$ that maps a linear vector space V into the real number field $\Re$ is called a *linear functional*.

Similarly, a linear transformation that maps pairs of vectors $(u, v) \in V \times V$ into real number field $\Re$, or $B(\cdot, \cdot) : V \times V \to \Re$, is called a *bilinear form*. Examples of linear and bilinear forms are provided by

$$l(u) = \int_a^b f u \, dx, \qquad B(u, v) = \int_a^b \left(\frac{du}{dx}\frac{dv}{dx} + uv \right) dx.$$

A bilinear form is said to be *symmetric* if it is symmetric in its arguments:

$$B(u, v) = B(v, u). \tag{7.29}$$

7.2.5 Minimum of a Quadratic Functional

Consider an operator equation of the form

$$Au = f, \tag{7.30}$$

where A is a certain operator (often a differential operator), $A : \mathcal{D}_A \to H$, and $f \in H$ is a given function. Here $\mathcal{D}_A$ denotes a set of elements from a Hilbert space H. The denseness of $\mathcal{D}_A$ in H is often assumed, but we will not discuss this topic in the present study.

The differential equations

$$-\frac{d}{dx}\left(EA \frac{du}{dx} \right) = f(x), \qquad EA > 0, \qquad 0 < x < L, \tag{7.31}$$

$$\frac{d^2}{dx^2}\left(EI \frac{d^2 u}{dx^2} \right) = f(x), \qquad EI > 0, \qquad 0 < x < L, \tag{7.32}$$

are special cases of the operator equation (7.30) with

$$A = -\frac{d}{dx}\left(EA \frac{d(\cdot)}{dx} \right), \qquad A = \frac{d^2}{dx^2}\left(EI \frac{d^2(\cdot)}{dx^2} \right),$$

respectively. In these cases, $H = L_2(0, L)$. The set $\mathcal{D}_A$ for Eq. (7.31) consists of functions from $C^2(0, L)$ and for (7.32) functions from $C^4(0, L)$.

An operator $A : \mathcal{D}_A \to H$ is called *symmetric* (or *self-adjoint*) if

$$(Au, v)_H = (u, Av)_H \tag{7.33}$$

holds for all $u, v \in \mathcal{D}_A$, where $(\cdot, \cdot)_H$ is the inner product in H. An operator A is called *strictly positive* in $\mathcal{D}_A$ if it is symmetric in $\mathcal{D}_A$ and if

$$(Au, u)_H > 0 \text{ holds for all } u \in \mathcal{D}_A \quad \text{and} \quad u \neq 0, \tag{7.34a}$$

$$(Au, u)_H = 0 \text{ if and only if } u \in \mathcal{D}_A \quad \text{and} \quad u = 0. \tag{7.34b}$$

A *quadratic functional* $Q: H \to \Re$ is one that is quadratic in its arguments, $Q(\alpha u) = \alpha^2 Q(u)$ for $\alpha \in \Re$. Every bilinear form $B(\cdot, \cdot)$ can be used to generate a quadratic form Q by setting

$$Q(u) = B(u, u), \qquad u \in H. \tag{7.35}$$

The following results are of fundamental importance for the present study.

Theorem 7.1 If A is a strictly positive operator in $\mathcal{D}_A$, then

$$Au = f \qquad \text{in } H$$

has at most one solution $u \in \mathcal{D}_A$ in H.

Proof: Suppose that there exist two solutions $u_1, u_2 \in \mathcal{D}_A$. Then

$$Au_1 = f \quad \text{and} \quad Au_2 = f \;\to\; A(u_1 - u_2) = 0 \quad \text{in } H$$

and

$$(A(u_1 - u_2), u_1 - u_2)_H = 0 \;\to\; u_1 - u_2 = 0 \quad \text{or} \quad u_1 = u_2 \in \mathcal{D}_A,$$

which was to be proved.

Theorem 7.2 Let A be a positive operator in $\mathcal{D}_A$, $f \in H$. Let Eq. (7.30) have a solution $u_0 \in \mathcal{D}_A$. Then the quadratic functional

$$I(u) = \tfrac{1}{2}(Au, u)_H - (f, u)_H \tag{7.36}$$

assumes its minimal value in $\mathcal{D}_A$ for the element u_0, i.e.,

$$I(u) \geq I(u_0), \qquad \text{and} \qquad I(u) = I(u_0) \qquad \text{only for } u = u_0.$$

Conversely, if $I(u)$ assumes its minimal value, among all $u \in \mathcal{D}_A$, for the element u_0, then u_0 is the solution of Eq. (7.30) (i.e., $Au_0 = f$).

Proof: First note that $I(u)$ is defined for all $u \in \mathcal{D}_A$. Let u_0 be the solution of Eq. (7.30). Then $f = Au_0$. Substituting for f into Eq. (7.36), we obtain for $u \in \mathcal{D}_A$:

$$\begin{aligned} I(u) &= \tfrac{1}{2}(Au, u)_H - (Au_0, u)_H \\ &= \tfrac{1}{2}\left[(Au, u)_H - (Au_0, u)_H - (u, Au_0)_H\right] \\ &= \tfrac{1}{2}\left[(Au, u)_H - (Au_0, u)_H - (Au, u_0)_H\right] \\ &= \tfrac{1}{2}\left[(Au, u)_H - (Au_0, u)_H - (Au, u_0)_H + (Au_0, u_0)_H - (Au_0, u_0)_H\right] \\ &= \tfrac{1}{2}\left[(A(u - u_0), u - u_0)_H - (Au_0, u_0)_H\right], \end{aligned} \tag{7.37}$$

where the linearity and symmetry of A, as well as the symmetry of the bilinear form, are used in arriving at the last step. From Eq. (7.37) it follows that

$$I(u_0) = -\tfrac{1}{2}(Au_0, u_0)_H. \tag{7.38}$$

Next, we use the strictly positive property of A to conclude that

$$I(u) \geq I(u_0) \quad \text{for } u \in \mathcal{D}_A, \quad \text{and} \quad I(u) = I(u_0)$$

if and only if $u = u_0$ in $\mathcal{D}_A$. Consequently, if the equation $Au_0 = f$ is satisfied, then the functional $I(u)$ assumes its minimal value in $\mathcal{D}_A$ precisely for the element $u = u_0$.

Now suppose that $I(u)$ assumes its minimal value in $\mathcal{D}_A$ for the element u_0. This implies that

$$I(u_0 + \alpha v) \geq I(u_0) \quad \text{for } \alpha \in \Re,\ v \in \mathcal{D}_A. \tag{7.39}$$

Using again the symmetry of A and the symmetry of the inner product, one obtains

$$\begin{aligned} I(u_0 + \alpha v) &= \tfrac{1}{2}(A(u_0 + \alpha v), u_0 + \alpha v)_H - (f, u_0 + \alpha v)_H \\ &= \tfrac{1}{2}\left[(Au_0 + \alpha Av, u_0 + \alpha v)_H - 2(f, u_0)_H - 2\alpha(f, v)_H\right] \\ &= \tfrac{1}{2}\Big[(Au_0, u_0)_H + \alpha(Av, u_0)_H + \alpha(Au_0, v)_H \\ &\qquad + \alpha^2(Av, v)_H - 2(f, u_0)_H - 2\alpha(f, v)_H\Big] \\ &= \tfrac{1}{2}\Big[(Au_0, u_0)_H + 2\alpha(Au_0, v)_H + \alpha^2(Av, v)_H \\ &\qquad - 2(f, u_0)_H - 2\alpha(f, v)_H\Big]. \end{aligned} \tag{7.40}$$

Since $u_0 \in \mathcal{D}_A$ and $f \in H$ are fixed elements, it is obvious that for arbitrarily fixed $v \in \mathcal{D}_A$ the function $I(u_0 + \alpha v)$ is a quadratic function in the variable α. From Eq. (7.39) it follows that the function has a local minimum at $\alpha = 0$, which implies that its first derivative is equal to zero at $\alpha = 0$ [or, equivalently, the first variation of I is zero; see Eq. (4.89)]:

$$\left[\frac{d}{d\alpha} I(u_0 + \alpha v)\right]_{\alpha=0} = 0,$$

or by Eq. (7.40) that

$$(Au_0, v)_H - (f, v)_H = 0 \quad \text{or} \quad (Au_0 - f, v)_H = 0.$$

Since $v \in H$ is arbitrary, by Lemma 7.1 it follows that $Au_0 - f = 0$ in H.

Example 7.4 Consider the differential equation

$$-\frac{d}{dx}\left(a(x)\frac{du}{dx}\right) = f(x), \qquad a(x) > 0, \qquad 0 < x < L, \tag{7.41}$$

subjected to the boundary conditions

$$u(0) = 0, \qquad u(L) = 0, \tag{7.42}$$

which arise in connection with the transverse deflection of cables. Here $u(x)$ denotes the deflection of a cable of original length L, tension $a = a(x)$, and subjected to distributed transverse load $f(x)$ (see Fig. 7.2). The boundary conditions in Eq. (7.42) indicate that the cable is fixed at $x = 0$ and $x = L$.

Let us choose $H = L_2(0, L)$, and define $\mathcal{D}_A$ as the linear set of functions that are continuous with their derivatives up to the second order inclusive in the interval $[0, L]$ and satisfy the end conditions in (7.42). Define the operator A on $\mathcal{D}_A$ by

$$Au = -\frac{d}{dx}\left(a(x)\frac{du}{dx}\right). \tag{7.43}$$

We now set out to prove that A is strictly positive on $\mathcal{D}_A$. First, we note that A is symmetric in $\mathcal{D}_A$: For every $u \in \mathcal{D}_A$ and $v \in \mathcal{D}_A$, we have

$$\begin{aligned}
(Au, v)_H &= \int_0^L \left[-\frac{d}{dx}\left(a\frac{du}{dx}\right)\right] v\,dx \\
&= -\left[a\frac{du}{dx}v\right]_0^L + \int_0^L \frac{dv}{dx}\left(a\frac{du}{dx}\right)dx \\
&= \int_0^L a(x)\frac{dv}{dx}\frac{du}{dx}\,dx \qquad (7.44)\\
&= \left[a\frac{dv}{dx}u\right]_0^L + \int_0^L \left[-\frac{d}{dx}\left(a\frac{dv}{dx}\right)\right]u\,dx \\
&= \int_0^L \left[-\frac{d}{dx}\left(a\frac{dv}{dx}\right)\right]u\,dx = (u, Av)_H, \qquad (7.45)
\end{aligned}$$

where we have used the fact that $u(0) = u(L) = v(0) = v(L) = 0$. Thus, A is symmetric on $\mathcal{D}_A$. From Eq. (7.44), it follows that

$$(Au, u)_H = \int_0^L a(x)\left(\frac{du}{dx}\right)^2 dx \qquad \text{for all } u \in \mathcal{D}_A. \tag{7.46}$$

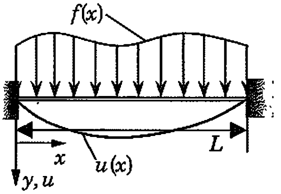

Figure 7.2 Transverse deflection of a cable fixed at its ends.

Due to the fact that $a(x) > 0$ in $[0, L]$, it follows from (7.46) that

$$(Au, u)_H \geq 0 \qquad \text{for every } u \in \mathcal{D}_A.$$

Moreover, if $(Au, u)_H = 0$, it follows that

$$\frac{du}{dx} = 0 \qquad \text{in } [0, L].$$

This in turn implies that

$$u(x) = c, \text{ constant in } [0, L].$$

Since $u(0) = 0$, it follows that $c = 0$ or $u(x) = 0$ in $[0, L]$. This proves that A is strictly positive in $\mathcal{D}_A$.

The quadratic functional associated with Eqs. (7.41) and (7.42) is given by

$$\Pi(u) = \frac{1}{2}(Au, u)_H - (f, u)_H = \frac{1}{2}\int_0^L a(x)\left(\frac{du}{dx}\right)^2 dx - \int_0^L fu\,dx, \qquad (7.47)$$

which represents the total potential energy of the cable. The first term is the elastic strain energy and the second term is the potential energy of the external load $f(x)$.

Let $u_0(x)$ be the solution of Eqs. (7.41) and (7.42). Then $f = Au_0$ and we have

$$\begin{aligned}
\Pi(u) &= \frac{1}{2}(Au, u)_H - (Au_0, u)_H \\
&= \frac{1}{2}\int_0^L a(x)\left(\frac{du}{dx}\right)^2 dx - \int_0^L \left[-\frac{d}{dx}\left(a(x)\frac{du_0}{dx}\right)\right] u\,dx \\
&= \frac{1}{2}\int_0^L a(x)\left(\frac{du}{dx}\right)^2 dx - \int_0^L a(x)\frac{du_0}{dx}\frac{du}{dx}\,dx \\
&= \frac{1}{2}\int_0^L a(x)\left(\frac{du}{dx} - \frac{du_0}{dx}\right)^2 dx - \frac{1}{2}\int_0^L a(x)\left(\frac{du_0}{dx}\right)^2 dx. \qquad (7.48)
\end{aligned}$$

Since $a(x)(u' - u_0')^2 \geq 0$, it is clear from Eq. (7.48) that $\Pi(u)$ is minimal in $\mathcal{D}_A$ if and only if $u' = u_0'$ in $\mathcal{D}_A$. Thus, if $u_0 \in \mathcal{D}_A$ is the solution of Eqs. (7.41) and (7.42), then the functional $\Pi(u)$ assumes its minimum for $u_0 \in \mathcal{D}_A$.

Conversely, let $u_0 \in \mathcal{D}_A$ be the element minimizing the functional $\Pi(u)$ in (7.47). Let $v \in \mathcal{D}_A$ be an arbitrary element from $\mathcal{D}_A$ and let α be an arbitrary real number. Then the minimum of $\Pi(u)$ implies that $(u = u_0 + \alpha v)$

$$\begin{aligned}
0 &= \left.\frac{d}{d\alpha}\Pi(u_0 + \alpha v)\right|_{\alpha=0} \\
&= \frac{d}{d\alpha}\left[\frac{1}{2}\int_0^L a(x)\left(\frac{du}{dx}\right)^2 dx - \int_0^L fu\,dx\right]_{\alpha=0}
\end{aligned}$$

$$= \int_0^L a(x)\frac{du_0}{dx}\frac{dv}{dx}\,dx - \int_0^L fv\,dx$$

$$= \int_0^L \left[-\frac{d}{dx}\left(a(x)\frac{du_0}{dx}\right) - f\right] v\,dx.$$

Since this result must hold for every v, it follows that

$$-\frac{d}{dx}\left(a(x)\frac{du_0}{dx}\right) - f = 0 \qquad \text{in } H = L_2(0, L).$$

The above arguments are equivalent to the principle of minimum total potential energy discussed in Chapter 5.

Recall that the operator $A : \mathcal{D}_A \to H$ is symmetric:

$$(Au, v)_H = (u, Av)_H \qquad \text{for } u, v \in \mathcal{D}_A$$

and positive definite on $\mathcal{D}_A$, i.e., there exists a constant $C > 0$ so that

$$(Au, u)_H \geq C\|u\|^2 \qquad \text{holds for every } u \in \mathcal{D}_A. \tag{7.49}$$

Hence, we can define a new inner product $(u, v)_A$ on $\mathcal{D}_A$ as follows:

$$(u, v)_A = (Au, v)_H \qquad \text{for all } u, v \in \mathcal{D}_A. \tag{7.50}$$

It can be easily verified that $(u, v)_A$ satisfies the axioms (1)–(3) in Eq. (7.21) of an inner product. The linear set $\mathcal{D}_A$ with the inner product (7.50) constitutes a linear vector space, called the *energy space*, and denoted by H_A. The norm and natural metric follow from the definition in (7.50):

$$\|u\|_A^2 = (u, v)_A, \qquad d(u, v) = \|u - v\|_A. \tag{7.51}$$

The energy space H_A can be shown to be complete with respect to the metric defined in (7.51), and hence it is a Hilbert space. Moreover, it can be shown that the functional $\Pi(u)$ can be extended to all elements of H_A, that the functional assumes its minimum at $u_0 \in H_A$, and that the element u_0 is uniquely determined by the element $f \in H$. These proofs are beyond the scope of the present study, and interested readers may consult Refs. [4], and [12–15].

7.3 THE RITZ METHOD

7.3.1 Introduction

As discussed in Chapters 5 and 6, the principles of virtual displacements and forces as applied to continuous systems can be used to determine the governing equations and natural boundary conditions of the problem. The energy methods (e.g., unit-dummy-force and unit-dummy-displacement methods and Castigliano's Theorems I and II) derived from these principles were used to determine deflections and forces at selected points. Here we consider a powerful method of determining approximate solutions to the governing equations of a problem by directly using the variational statements (i.e., virtual work principles, the principle of total potential energy, or the principle of complementary energy). The method bypasses the derivation of the Euler equations and goes directly from a variational statement of the problem to the solution of the Euler equations. One such direct method was proposed by German engineer W. Ritz (1878–1909).

7.3.2 Description of the Method

Consider the linear operator equation

$$Au = f \qquad \text{in } \Omega, \tag{7.52}$$

where A is a strictly positive operator from H_A into H, and $f \in H$. The solution u_0 of Eq. (7.52) is the element $u_0 \in H_A$ that minimizes the quadratic functional

$$I(u) = \frac{1}{2}(u, u)_A - l(u), \tag{7.53}$$

where $l(\cdot)$ denotes a linear functional. In structural mechanics problems, the functional $I(u)$ represents the total potential energy and $\delta I(u) = 0$ yields Eq. (7.52) as the Euler equation.

We seek an approximation $U_N(x)$ of $u_0(x)$, for a fixed and preselected N, in the form

$$u_0(x) \approx U_N(x) = \sum_{i=1}^{N} c_i \phi_i(x) + \phi_0(x), \tag{7.54}$$

where $\phi_i(x)$ are the elements of a base in H_A, and c_i are as yet unknown real constants. These constants are determined by the condition that $I(U_N)$ is the minimum. Since $\{\phi_i\}$ is a base in H_A, the solution $u_0 \in H_A$ can be approximated to arbitrary accuracy by a suitable linear combination of its elements. Therefore, it can be expected that the approximate solution U_N, with constants determined by minimizing the functional $I(U_N)$, will differ sufficiently slightly in H_A from the actual solution u_0 if N is selected sufficiently large. This process of determining U_N is known as the Ritz method.

To fully illustrate the basic idea of the Ritz method described above, we consider the axial deformation of a nonuniform bar with an end spring. The governing equation is

$$-\frac{d}{dx}\left(EA(x)\frac{du}{dx}\right) = f(x), \qquad 0 < x < L, \tag{7.55a}$$

subjected to the boundary conditions

$$u(0) = 0, \qquad \left[\left(EA(x)\frac{du}{dx}\right) + ku(x)\right]_{x=L} = P, \tag{7.55b}$$

where E denotes Young's modulus, $A = A(x)$ the area of cross section, L the length, k the spring constant, $f(x)$ the distributed axial load, and P the axial load at $x = L$ (see Fig. 7.3). The space H_A in this case is the completion of the set $\mathcal{D}_A$ of functions that are continuous with their derivatives up to the second order in $[0, L]$.

As discussed earlier, the problem is equivalent to minimizing the total potential energy functional Π:

$$\Pi(u) = \int_0^L \left[\frac{EA}{2} \left(\frac{du}{dx} \right)^2 - fu \right] dx + \frac{k}{2} [u(L)]^2 - Pu(L). \tag{7.56}$$

The necessary condition for the minimum of Π is

$$0 = \delta\Pi = B(\delta u, u) - l(\delta u) \quad \text{or} \quad B(\delta u, u) = l(\delta u), \tag{7.57a}$$

where $B(\cdot, \cdot)$ is the bilinear form and $l(\cdot)$ is the linear functional. Equation (7.57a) is known as the *variational problem* associated with Eq. (7.52). The inner product in H_A is defined by

$$(u, v)_A = B(u, v). \tag{7.57b}$$

For the specific case of $\Pi(u)$ in Eq. (7.56), the bilinear and linear forms are

$$B(u, v) = \int_0^L EA \frac{du}{dx}\frac{dv}{dx} dx + ku(L)v(L), \qquad l(u) = \int_0^L fu\, dx + Pu(L). \tag{7.58}$$

The Euler equations and natural boundary conditions associated with the minimization of $\Pi(u)$ in Eq. (7.56) are

$$-\frac{d}{dx}\left(EA \frac{du}{dx} \right) - f = 0 \quad \text{in } 0 < x < L, \tag{7.59a}$$

$$EA \frac{du}{dx} + ku - P = 0 \quad \text{at } x = L. \tag{7.59b}$$

The essential boundary condition of the problem is provided by the geometric constraint

$$u(0) = 0. \tag{7.59c}$$

The exact solution u to the problem is one that satisfies Eq. (7.59a) at every $x \in (0, L)$ and the boundary conditions in Eqs. (7.59b,c). Thus, the solution of Eqs. (7.59a–c)

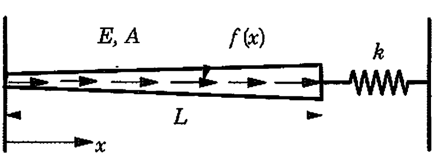

Figure 7.3 Axial deformation of a nonuniform bar with an end spring.

is equivalent to minimizing $\Pi(u)$ over a set of functions that satisfy the condition in (7.59c).

In the Ritz method, we seek an approximate solution U_N (which may be exact if we choose the right kind of approximate solution) to the problem as a finite linear combination of the form (7.54). The reason for selecting the particular form of the approximate solution will be apparent in the sequel. If we select ϕ_0 and ϕ_i such that U_N satisfies the specified essential boundary condition, $u(0) = 0$, and substitute U_N into the total potential energy functional Π in Eq. (7.56), we obtain Π as a function of the parameters $c_1, c_2, \ldots, c_N$ (after carrying out the indicated integration with respect to x):

$$\Pi = \Pi(c_1, c_2, \ldots, c_N).$$

Then c_i are determined (or adjusted) such that $\delta\Pi = 0$; in other words, we minimize Π with respect to c_i, $i = 1, 2, \ldots, N$:

$$0 = \delta\Pi = \frac{\partial\Pi}{\partial c_1}\delta c_1 + \frac{\partial\Pi}{\partial c_2}\delta c_2 + \cdots + \frac{\partial\Pi}{\partial c_N}\delta c_N = \sum_{j=1}^{N} \frac{\partial\Pi}{\partial c_j}\delta c_j.$$

Since the set $\{c_i\}$ is linearly independent, it follows that

$$\frac{\partial\Pi}{\partial c_i} = 0 \qquad \text{for } i = 1, 2, \ldots, N \tag{7.60a}$$

or

$$[A]\{c\} = \{b\}. \tag{7.60b}$$

Equations (7.60a,b) represents a set of N linear equations among $c_1, c_2, \ldots, c_N$, whose solution together with Eq. (7.54) yields the approximate solution $U_N(x)$. This completes the description of the Ritz method.

Since the natural boundary conditions of the problem are included in the variational statement, we require the approximate solution U_N to satisfy only the essential boundary conditions. In order for U_N to satisfy the essential boundary conditions *for any* c_i, it is convenient to choose the approximation in the form (7.54) and require $\phi_0(x)$ to satisfy the specified essential boundary conditions. For instance, if $u(x)$ is specified to be $\hat{u}$ at $x = 0$, we require $\phi_0(x)$ be such that $\phi_0(0) = \hat{u}$. Then

$$U_N(0) = \sum_{i=1}^{N} c_i\phi_i(0) + \phi_0(0) = \sum_{i=1}^{N} c_i\phi_i(0) + \hat{u}. \tag{7.61a}$$

Since $U_N(0) = \hat{u}$, it follows that

$$\sum_{i=1}^{N} c_i\phi_i(0) = 0 \rightarrow \phi_i(0) = 0 \qquad \text{for all } i = 1, 2, \ldots, N. \tag{7.61b}$$

Thus, $\phi_i(x)$ must satisfy the *homogeneous form* of specified essential boundary conditions.

Equation (7.54) can be viewed as a representation of u in a component form; the parameters c_i are the *components* (or coordinates) and $\{\phi_i\}$ are the *coordinate functions*. Another interpretation of Eq. (7.54) is provided by the finite Fourier series, in which c_i are known as the *Fourier coefficients*.

7.3.3 Properties of Approximation Functions

The *approximation functions* ϕ_0 and ϕ_i should be such that the substitution of Eq. (7.54) into $\delta\Pi$ or its equivalent results in N linearly independent equations for the parameters c_j $(j = 1, 2, \ldots, N)$ so that the system has a solution. To ensure that the algebraic equations resulting from the Ritz approximation have a solution, and the approximate solution $U_N(x)$ converges to the true solution $u(x)$ of the problem as the value of N is increased, ϕ_i $(i = 1, 2, \ldots, N)$ and ϕ_0 must satisfy certain requirements. Before we list the requirements, it is informative to discuss the concepts of completeness of a set of functions and convergence of a sequence of approximations. To make the ideas presented simple to understand, mathematical rigor is sacrificed.

Convergence A sequence $\{U_N\}$ of functions is said to *converge* to u if for each $\epsilon > 0$ there is a number $M > 0$, depending on ϵ, such that

$$\|U_N(\mathbf{x}) - u(\mathbf{x})\| < \epsilon \qquad \text{for all } N > M,$$

where $\|\cdot\|$ denotes a norm of the enclosed quantity and u is called the limit of the sequence. In the above statement, U_N represents the approximate solution and u the true solution. The statement implies that the N-parameter solution U_N can be made as close to u as we wish, say within ϵ, by choosing N to be greater than $M = M(\epsilon)$, provided that the approximate solution is convergent. While there is no formula to determine M, a series of trials will help determine the value of N for which the approximate solution U_N is within the tolerance.

Completeness The concept of convergence of a sequence involves a limit of the sequence. If the limit is not a part of the sequence $\{U_N\}_{N=1}^{\infty}$, then there is no hope of attaining convergence. For example, if the true solution to a certain problem is of the form $u(x) = ax^2 + bx^3 + cx^5$, where a, b, and c are constants, then the sequence of approximations

$$U_1 = c_1x^3,\ U_2 = c_1x^3 + c_2x^4, \ldots, \qquad U_N = c_1x^3 + c_2x^4 + \cdots + c_Nx^{N+2}$$

will not converge to the true solution because the sequence does not contain the x^2 term. The sequence is said to be incomplete. As a rule, in selecting an approximate solution one should include all terms up to the highest-order term. If a certain term is not a part of the true solution, like the x^4 term, its coefficient will turn out to be zero by the time all terms of the true solution are included in the approximation.

We now list the requirements of a convergent Ritz approximation (7.54):

1. ϕ_0 must satisfy the *specified* essential boundary conditions. When the specified essential boundary conditions are homogeneous, then $\phi_0(x) = 0$.
2. $\phi_i \in H_A$ $(i = 1, 2, \ldots, N)$ must satisfy the following three conditions:

   (a) be continuous, as required by the variational statement being used;

   (b) satisfy the *homogeneous form* of the specified essential boundary conditions; and

   (c) the set $\{\phi_i\}$ must be linearly independent and complete. (7.62)

7.3.4 Ritz Equations for the Parameters

Returning to the problem of determining the approximate solution $U_N(x)$ of the bar problem described by Eqs. (7.59a,b), we substitute Eq. (7.54) into the total potential energy functional Π:

$$\Pi(c_1, c_2, \ldots, c_N) = \int_0^L \left[\frac{EA}{2} \left(\sum_{j=1}^N c_j \frac{d\phi_j}{dx} + \frac{d\phi_0}{dx} \right)^2 - f \left(\sum_{j=1}^N c_j \phi_j + \phi_0 \right) \right] dx$$
$$+ \frac{k}{2} \left(\sum_{j=1}^N c_j \phi_j(L) + \phi_0(L) \right)^2 - P \left(\sum_{j=1}^N c_j \phi_j(L) + \phi_0(L) \right).$$

Now differentiating Π with respect to c_i $(i = 1, 2, \ldots, N)$, we obtain N linearly independent equations for the unknowns, $c_1, c_2, \ldots, c_N$. We have

$$0 = \frac{\partial \Pi}{\partial c_1} = \int_0^L \left[EA \frac{d\phi_1}{dx} \left(\sum_{j=1}^N c_j \frac{d\phi_j}{dx} + \frac{d\phi_0}{dx} \right) - f \phi_1 \right] dx$$
$$+ k\phi_1(L) \left(\sum_{j=1}^N c_j \phi_j(L) + \phi_0(L) \right) - P\phi_1(L), \quad \text{(Eq. 1)}$$

$$0 = \frac{\partial \Pi}{\partial c_2} = \int_0^L \left[EA \frac{d\phi_2}{dx} \left(\sum_{j=1}^N c_j \frac{d\phi_j}{dx} + \frac{d\phi_0}{dx} \right) - f \phi_2 \right] dx$$
$$+ k\phi_2(L) \left(\sum_{j=1}^N c_j \phi_j(L) + \phi_0(L) \right) - P\phi_2(L), \quad \text{(Eq. 2)}$$

....

$$0 = \frac{\partial \Pi}{\partial c_i} = \int_0^L \left[EA \frac{d\phi_i}{dx} \left(\sum_{j=1}^{N} c_j \frac{d\phi_j}{dx} + \frac{d\phi_0}{dx} \right) - f\phi_i \right] dx + k\phi_i(L) \left(\sum_{j=1}^{N} c_j \phi_j(L) + \phi_0(L) \right) - P\phi_i(L), \qquad \text{(Eq. } i\text{)}$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$0 = \frac{\partial \Pi}{\partial c_N} = \int_0^L \left[EA \frac{d\phi_N}{dx} \left(\sum_{j=1}^{N} c_j \frac{d\phi_j}{dx} + \frac{d\phi_0}{dx} \right) - f\phi_N \right] dx + k\phi_N(L) \left(\sum_{j=1}^{N} c_j \phi_j(L) + \phi_0(L) \right) - P\phi_N(L). \qquad \text{(Eq. } N\text{)}$$

Note that Eq. i is the ith equation of the set of N equations. The ith equation can be written in the short form

$$0 = \sum_{j=1}^{N} a_{ij} c_j - b_i, \tag{7.63a}$$

where the coefficients a_{ij} and b_i are defined by

$$a_{ij} = \int_0^L EA \frac{d\phi_i}{dx} \frac{d\phi_j}{dx}\, dx + k\phi_i(L)\phi_j(L),$$

$$b_i = -\int_0^L EA \frac{d\phi_0}{dx} \frac{d\phi_i}{dx}\, dx - k\phi_0(L)\phi_i(L) + \int_0^L f\phi_i\, dx + P\phi_i(L). \tag{7.63b}$$

The N equations can be written in matrix form as

$$[A]\{c\} = \{b\}. \tag{7.64}$$

Equations (7.64) are called the *equations for the Ritz parameters* c_i. Once $c_i (i = 1, 2, \ldots, N)$ are determined from Eq. (7.64), the approximate solution of the problem is given by Eq. (7.54). This displacement can be used to evaluate strains and stresses:

$$\varepsilon_{xx} = \frac{du}{dx} \approx \sum_{i=1}^{N} c_i \frac{d\phi_i}{dx} + \frac{d\phi_0}{dx}, \qquad \sigma_{xx} = E\varepsilon_{xx} \approx E \left(\sum_{i=1}^{N} c_i \frac{d\phi_i}{dx} + \frac{d\phi_0}{dx} \right).$$

Equations (7.63a,b) can also be arrived at by substituting the Ritz approximation (7.54) and its variation

$$\delta u \approx \delta U_N = \sum_{i=1}^{N} \delta c_i \phi_i(x) \tag{7.65}$$

in the variational statement $\delta\Pi = 0$, instead of substituting (7.54) in Π and then taking the variation with respect to c_i. This results in

$$0 = \sum_{i=1}^{N} \delta c_i \left\{ \int_0^L \left[EA \frac{d\phi_i}{dx} \left(\sum_{j=1}^{N} c_j \frac{d\phi_j}{dx} + \frac{d\phi_0}{dx} \right) - f\phi_i \right] dx \right.$$
$$\left. + k\phi_i(L) \left(\sum_{j=1}^{N} c_j \phi_j(L) + \phi_0(L) \right) - P\phi_i(L) \right\}.$$

Since δc_i are arbitrary, we obtain the result in Eq. (7.63a).

Next we discuss the task of selecting the approximation functions ϕ_0 and ϕ_i. The properties listed in Eq. (7.62) provide guidelines for selecting the coordinate functions $\phi_0(x)$ and $\phi_i(x)$; they do not, however, give any formulae for generating the functions. Thus, apart from the guidelines, the selection of the coordinate functions is largely arbitrary. As a general rule, the coordinate functions ϕ_i should be selected from an admissible set [i.e., those meeting the conditions in Eq. (7.62)], from the lowest order to a desirable order, without missing any intermediate terms (i.e., the completeness property). Also, ϕ_0 should be any lowest order (including zero) that satisfied the specified essential boundary conditions of the problem; it has no continuity (differentiability) requirement.

For the problem at hand, $\phi_0 = 0$ since the specified essential boundary condition is homogeneous. Next, we find $\phi_1(x)$ such that $\phi_1(0) = 0$ and differentiable at least once with respect to x because Π involves the first derivatives of $u \approx U_N$. If an algebraic polynomial is to be selected, the lowest-order polynomial that has a nonzero first derivative is

$$\phi_1(x) = a + bx,$$

where a and b are constants to be determined. The condition $\phi_1(0) = 0$ gives $a = 0$. Since b is arbitrary, we take it to be unity (any nonzero constant will be absorbed into c_1). When $N > 1$, property 2(c) in Eq. (7.62) requires that ϕ_i, $i > 1$, should be selected such that the set $\{\phi_i\}_{i=1}^{N}$ is linearly independent and makes the set complete. In the present case, this is done by choosing ϕ_2 to be x^2. Clearly, $\phi_2(x) = x^2$ meets the conditions $\phi_2(0) = 0$, linearly independent of $\phi_1(x) = x$ (i.e., ϕ_2 is not a constant multiple of ϕ_1), and the set $\{x, x^2\}$ is complete (i.e., no other admissible term up to quadratic is omitted). In other words, in selecting coordinate functions of a given degree, one should not omit any lower-order terms that are admissible. Otherwise the approximate solution will never converge to the exact solution, no matter how many terms are used in the Ritz approximation, as the exact solution may have those lower-order terms that were omitted in the approximate solution. Note that $\hat{\phi}_2(x) = x + x^2$ is also an admissible function that meets all requirements. Then

$$U_2(x) = c_1\phi_1 + c_2\hat{\phi}_2 = \hat{c}_1 x + \hat{c}_2 x^2, \qquad \text{with } \hat{c}_1 = c_1 + c_2,\ \hat{c}_2 = c_2,$$

which is equivalent to

$$U_2(x) = c_1\phi_1 + c_2\phi_2.$$

Thus, one may select $\phi_i = x^i$, $i = 1, 2, \ldots, N$.

If trigonometric functions are to be selected, one may be tempted to select $\phi_1 = \sin(\pi x/L)$, which satisfies the condition $\phi_1(0) = 0$. However, this choice also gives $U_1(L) = 0$ since $\phi_1(L) = 0$. A better choice would be to select $\phi_1(x) = \sin(\pi x/2L)$, or for $N > 1$, select $\phi_i = \sin[(2i-1)\pi x/2L]$; this choice will yield a good solution.

For the choice of algebraic polynomials, the N-parameter Ritz approximation for the bar problem is

$$U_N(x) = \sum_{i=1}^{N} c_i\phi_i(x), \qquad \phi_i(x) = x^i, \tag{7.66}$$

and the coefficients a_{ij} and b_i for

$$EA = a_0\left(2 - \frac{x}{L}\right), \qquad f = f_0 \text{ (a constant)}, \qquad P = P_0, \tag{7.67}$$

are given by

$$\begin{aligned} a_{ij} &= a_0 \int_0^L \left(1 - \frac{x}{L}\right)\frac{d\phi_i}{dx}\frac{d\phi_j}{dx}\,dx + k\phi_i(L)\phi_j(L) \\ &= a_0 ij \int_0^L \left(1 - \frac{x}{L}\right) x^{i+j-2}\,dx + k(L)^{i+j} \\ &= a_0 \frac{ij(1+i+j)}{(i+j-1)(i+j)}(L)^{i+j-1} + k(L)^{i+j}, \end{aligned} \tag{7.68a}$$

$$b_i = \int_0^L f\phi_i\,dx + P\phi_i(L) = \frac{f_0}{i+1}(L)^{i+1} + P_0(L)^i. \tag{7.68b}$$

For one-term approximation ($N = 1$ and $k = 0$), we have

$$a_{11} = \frac{3}{2}a_0 L, \qquad b_1 = \frac{1}{2}f_0 L^2 + P_0 L,$$

$$c_1 = \frac{b_1}{a_{11}} = \frac{6}{9a_0 L}\left(\frac{3}{6}f_0 L^2 + PL\right) = \frac{f_0 L + 2P_0}{3a_0},$$

and

$$U_1(x) = \frac{f_0 L + 2P_0}{3a_0} x.$$

For $N = 2$ and $k = 0$, we have

$$a_{11} = \frac{3}{2} a_0 L, \qquad a_{12} = a_{21} = \frac{4}{3} a_0 L^2, \qquad a_{22} = \frac{5}{3} a_0 L^3,$$

$$b_1 = \frac{1}{2} f_0 L^2 + P_0 L, \qquad b_2 = \frac{1}{3} f_0 L^3 + P_0 L^2.$$

The Ritz equations can be written in matrix form as

$$\frac{a_0 L}{6} \begin{bmatrix} 9 & 8L \\ 8L & 10L^2 \end{bmatrix} \begin{Bmatrix} c_1 \\ c_2 \end{Bmatrix} = \frac{f_0 L^2}{6} \begin{Bmatrix} 3 \\ 2L \end{Bmatrix} + P_0 L \begin{Bmatrix} 1 \\ L \end{Bmatrix},$$

whose solution by Cramer's rule is

$$c_1 = \frac{1}{a_0} \left(\frac{7}{13} f_0 + \frac{6}{13} P_0 \right), \qquad c_2 = \frac{3}{13 a_0 L} (-f_0 L + P_0).$$

Hence the two-parameter Ritz solution is

$$U_2(x) = \frac{7 f_0 L + 6P_0}{13 a_0} x + \frac{3(P_0 - f_0 L)}{13 a_0 L} x^2.$$

The exact solution of Eqs. (7.55a,b) with $u(0) = 0$, $k = 0$, $EA = a_0[2 - (x/L)]$, and $f = f_0$ is

$$u(x) = \frac{f_0 L}{a_0} x + \frac{(f_0 L - P_0) L}{a_0} \log\left(1 - \frac{x}{2L}\right) \tag{7.69a}$$

$$\approx \frac{f_0 L + P_0}{2a_0} x + \frac{P_0 - f_0 L}{8 a_0 L} x^2 + \frac{P_0 - f_0 L}{24 a_0 L^2} x^3 + \cdots \tag{7.69b}$$

Table 7.1 contains a comparison of the Ritz coefficients c_i for $N = 1, 2, \ldots, 8$ with the exact coefficients in Eq. (7.69b) for $L = 10$ ft, $a_0 = 180 \times 10^6$ lb, $f_0 = 0$, and $P_0 = 10$ kip. Clearly the Ritz coefficients c_i converge to the exact ones as N goes from 1 to 8.

7.3.5 General Features of the Method

Some general features of the Ritz method are listed below:

1. If the approximate functions $\phi_i(x)$ are selected to satisfy Eq. (7.62), the assumed approximation $U_N(x)$ normally converges to the actual solution $u(x)$ with an increase in the number of parameters (i.e., $N \to \infty$). A mathematical proof of such an assertion is not given here, but interested readers may consult the references at the end of the chapter.

Table 7.1 The Ritz coefficients[a] for the axial deformation of an isotropic elastic bar subjected to axial force

n	$\bar{c}_1$	$\bar{c}_2$	$\bar{c}_3$	$\bar{c}_4$	$\bar{c}_5$	$\bar{c}_6$	$\bar{c}_7$	$\bar{c}_8$
1	37.037							
2	25.641	12.821						
3	28.219	4.409	4.879					
4	27.691	7.788	0.000	3.029				
5	27.794	6.701	3.389	−1.040	1.664			
6	27.775	7.009	1.904	2.012	−1.142	0.952		
7	27.778	6.929	2.453	0.320	1.447	−0.980	0.560	
8	27.778	6.948	2.272	1.094	−0.287	1.136	−0.769	0.336
Exact	27.778	6.944	2.315	0.868	0.347	0.145	0.062	0.027

[a] $\bar{c}_i = c_i \times 10^{5+i}$.

2. For increasing values of N, the previously computed coefficients of the algebraic equations (7.60b) remain unchanged (provided the previously selected coordinate functions are not changed), and one must add newly computed coefficients to the system of equations.
3. If the set of approximation functions $\{\phi_i\}$ chosen is an orthogonal set in the sense $B(\phi_i, \phi_j) = a_{(ij)}\delta_{ij}$ (no sum on i and j), then one need not invert the system of equations, and the solution is obtained as $c_i = b_i/a_{ii}$.
4. The Ritz method applies to all problems, linear or nonlinear, as long as the variational problem

$$B(\delta u, u) = l(\delta u) \tag{7.70}$$

is equivalent to the governing equation and natural boundary conditions. In general, $B(\delta u, u)$ is linear in δu but may be nonlinear in u (and thus $B(\cdot, \cdot)$ may not be symmetric), and $l(\delta u)$ is a linear functional.
5. If the variational problem used in the Ritz approximation is such that its bilinear form is symmetric (in u and δu), the resulting algebraic equations are also symmetric and, therefore, only elements above or below the main diagonal of the coefficient matrix need to be computed.
6. If $\delta\Pi$ (or $B(\delta u, u)$) is nonlinear in u, the resulting algebraic equations $[A(\{c\})]\{c\} = \{b\}$ will also be nonlinear in the parameters c_i. To solve such nonlinear equations, a variety of numerical methods are available (e.g., the Newton–Raphson method). Generally, there is more than one solution to the set of nonlinear equations.
7. Since the strains are computed from the approximate displacements, the strains and stresses are generally less accurate than the displacements.
8. The governing equation and natural boundary conditions of the problem are satisfied only in the variational sense, and not in the differential equation sense.

Therefore, the displacements obtained from the Ritz approximation generally do not satisfy the equations of equilibrium pointwise.

9. Since a continuous system is approximated by a finite number of coordinates (or degrees of freedom), the approximate system is less flexible than the actual system (recall that $\Pi(u) < \Pi(U)$ for any U that is not exact). Consequently, the displacements obtained by the Ritz method using the principle of minimum total potential energy converge to the exact displacement from below:

$$U_1 < U_2 < \cdots < U_n < U_m \cdots < u, \qquad \text{for } m > n, \tag{7.71}$$

where U_N denotes the N-parameter Ritz approximation of u obtained from the principle of minimum total potential energy. The displacements obtained from the Ritz approximations based on the total complementary energy principle provide an upper bound for the exact solution.

10. Although the discussion of the Ritz method in this section thus far is confined to a linear solid mechanics problem, the method can be employed for any equation that admits a variational formulation (in the sense discussed in Comment 4), as will be illustrated through several examples shortly. However, the bounds mentioned above do not hold unless the variational problem is based on a minimum variational principle.

7.3.6 Examples

In this section we illustrate the application of the Ritz method to a variety of problems. These include static, eigenvalue, and transient problems. As will be shown in the sequel, the Ritz method can also be applied to problems that either do not admit a quadratic functional or where one knows only the governing equations of the problem. In the latter case, a way to develop the so-called weak form is discussed in Section 7.4.

Example 7.5 Consider a uniform cross-section bar of length L, with the left end fixed and the right end connected to a rigid support via a linear elastic spring (with spring constant k), as shown in Fig. 7.4. We wish to determine the first two natural axial frequencies of the bar using the Ritz method.

The kinetic energy K and the strain energy U associated with the axial motion of the member are given by

$$K = \int_0^L \frac{\rho A}{2}\left(\frac{\partial u}{\partial t}\right)^2 dx, \qquad U = \int_0^L \frac{EA}{2}\left(\frac{\partial u}{\partial x}\right)^2 dx + \frac{k}{2}[u(L,t)]^2. \tag{7.72}$$

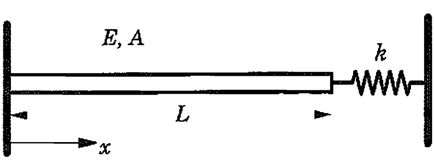

Figure 7.4 Natural vibrations of a bar with an end spring.

Substituting for K and U from Eq. (7.72), and $V = 0$ in Hamilton's principle, we obtain $[\delta u(x, t_1) = \delta u(x, t_2) = 0$ and $\delta u(0, t) = 0]$:

$$
\begin{aligned}
0 &= \int_{t_1}^{t_2} \delta(K - U)\, dt \\
&= \frac{1}{2}\int_{t_1}^{t_2} \delta\left\{\int_0^L \left[\rho A\left(\frac{\partial u}{\partial t}\right)^2 - EA\left(\frac{\partial u}{\partial x}\right)^2\right] dx - k[u(L,t)]^2\right\} dt \qquad (7.73)\\
&= \int_{t_1}^{t_2}\left[\int_0^L \left(\rho A\frac{\partial u}{\partial t}\frac{\partial \delta u}{\partial t} - EA\frac{\partial u}{\partial x}\frac{\partial \delta u}{\partial x}\right) dx - ku(L,t)\delta u(L,t)\right] dt \\
&= \int_{t_1}^{t_2}\left[\int_0^L \left(-\rho A\frac{\partial^2 u}{\partial t^2}\delta u - EA\frac{\partial u}{\partial x}\frac{\partial \delta u}{\partial x}\right) dx - ku(L,t)\delta u(L,t)\right] dt. \qquad (7.74)
\end{aligned}
$$

We seek the periodic motion of the form

$$
u(x,t) = u_0(x)e^{i\omega t}, \qquad i = \sqrt{-1}, \qquad (7.75)
$$

where ω is the frequency of natural vibration, and $u_0(x)$ is the amplitude. Substituting Eq. (7.75) into Eq. (7.74), we obtain

$$
0 = \int_0^L \left(\rho A\omega^2 u_0 \delta u_0 - EA\frac{du_0}{dx}\frac{d\delta u_0}{dx}\right) dx - ku_0(L)\delta u_0(L), \qquad (7.76)
$$

where $(i\omega)^2 = -\omega^2$, and $\int_{t_1}^{t_2} e^{2i\omega t}\,dt$, being nonzero, is factored out. We use Eq. (7.76) to determine the values of ω. Note that the Rayleigh quotient for the problem at hand is given by

$$
\omega^2 = \frac{\int_0^L EA(du_0/dx)(d\delta u_0/dx)\,dx + ku_0(L)\delta u_0(L)}{\int_0^L \rho A u_0 \delta u_0\, dx}.
$$

The Euler equation and natural boundary condition associated with Eq. (7.76) are

$$
-\frac{d}{dx}\left(EA\frac{du_0}{dx}\right) - \rho A\omega^2 u_0 = 0, \qquad 0 < x < L, \qquad (7.77a)
$$

$$
EA\frac{du_0}{dx} + ku_0 = 0 \qquad \text{at } x = L. \qquad (7.77b)
$$

The essential boundary condition is $u_0(0) = 0$.

A nondimensionalization of the variables is used for simplicity:

$$
\bar{x} = \frac{x}{L}, \qquad \bar{u} = \frac{u_0}{L}, \qquad \alpha = \frac{kL}{EA}, \qquad \lambda = \frac{\omega^2 \rho L^2}{E}. \qquad (7.78)
$$

Then Eq. (7.76) becomes

$$0 = \int_0^1 \left(\lambda \bar{u} \delta \bar{u} - \frac{d\bar{u}}{dx} \frac{d\delta \bar{u}}{dx} \right) dx - \alpha \bar{u}(1) \delta \bar{u}(1). \tag{7.79}$$

The bar over the nondimensional variables will be omitted in the interest of brevity. Further, in the following discussions, we shall assume that $\alpha = 1$.

Substituting an N-parameter Ritz approximation (obviously, we have $\phi_0 = 0$):

$$\bar{u}(x) \approx U_N(x) = \sum_{i=1}^{N} c_i \phi_i(x)$$

into Eq. (7.79), we obtain

$$0 = \sum_{i=1}^{N} \left\{ \sum_{j=1}^{N} \left[\lambda \int_0^1 \phi_i \phi_j \, dx - \left(\int_0^1 \frac{d\phi_i}{dx} \frac{d\phi_j}{dx} \, dx + \alpha \phi_i(1) \phi_j(1) \right) \right] c_j \right\} \delta c_i.$$

Because of the independent nature of δc_i, we obtain

$$0 = \sum_{j=1}^{N} \left[\lambda \int_0^1 \phi_i \phi_j \, dx - \left(\int_0^1 \frac{d\phi_i}{dx} \frac{d\phi_j}{dx} \, dx + \alpha \phi_i(1) \phi_j(1) \right) \right] c_j, \tag{7.80a}$$

and in matrix form

$$([A] - \lambda [M]) \{c\} = \{0\}, \tag{7.80b}$$

where

$$a_{ij} = \int_0^1 \frac{d\phi_i}{dx} \frac{d\phi_j}{dx} \, dx + \alpha \phi_i(1) \phi_j(1), \qquad m_{ij} = \int_0^1 \phi_i \phi_j \, dx. \tag{7.80c}$$

Equation (7.80b) represents a matrix eigenvalue problem, and we obtain N eigenvalues, λ_i, $i = 1, 2, \ldots, N$. An analytical method for finding eigenvalues and eigenvectors was discussed in Chapter 2 (see Section 2.3.4).

For the problem at hand, the approximation functions can be taken as

$$\phi_i(x) = x^i. \tag{7.81a}$$

Substituting $\phi_i = x^i$ into Eq. (7.80c), we obtain

$$m_{ij} = \int_0^1 \phi_i \phi_j \, dx = \frac{1}{i + j + 1},$$

$$a_{ij} = \int_0^1 \frac{d\phi_i}{dx} \frac{d\phi_j}{dx} \, dx + \phi_i(1) \phi_j(1) = \frac{ij}{i + j - 1} + 1. \tag{7.81b}$$

Since we wish to determine two eigenvalues, we take $N = 2$ and obtain

$$m_{11} = \tfrac{1}{3}, \qquad m_{12} = \tfrac{1}{4}, \qquad m_{22} = \tfrac{1}{5}, \qquad a_{11} = 2, \qquad a_{12} = 2, \qquad a_{22} = \tfrac{7}{3},$$

and the matrix eigenvalue problem (7.80b) becomes

$$\left(\begin{bmatrix} 2 & 2 \\ 2 & \frac{7}{3} \end{bmatrix} - \lambda \begin{bmatrix} \frac{1}{3} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{5} \end{bmatrix}\right) \begin{Bmatrix} c_1 \\ c_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}. \tag{7.82}$$

For a nontrivial solution (i.e., $c_1 \neq 0, c_2 \neq 0$), the determinant of the coefficient matrix in Eq. (7.82) is set to zero:

$$\begin{vmatrix} 2 - \frac{\lambda}{3} & 2 - \frac{\lambda}{4} \\ 2 - \frac{\lambda}{4} & \frac{7}{3} - \frac{\lambda}{5} \end{vmatrix} = 0$$

or

$$15\lambda^2 - 640\lambda + 2400 = 0.$$

The quadratic equation has two roots:

$$\lambda_1 = 4.1545, \qquad \lambda_2 = 38.512 \rightarrow \omega_1 = \frac{2.038}{L}\sqrt{\frac{E}{\rho}}, \qquad \omega_2 = \frac{6.206}{L}\sqrt{\frac{E}{\rho}}. \tag{7.83}$$

The eigenvectors are given by

$$U_2^{(i)} = c_1^{(i)} x + c_2^{(i)} x^2,$$

where $c_1^{(i)}$ and $c_2^{(i)}$ are calculated from Eq. (7.82) for $\lambda = \lambda_i$, $i = 1, 2$ (see Example 2.8).

The exact values of λ are the roots of the transcendental equation

$$\lambda + \tan \lambda = 0, \tag{7.84}$$

whose first two roots are ($\omega^2 = \lambda$):

$$\omega_{01} = \frac{2.02875}{L}\sqrt{\frac{E}{\rho}}, \qquad \omega_{02} = \frac{4.91318}{L}\sqrt{\frac{E}{\rho}}. \tag{7.85}$$

Note that the first approximate frequency is closer to the exact than the second.

If one selects ϕ_0 and ϕ_i to satisfy the natural boundary condition also, the degree of polynomials will inevitably go up. For example, the lowest-order function that satisfies the homogeneous form (we still have $\phi_0 = 0$) of the natural boundary condition $u'(1) + u(1) = 0$ is

$$\hat{\phi}_1 = 3x - 2x^2. \tag{7.86}$$

The one-parameter solution with the choice of $\hat{\phi}_1$ in Eq. (7.84) gives $\lambda_1 = 50/12 = 4.1667$, which is no better than the two-parameter solution computed using $\phi_1 = x$ and $\phi_2 = x^2$. Of course, solution $c_1\hat{\phi}_1$ would yield a more accurate value for λ_1 than the solution $c_1\phi_1$. Although $c_1\hat{\phi}_1$ and $c_1\phi_1 + c_2\phi_2$ are of the same degree (polynomials), the latter gives better accuracy for λ_1 because the number of parameters is greater, which provides greater freedom to adjust the parameters.

Example 7.6 Consider a uniform cross-section bar of length L with the left end fixed and the right end connected to a rigid support via a linear elastic spring with spring constant k. Suppose that the bar is subjected to a body force $f(x, t)$ (see Fig. 7.4). We wish to determine the transient response of the bar under the assumption that the motion starts from rest, i.e., the initial conditions of the problem are

$$u(x, 0) = 0, \qquad \dot{u}(x, 0) = 0. \tag{7.87}$$

The kinetic and strain energies associated with the axial motion of the bar are given in Eq. (7.70). The potential energy due to $f(x, t)$ is

$$V = -\int_{t_1}^{t_2} \int_0^L f u \, dx.$$

Then Eq. (7.73) becomes

$$0 = \int_{t_1}^{t_2} \left[\int_0^L \left(-\rho A \frac{\partial^2 u}{\partial t^2} \delta u - EA \frac{\partial u}{\partial x} \frac{\partial \delta u}{\partial x} + f \delta u \right) dx - ku(L, t) \delta u(L, t) \right] dt. \tag{7.88}$$

The Euler–Lagrange equations associated with Eq. (7.88) are

$$\frac{\partial}{\partial x} \left(EA \frac{\partial u}{\partial x} \right) - \frac{\partial}{\partial t} \left(\rho A \frac{\partial u}{\partial t} \right) + f = 0, \qquad 0 < x < L; \;\; t > 0, \tag{7.89a}$$

$$EA \frac{\partial u}{\partial x} + ku = 0, \qquad \text{at } x = L; \;\; t \geq 0. \tag{7.89b}$$

When one is interested in determining the time-dependent solution $u(x, t)$ under applied load $f(x, t)$, the Ritz solution is sought in the form

$$u(x, t) \approx \sum_{j=1}^{n} c_j(t) \phi_j(x), \qquad \phi_i = x^i, \tag{7.90}$$

where c_j are now time-dependent parameters to be determined for all times $t > 0$. Substituting Eq. (7.90) into (7.88), we obtain

$$0 = -\sum_{j=1}^{n}\left[\left(\int_0^1 \phi_i\phi_j\,dx\right)\frac{d^2c_j}{dt^2} + \left(\int_0^1 \frac{d\phi_i}{dx}\frac{d\phi_j}{dx}\,dx + \phi_i(1)\phi_j(1)\right)c_j\right] + \int_0^1 \phi_i\bar{f}\,dx$$

$$= -\sum_{j=1}^{n}\left(m_{ij}\frac{d^2c_j}{dt^2} + a_{ij}c_j\right) + b_i, \tag{7.91a}$$

where x, u, f, and t are nondimensionalized as

$$\bar{x} = \frac{x}{L}, \qquad \bar{u} = \frac{u}{(f_0L^2/EA)}, \qquad \bar{f} = \frac{f}{f_0}, \qquad \bar{t} = \frac{t}{L\sqrt{\rho/E}}, \tag{7.91b}$$

f_0 being a constant, and

$$m_{ij} = \int_0^1 \phi_i\phi_j\,dx,$$

$$a_{ij} = \int_0^1 \frac{d\phi_i}{dx}\frac{d\phi_j}{dx}\,dx + \phi_i(1)\phi_j(1), \tag{7.91c}$$

$$b_i = \int_0^1 \phi_i\bar{f}(x,t)\,dx.$$

For $N = 1$ and $f = f_0$ (or $\bar{f} = 1$), we have

$$m_{11}\frac{d^2c_1}{dt^2} + a_{11}c_1 = b_1 \qquad \text{or} \qquad \frac{1}{3}\frac{d^2c_1}{dt^2} + 2c_1 = \frac{1}{2}.$$

The solution to the second-order differential equation is

$$c_1(t) = A\sin\sqrt{6}t + B\cos\sqrt{6}t + \tfrac{1}{4}.$$

Hence the one-parameter Ritz solution is given by

$$U_1(x,t) = \left(A\sin\sqrt{6}t + B\cos\sqrt{6}t + \tfrac{1}{4}\right)x,$$

where A and B are constants to be determined using the initial conditions. For zero initial conditions

$$u(x,0) = 0, \qquad \dot{u}(x,0) = 0, \tag{7.92}$$

we can determine the constants as $B = -1/4$ and $A = 0$. The solution becomes

$$U_1(x,t) = \tfrac{1}{4}\left(1 - \cos\sqrt{6}t\right)x.$$

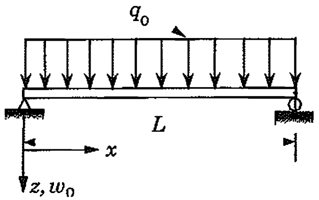

Figure 7.5 A simply supported beam under uniform load.

For $N \geq 2$, the resulting system of differential equations in time, Eq. (7.91a), may be solved for $c_i(t)$ using the Laplace transform method or a numerical method, the latter being more practical. In general, the initial conditions (7.92) cannot be satisfied exactly, requiring approximation. Further, the numerical solution may be obtained only for discrete values of time. For example, at time $t_s = s\Delta t$ (i.e., the total time interval is divided into a finite number of time steps of size Δt), we would have

$$u_N(x, t_s) = \sum_{j=1}^{N} c_j(t_s)\phi_j(x). \tag{7.93}$$

For more details of the ideas discussed here, see Example 7.17.

Example 7.7 Consider a simply supported beam of length L. We wish to find the transverse deflection of the beam under uniformly distributed transverse load q_0 (see Fig. 7.5) using the Euler–Bernoulli beam theory. The principle of virtual displacements for the problem becomes

$$0 = \int_0^L \left(EI \frac{d^2\delta w_0}{dx^2} \frac{d^2 w_0}{dx^2} - \delta w_0 q_0 \right) dx. \tag{7.94a}$$

The essential boundary conditions are

$$w_0(0) = w_0(L) = 0. \tag{7.94b}$$

We choose a two-parameter approximation of the form

$$w_0 \approx W_2(x) = c_1\phi_1 + c_2\phi_2 + \phi_0, \qquad \delta W_2 = \delta c_1\phi_1 + \delta c_2\phi_2, \tag{7.95a}$$

where

$$\phi_0 = 0, \qquad \phi_1 = x(L - x), \qquad \phi_2 = x^2(L - x). \tag{7.95b}$$

Substituting Eq. (7.95a) into Eq. (7.94a), we obtain

$$0 = \int_0^L \left[EI\left(\delta c_1\phi_1'' + \delta c_2\phi_2''\right)\left(c_1\phi_1'' + c_2\phi_2''\right) - \left(\delta c_1\phi_1 + \delta c_2\phi_2\right)\right] q_0\, dx$$

$$= \sum_{i=1}^{2} \left(\sum_{j=1}^{2} a_{ij} c_j - b_i \right) \delta c_i \qquad \text{or} \qquad 0 = \sum_{j=1}^{2} a_{ij} c_j - b_i,$$

where

$$a_{ij} = \int_0^L EI \frac{d^2\phi_i}{dx^2} \frac{d^2\phi_j}{dx^2} \, dx, \qquad b_i = \int_0^L q_0 \phi_i \, dx.$$

For the particular choice of ϕ_i in Eq. (7.95b), we have

$$EIL \begin{bmatrix} 4 & 2L \\ 2L & 4L^2 \end{bmatrix} \begin{Bmatrix} c_1 \\ c_2 \end{Bmatrix} = \frac{q_0 L^3}{12} \begin{Bmatrix} 2 \\ L \end{Bmatrix},$$

and the solution of these equations yields the result

$$c_1 = \frac{q_0 L^2}{24EI}, \qquad c_2 = 0,$$

so that the two-parameter Ritz solution becomes

$$W_2(x) = \frac{q_0 L^4}{24EI} \left(\frac{x}{L} - \frac{x^2}{L^2} \right). \tag{7.96}$$

The exact solution of the problem is given by

$$w_0(x) = \frac{q_0 L^4}{24EI} \left(\frac{x}{L} - 2\frac{x^3}{L^3} + \frac{x^4}{L^4} \right). \tag{7.97}$$

The maximum deflections according to the exact and Ritz solutions are

$$w_0\left(\frac{L}{2}\right) = \frac{5}{384} \frac{q_0 L^4}{EI} = 0.01302 \frac{q_0 L^4}{EI}, \quad W_2\left(\frac{L}{2}\right) = \frac{37}{2352} \frac{q_0 L^4}{EI} = 0.01042 \frac{q_0 L^4}{EI}.$$

Thus the two-parameter Ritz approximation is about 20% in error.

The three-parameter Ritz approximation with $\phi_3 = x^3(L - x)$ yields

$$EIL \begin{bmatrix} 4 & 2L & 2L^2 \\ 2L & 4L^2 & 4L^3 \\ 2L^2 & 4L^3 & 4.8L^4 \end{bmatrix} \begin{Bmatrix} c_1 \\ c_2 \\ c_3 \end{Bmatrix} = \frac{q_0 L^3}{12} \begin{Bmatrix} 2 \\ L \\ 0.6L^2 \end{Bmatrix}, \tag{7.98}$$

and we obtain

$$c_1 = c_2 L = -c_3 L^2 = \frac{q_0 L^2}{24EI}.$$

The three-parameter Ritz solution coincides with the exact solution in Eq. (7.97).

Remark 1 If the beam is subjected to a point load F_0 at $x = L/2$, instead of a uniform load throughout the span of the beam (see Fig. 7.6), the exact solution will be in two parts:

$$w_0(x) = \begin{cases} \dfrac{F_0 L^3}{48EI}\left(3\dfrac{x}{L} - 4\dfrac{x^3}{L^3}\right), & 0 \le x \le \dfrac{L}{2}, \\ \dfrac{F_0 L^3}{48EI}\left[3\dfrac{x}{L} - 4\dfrac{x^3}{L^3} + \left(2\dfrac{x}{L} - 1\right)^3\right], & \dfrac{L}{2} \le x \le L. \end{cases} \tag{7.99}$$

Then it is clear that we must seek the Ritz solution also in two parts.

Suppose that we use the virtual work statement (or $\delta\Pi = 0$)

$$0 = \int_0^L EI \frac{d^2\delta w_0}{dx^2}\frac{d^2 w_0}{dx^2}\, dx - F_0 \delta w_0\left(\frac{L}{2}\right) \tag{7.100}$$

with the three-parameter approximation

$$W_3(x) = c_1 x(L - x) + c_2 x^2 (L - x) + c_3 x^3 (L - x), \tag{7.101}$$

we obtain the same coefficient matrix as in Eq. (7.94), because the bilinear form did not change but the linear form changed, and the right-hand side is given by

$$\frac{F_0 L^2}{4}\begin{Bmatrix} 1 \\ \dfrac{L}{2} \\ \dfrac{L^2}{4} \end{Bmatrix}.$$

The solution for the Ritz coefficients gives

$$c_1 = \frac{F_0 L}{12EI}, \qquad c_2 = -\frac{F_0}{12EI}, \qquad c_3 = -\frac{5F_0}{64EIL}.$$

The Ritz solution *does not* coincide with the exact solution. In particular, the maximum deflection predicted by the three-parameter Ritz approximation (7.101) is

$$W_3\left(\frac{L}{2}\right) = \frac{7}{384}\frac{F_0 L^3}{EI} = \frac{F_0 L^3}{54.86EI},$$

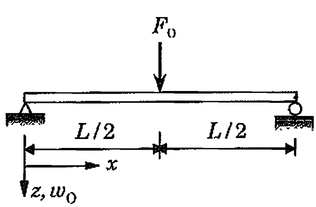

Figure 7.6 A simply supported beam under center point load.

whereas the exact value from Eq. (7.99) is

$$w_0\left(\frac{L}{2}\right) = \frac{F_0 L^3}{48EI}. \tag{7.102}$$

The reason for the Ritz solution based on the variational problem (7.100) not being exact even for a point load is that (7.100) does not account for the discontinuity in the load. Note that the exact shear force, $Q = EI(d^3w_0/dx^3)$, is discontinuous at $x = L/2$, but the approximate one is continuous. Thus, we must modify the variational problem, as discussed in Example 7.8.

Remark 2 One can also use trigonometric polynomials in place of algebraic polynomials for the approximation functions ϕ_i. For instance, the deflection of a simply supported beam subjected to continuous distributed load $q(x)$ can be represented by

$$w_0 \approx c_1 \sin\frac{\pi x}{L} + c_2 \sin\frac{3\pi x}{L} + \cdots + c_N \sin(2N-1)\frac{\pi x}{L}. \tag{7.103}$$

The functions $\phi_i = \sin(2i-1)(\pi x/L)$ are linearly independent, and are complete if all lower functions up to $\sin(2N-1)(\pi x/L)$ are included. When the load is sinusoidal,

$$q(x) = q_0 \sin\frac{m\pi x}{L} \quad \text{(for fixed } m\text{)}, \tag{7.104}$$

we obtain the exact solution ($c_1 = c_2 = \cdots = c_{m-1} = c_{m+1} = \cdots = c_N = 0$):

$$w_0(x) = c_m \sin\frac{m\pi x}{L}, \qquad c_m = \frac{q_0 L^4}{EIm^4\pi^4}. \tag{7.105}$$

This solution cannot be represented using a finite set of algebraic functions $\{\phi_i\}$, although the Ritz solution with a finite number of terms may be very close to the exact when evaluated at a point.

On the other hand, if the load is representable by an algebraic polynomial (e.g., $q(x)$ is a constant, linear, or higher-order function of x), then the Ritz solution (7.103) will not coincide with the exact solution (7.97) for any finite value of N, because the sine-series representation of such a load is an infinite series. For example, when $q = q_0$, a constant, then

$$q(x) = q_0 = \sum_{i=1,3,\ldots}^{\infty} \frac{16q_0}{\pi i} \sin\frac{i\pi x}{L}. \tag{7.106}$$

However, the Ritz solution (7.103) converges rapidly, giving an accurate solution, especially away from the ends, for a finite value of N.

Thus, in general, a judicious choice of approximation functions ϕ_i based on the source term $q(x)$ will not only make the computational effort minimal, but also gives an accurate solution.

Example 7.8 Here we consider a beam with discontinuous loading. As an example, consider the beam shown in Fig. 7.6. We divide the beam into as many parts as there are regions with continuous loading. Consider the ith part, located between $x = x_i$ and $x = x_{i+1}$. Within each part of the beam, the differential equation for $w_0(x)$ is

$$\frac{d^2}{dx^2}\left(EI\frac{d^2w_0}{dx^2}\right) = q(x), \qquad x_i < x < x_{i+1}, \tag{7.107}$$

where $q(x)$ is any distributed load in the part. We isolate the ith part and set up a free-body diagram depicting the internal forces $(Q_1^{(i)}, Q_2^{(i)})$ and moments $(M_1^{(i)}, M_2^{(i)})$, as shown in Fig. 7.7. The total potential energy for the ith part is

$$\begin{aligned}\Pi_i(w_0) = &\int_{x_i}^{x_{i+1}}\left[\frac{EI}{2}\left(\frac{d^2w_0}{dx^2}\right)^2 - qw_0\right]dx - M_1^{(i)}\left(-\frac{dw_0}{dx}\right)_{x_i}\\ &- M_2^{(i)}\left(-\frac{dw_0}{dx}\right)_{x_{i+1}} - Q_1^{(i)}w_0(x_i) - Q_2^{(i)}w_0(x_{i+1}).\end{aligned} \tag{7.108}$$

where the left end is labeled as 1 and the right end as 2.

To use the Ritz method, we must select approximation functions based on specified boundary conditions. If we assume for the moment that all of the boundary conditions are of natural type, then the Ritz approximation over the ith part may be assumed in the form (see Example 5.6):

$$\begin{aligned}w_0^i(x) &= c_1 + c_2x + c_3x^2 + c_4x^3\\ &= \varphi_1^{(i)}(x)w_1^{(i)} + \varphi_2^{(i)}(x)\theta_1^{(i)} + \varphi_3^{(i)}(x)w_2^{(i)} + \varphi_4^{(i)}(x)\theta_4^{(i)},\end{aligned} \tag{7.109}$$

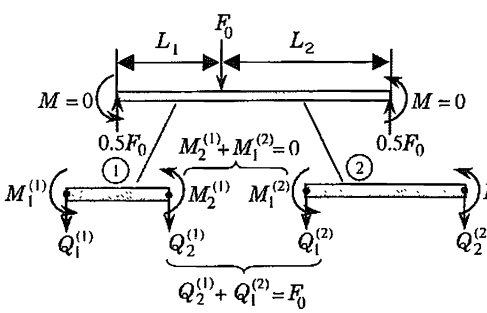

Figure 7.7 Boundary, continuity, and equilibrium conditions for the beam of Example 7.8.

where ($\bar{x} = x - x_i$ and $L_i = x_{i+1} - x_i$)

$$\varphi_1^{(i)} = 1 - 3\left(\frac{\bar{x}}{L_i}\right)^2 + 2\left(\frac{\bar{x}}{L_i}\right)^3, \qquad \varphi_2^{(i)} = -\bar{x}\left(1 - \frac{\bar{x}}{L_i}\right)^2,$$
$$\varphi_3^{(i)} = 3\left(\frac{\bar{x}}{L_i}\right)^2 - 2\left(\frac{\bar{x}}{L_i}\right)^3, \qquad \varphi_4^{(i)} = -\bar{x}\left[\left(\frac{\bar{x}}{L_i}\right)^2 - \frac{\bar{x}}{L_i}\right]. \tag{7.110}$$

Substitution of Eq. (7.109) into $\delta\Pi_i = 0$ and carrying out the integration yields the Ritz equations for the ith part:

$$\frac{2E_iI_i}{L_i^3}\begin{bmatrix} 6 & -3L_i & -6 & -3L_i \\ -3L_i & 2L_i^2 & 3L_i & L_i^2 \\ -6 & 3L_i & 6 & 3L_i \\ -3L_i & L_i^2 & 3L_i & 2L_i^2 \end{bmatrix} \begin{Bmatrix} w_1^{(i)} \\ \theta_1^{(i)} \\ w_2^{(i)} \\ \theta_2^{(i)} \end{Bmatrix} = \begin{Bmatrix} q_1^{(i)} \\ q_2^{(i)} \\ q_3^{(i)} \\ q_4^{(i)} \end{Bmatrix} + \begin{Bmatrix} Q_1^{(i)} \\ M_1^{(i)} \\ Q_2^{(i)} \\ M_2^{(i)} \end{Bmatrix}, \tag{7.111a}$$

where

$$q_j^{(i)} = \int_{x_i}^{x_{i+1}} q^{(i)}(x)\varphi_j^{(i)}(x)\,dx. \tag{7.111b}$$

Here the superscript or subscript i on variables indicates that they belong to the ith part. Note that the modulus of elasticity and moment of inertia are assumed to be constant within each part, while the load is arbitrary but continuous in each part. For uniformly distributed load $q^{(i)}(x) = q_0^{(i)}$ on part i, we have

$$\begin{Bmatrix} q_1^{(i)} \\ q_2^{(i)} \\ q_3^{(i)} \\ q_4^{(i)} \end{Bmatrix} = \frac{q_0^{(i)}L_i}{12}\begin{Bmatrix} 6 \\ -L_i \\ 6 \\ L_i \end{Bmatrix}. \tag{7.111c}$$

Equations (7.111a) relate the four generalized displacements $(w_1^{(i)}, \theta_1^{(i)}, w_2^{(i)}, \theta_2^{(i)})$ to the four generalized forces $(Q_1^{(i)}, M_1^{(i)}, Q_2^{(i)}, M_2^{(i)})$. Obviously, four of the eight variables should be known in order to solve the four equations. If there are n parts, there will be a total of $4n$ equations in $8n$ variables. Hence the remaining $4n$ variables should be eliminated through known conditions (e.g., boundary conditions, continuity conditions, and equilibrium conditions).

To illustrate the ideas, consider a simply supported beam with a point load F_0 at the center. Obviously, the beam needs to be divided into two parts, $0 \le x \le L/2$ and $L/2 \le x \le L$. The Ritz equations for the two parts are given below ($E_1 = E_2 = E$, $I_1 = I_2 = I$, $L_1 = L_2 = L/2$, and $q(x) = 0$).

Part 1:

$$\frac{16EI}{L^3}\begin{bmatrix} 6 & -1.5L & -6 & -1.5L \\ -1.5L & 0.5L^2 & 1.5L & 0.25L \\ -6 & 1.5L & 6 & 1.5L \\ -1.5L & 0.25L & 1.5L & 0.5L^2 \end{bmatrix}\begin{bmatrix} w_1^{(1)} \\ \theta_1^{(1)} \\ w_2^{(1)} \\ \theta_2^{(1)} \end{bmatrix} = \begin{bmatrix} Q_1^{(1)} \\ M_1^{(1)} \\ Q_2^{(1)} \\ M_2^{(1)} \end{bmatrix}. \tag{7.112a}$$

Part 2:

$$\frac{16EI}{L^3}\begin{bmatrix} 6 & -1.5L & -6 & -1.5L \\ -1.5L & 0.5L^2 & 1.5L & 0.25L \\ -6 & 1.5L & 6 & 1.5L \\ -1.5L & 0.25L & 1.5L & 0.5L^2 \end{bmatrix}\begin{Bmatrix} w_1^{(2)} \\ \theta_1^{(2)} \\ w_2^{(2)} \\ \theta_2^{(2)} \end{Bmatrix} = \begin{Bmatrix} Q_1^{(2)} \\ M_1^{(2)} \\ Q_2^{(2)} \\ M_2^{(2)} \end{Bmatrix}. \tag{7.112b}$$

There are a total of eight equations in 16 variables. However, some of the variables are duplicative and others are related. In particular, we have the following eight conditions (see Fig. 7.7):

Boundary Conditions

$$w_1^{(1)} = 0, \qquad w_2^{(2)} = 0, \qquad M_1^{(1)} = 0, \qquad M_2^{(2)} = 0. \tag{7.113a}$$

Continuity Conditions

$$w_2^{(1)} = w_1^{(2)}, \qquad \theta_2^{(1)} = \theta_1^{(2)}. \tag{7.113b}$$

Equilibrium Conditions

$$Q_2^{(1)} + Q_1^{(2)} = F_0, \qquad M_2^{(1)} + M_1^{(2)} = 0. \tag{7.113c}$$

Imposition of conditions in Eqs. (7.113a,b) is straightforward. However, to impose the equilibrium conditions (7.113c), we must add the third equation in (7.112a) to the first equation in (7.112b) and then, using the first condition of Eq. (7.113c) [conditions in Eqs. (7.113a,b) are also used], we obtain

$$\frac{16EI}{L^3}(1.5L\Theta_1 + 12W_2 - 1.5L\Theta_3) = F_0, \tag{7.114a}$$

where $w_2^{(1)} = W_2$, $\theta_1^{(1)} = \Theta_1$, $\theta_2^{(1)} = \Theta_2$, and $\theta_2^{(2)} = \Theta_3$. Similarly, adding the fourth equation in (7.112a) to the second equation in (7.112b) and then using the second condition in (7.113c), we obtain

$$\frac{16EI}{L^3}\left(0.25L\Theta_1 + L^2W_2 + 1.5L\Theta_3\right) = 0. \tag{7.114b}$$

Thus, we have a total of six equations: first two equations of Eq. (7.112a), the last two equations of Eq. (7.112b), and two equations in Eqs. (7.114a,b) in six unknowns $Q_1^{(1)}$, Θ_1, W_2, Θ_2, Θ_3, and $Q_2^{(2)}$. Solving the middle four equations for (Θ_1, W_2, Θ_2, Θ_3), we obtain

$$\Theta_1 = -\frac{F_0 L^2}{16EI}, \qquad W_2 = \frac{F_0 L^3}{48EI}, \qquad \Theta_2 = 0.0, \qquad \Theta_3 = \frac{F_0 L^2}{16EI}, \tag{7.115}$$

and $Q_1^{(1)}$ and $Q_2^{(2)}$ can be computed to be $-0.5F_0$ from the first equation of (7.112a) and the last equation of (7.112b). The rotation Θ_2 at the center of the beam is correctly predicted to be zero (why?). The four-parameter Ritz solution [but with the same polynomial degree as in the two-parameter solution (7.95a,b)] becomes

$$W_4(x) = \begin{cases} \dfrac{F_0 L^3}{16EI}\left(\dfrac{x}{L} - 4\dfrac{x^2}{L^2} + 4\dfrac{x^3}{L^3}\right) + \dfrac{F_0 L^3}{48EI}\left(12\dfrac{x^2}{L^2} - 16\dfrac{x^3}{L^3}\right), & 0 \le x \le \dfrac{L}{2}, \\ \dfrac{F_0 L^3}{48EI}\left[1 - 12\left(\dfrac{\bar{x}}{L}\right)^2 + 16\left(\dfrac{\bar{x}}{L}\right)^3\right] + \dfrac{F_0 L^3}{16EI}\left[2\left(\dfrac{\bar{x}}{L}\right)^2 - 4\left(\dfrac{\bar{x}}{L}\right)^3\right], & \dfrac{L}{2} \le x \le L, \end{cases} \tag{7.116}$$

where $\bar{x} = x - L/2$. Simplification of Eq. (7.116) gives the expression in Eq. (7.99); thus, the Ritz solution matches with the exact solution.

The procedure described in this example closely resembles that of the finite element method, which will be discussed in detail in Chapter 9. The procedure is valid for *all* beam problems irrespective of the nature of the loading and boundary conditions. In addition, the procedure gives exact values of the deflection $w_0(x)$ and rotation $-(dw_0/dx)$ at the end points of each part for all loads and boundary conditions, provided that the flexural rigidity EI is constant within each part.

7.4 GENERAL BOUNDARY-VALUE PROBLEMS

All the examples presented in Section 7.3.3 utilized the minimum total potential energy principle or Hamilton's principle. For problems outside the field of solid and structural mechanics, the construction of an analog of the minimum total potential energy principle is needed to derive the Ritz equations. Here the procedure for constructing *weak forms* from differential equations and use of these statements in the Ritz approximation are studied.

7.4.1 Variational Formulations

The steps involved in the weak formulation of differential equations are described with the aid of three model equations: (1) a second-order equation in one dimension, (2) a fourth-order equation in one dimension, and (3) a second-order equation in two dimensions. These equations are quite general and they arise in a number of fields in engineering and applied sciences. Most of the ideas in developing a weak form are presented in connection with Model Equation 1. Model Equation 2 is used to illustrate the weak form development for a higher-order equation, and Model Equation 3 is used to extend the ideas to two dimensions.

Model Equation 1 Consider the problem of finding the function $u(x)$ that satisfies the differential equation

$$-\frac{d}{dx}\left(a(x)\frac{du}{dx}\right) + c(x)u - f = 0 \qquad \text{for } 0 < x < L, \tag{7.117a}$$

and the boundary conditions

$$u(0) = u_0, \qquad \left(a\frac{du}{dx}\right)\bigg|_{x=L} = P, \tag{7.117b}$$

where $a = a(x)$, $c = c(x)$, $f = f(x)$, u_0, and P are the data (i.e., known quantities) of the problem. Equation (7.117a) arises in connection with the analytical description of many physical processes. For example, conduction and convection heat transfer in a plane wall or fin (1D heat transfer), flow through channels and pipes, transverse deflection of cables, axial deformation of bars, and many others. Table 7.2 contains a list of several field problems described by (7.117a) when $c(x) = 0$. The mathematical structure common to different fields is brought out in this table. Thus, if we can develop a numerical procedure by which Eq. (7.117a) can be solved for all physically possible boundary conditions, the procedure can be used to solve all field problems listed in Table 7.2, as well as many others. This fact provides us with the motivation to use Eq. (7.2) as a model second-order equation in one dimension. The motivation for the development of the weak form and a step-by-step procedure for the weak form development are discussed next.

In the Ritz method, we seek an approximate solution to Eq. (7.117a) in the form

$$U_N(x) = \sum_{j=1}^{N} c_j\phi_j(x) + \phi_0(x), \tag{7.118}$$

and determine the unknown parameters c_j such that Eqs. (7.117a) and (7.117b) are satisfied by the N-parameter approximate solution U_N. For example, suppose that $L = 1$, $a = x$, $c = 1$, $u_0 = 1$, $f = 0$, and $P = 0$; then we could take $N = 2$ and write the approximate solution of (7.117a) in the form ($\phi_1 = x^2 - 2x$, $\phi_2 = x^3 - 3x$, $\phi_0 = 1$):

$$u \approx U_N = c_1(x^2 - 2x) + c_2(x^3 - 3x) + 1,$$

which satisfies the boundary conditions (7.117b) of the problem for any values of c_1 and c_2. Then the constants c_1 and c_2 must be determined such that the approximate solution U_N satisfies Eq. (7.117a):

$$-\frac{d}{dx}\left(x\frac{du}{dx}\right) + u = 0 \tag{a}$$

Table 7.2 Some examples of second-order equation (7.117a) in one dimension:

$$-\frac{d}{dx}\left(a\frac{du}{dx}\right) = f \qquad \text{for } 0 < x < L$$

$$\text{EBC:} \quad u(0) = u_0; \qquad \text{NBC:} \quad \left(a\frac{du}{dx}\right)_{x=L} = P$$

Field	Primary Variable u	Coefficient[a] a	Source Term f	Secondary Variable P
1. Cables	Transverse deflection	T	Distributed vertical force	Axial load
2. Bars	Longitudinal displacement	EA	Distributed axial force	Axial load
3. Heat transfer	Temperature	k	Internal heat generation	Heat flux Heat
4. Pipe flows	Hydrostatic pressure	$\frac{\pi D^4}{128\mu}$	Flow source	Flow rate
5. Viscous flows	Velocity	μ	Pressure gradient	Stress
6. Seepage	Fluid head	ε	Fluid flux	Flow
7. Electrostatics	Electrical potential	ϵ	Charge density	Electric flux

[a] E = Young's modulus; A = area of cross section; D = diameter of the pipe; k = thermal conductivity; μ = viscosity; T = tension; ε = permeability; ϵ = dielectric constant.

in some sense. If we require U_N to satisfy the above equation in the exact sense, we obtain

$$-\frac{dU_N}{dx} - x\frac{d^2U_N}{dx^2} + U_N = -2c_1(x-1) - 3C_2(x^2-1) - 2c_1x - 6c_2x^2 + c_1(x^2-2x) + c_2(x^3-3x) + 1 = 0.$$

Since this expression must be zero at all x, the coefficients of the various powers of x must be zero:

$$1 + 2c_1 + 3c_2 = 0,$$
$$-(6c_1 + 3c_2) = 0,$$
$$c_1 - 9c_2 = 0,$$
$$c_2 = 0.$$

The above relations are inconsistent; hence there is *no solution* to the equations. If we were able to find a unique solution to these equations, then $U_N = c_1(x^2 - 2x) + c_2(x^3 - 3x) + 1$ is the exact solution of the problem. It is not always possible for arbitrary data of the problem to find the exact solution. An alternative is that we may require the approximate solution U_N to satisfy the differential equation (a) in a weighted-integral sense,

$$\int_0^1 w(x) R\, dx = 0, \tag{b}$$

where R is the *residual* (i.e., error) in the differential equation,

$$R \equiv -\frac{dU_N}{dx} - x\frac{d^2U_N}{dx^2} + U_N \neq 0,$$

and $w(x)$ is a function, called the *weight function*, which is introduced to provide as many independent relations among c_j as there are unknown parameters c_j ($j = 1, 2, \ldots, N$). This is accomplished by selecting N independent functions for $w(x)$. For example, if we take the two choices $w(x) = 1$ and $w(x) = x$, which are linearly independent, we obtain

$$0 = \int_0^1 1 \cdot R\, dx = (1 + 2c_1 + 3c_2) + \tfrac{1}{2}(-6c_1 - 3c_2) + \tfrac{1}{3}(c_1 - 9c_2) + \tfrac{1}{4}c_2,$$

$$0 = \int_0^1 x \cdot R\, dx = \tfrac{1}{2}(1 + 2c_1 + 3c_2) + \tfrac{1}{3}(-6c_1 - 3c_2) + \tfrac{1}{4}(c_1 - 9c_2) + \tfrac{1}{5}c_2,$$

or

$$\tfrac{2}{3}c_1 + \tfrac{5}{4}c_2 = 1, \qquad \tfrac{3}{4}c_1 + \tfrac{31}{20}c_2 = \tfrac{1}{2}. \tag{c}$$

These equations provide two linearly independent relations for c_1 and c_2 that can be solved to obtain unique solution, $c_1 = 222/23$ and $c_2 = -100/23$, and the approximate solution $U_2(x)$ becomes

$$U_2(x) = c_1(x^2 - 2x) + c_2(x^3 - 3x) + 1 = \tfrac{222}{23}(x^2 - 2x) - \tfrac{100}{23}(x^3 - 3x) + 1. \tag{d}$$

Thus, integral statements of the type in (b) provide means for obtaining as many algebraic equations as there are unknown coefficients in the approximation. There are several variational methods, in addition to the Ritz method, in which approximate solutions of the type $u \approx \sum c_j\phi_j + \phi_0$ are sought, and the coefficients c_j are determined, as shown above, using an integral statement. These methods differ from each other in the choice of the weight function $w(x)$ and the integral statement used, which in turn dictates the choice of the approximation functions ϕ_j and ϕ_0. For the moment we deal with the Ritz method of approximation.

As discussed above, the necessary and sufficient number of algebraic relations among the c_j's can be obtained by recasting the differential equation (7.117) in a weighted-integral form:

$$0 = \int_0^L w(x)\left[-\frac{d}{dx}\left(a\frac{du}{dx}\right) + cu - f\right] dx, \tag{7.119}$$

where $w(x)$ denotes the weight function, which for the moment is arbitrary. For $u(x) \approx U_N(x)$ and each independent choice of $w(x)$, we obtain an independent algebraic equation relating all c_j. A total of N independent equations are required to solve for the N parameters c_j. When a weighted-integral statement like Eq. (7.119) is used to obtain the N equations among c_j, the method is known as the *weighted-residual method*, and it will be discussed in Section 7.5. Note that the use of Eq. (7.119) precludes that $U_N(x)$ satisfies *all* specified boundary conditions and is differentiable as many times as required in the differential equation.

To weaken the continuity (i.e., differentiability) required of $U_N(x)$ and therefore of $\phi_j(x)$, we trade the differentiation in (7.119) from u to w such that both u and w are differentiated equally—once each in the present case. The resulting integral form is termed the *weak form* of (7.117). This form is not only equivalent to (7.117) but it also contains the natural boundary conditions of the problem, and therefore $U_N(x)$ need not satisfy the natural boundary conditions. The three-step procedure of constructing the weak form of (7.117) is discussed next.

The first step is to multiply the governing differential equation with a weight function w and integrate over domain $(0, L)$, giving Eq. (7.119). The second step is to trade differentiation from u to w, using integration by parts. This is achieved as follows. Consider the identity

$$-w\left[\frac{d}{dx}\left(a\frac{du}{dx}\right)\right] = -\frac{d}{dx}\left(wa\frac{du}{dx}\right) + a\frac{dw}{dx}\frac{du}{dx}, \tag{7.120a}$$

which is simply the product rule of differentiation applied to the product of two functions, $a(du/dx)$ and w. Integrating this identity over the domain, we obtain

$$\begin{aligned} -\int_0^L w\left[\frac{d}{dx}\left(a\frac{du}{dx}\right)\right]dx &= -\int_0^L \frac{d}{dx}\left(wa\frac{du}{dx}\right)dx + \int_0^L a\frac{dw}{dx}\frac{du}{dx}\,dx \\ &= -\left[wa\frac{du}{dx}\right]_0^L + \int_0^L a\frac{dw}{dx}\frac{du}{dx}\,dx. \end{aligned} \tag{7.120b}$$

Substituting (7.120b) into (7.119), we arrive at the result

$$0 = \int_0^L \left(a\frac{dw}{dx}\frac{du}{dx} + cwu - wf\right)dx - \left[w \cdot a\frac{du}{dx}\right]_0^L. \tag{7.121}$$

The third and last step is to identify the primary and secondary variables of the variational (or weak) form. This requires us to classify the boundary conditions of each differential equation into *essential* (or geometric) and *natural* (or force) boundary conditions. The classification is made uniquely by examining the boundary term appearing in the second step of the weak form development, namely, Eq. (7.121):

$$\left[w \cdot a\frac{du}{dx}\right]_0^L.$$

As a rule, the coefficient of the weight function in the boundary expression is called the *secondary variable*, and its specification constitutes the *natural* or *Neumann* boundary condition. The dependent unknown *in the same form as the weight function* in the boundary expression is termed the *primary variable*, and its specification constitutes the *essential* or *Dirichlet* boundary condition. For the model equation at hand, the primary and secondary variables are

$$u \quad \text{and} \quad a\frac{du}{dx} \equiv Q.$$

Next, we denote the secondary variables at the end points by some symbols:

$$-Q_0 = \left(a\frac{du}{dx}\right)\bigg|_0, \qquad Q_L = \left(a\frac{du}{dx}\right)\bigg|_L = P. \tag{7.122a}$$

With the notation in (7.122a), the variational form becomes

$$0 = \int_0^L \left(a\frac{dw}{dx}\frac{du}{dx} + cwu - wf\right)dx - w(0)Q_0 - w(L)Q_L. \tag{7.122b}$$

Now is the time to discuss the conditions on the weight function $w(x)$. Clearly, it should be differentiable at least once (like $u(x)$ is). The Ritz method uses the weak form, with $w = \phi_i$ to obtain the ith equation of the set of N relations among c_j's. Thus, we require $w(x)$ to satisfy the homogeneous form of specified essential boundary conditions (i.e., to belong to the set of admissible variations). In the present case, $w(x)$ should be once differentiable and vanish at $x = 0$. Hence, the final weak form is

$$0 = \int_0^L \left(a\frac{dw}{dx}\frac{du}{dx} + cwu - wf\right)dx - w(L)P. \tag{7.123}$$

This completes the three-step procedure of constructing the weak form. The weak form in (7.123) contains two types of expressions: those containing both w and u; and those containing only w. We group the former type into a single expression, called the *bilinear form*:

$$B(w, u) \equiv \int_0^L \left(a\frac{dw}{dx}\frac{du}{dx} + cwu\right)dx. \tag{7.124a}$$

We denote all terms containing only w (but not u) by $l(w)$, called the *linear form*:

$$l(w) = \int_0^L wf\,dx + w(L)P. \tag{7.124b}$$

The statement (7.123) can now be expressed as one of finding u from the set of admissible functions (i.e., differentiable at least once and satisfying the essential boundary conditions) such that the variational problem

$$B(w, u) = l(w) \tag{7.125}$$

is satisfied for all w from the set of admissible variations. As seen before, the bilinear form results directly in the coefficient matrix, while the linear form gives rise to the right-hand-side column vector of the Ritz equations. Since $B(\cdot,\cdot)$ is symmetric, the functional associated with the variational problem (7.125) is given by Eq. (7.53) (which represents the total potential energy in the case of bars or cables):

$$\begin{aligned} I(u) &= \frac{1}{2}B(u,u) - l(u) \\ &= \int_0^L \left[\frac{a}{2}\left(\frac{du}{dx}\right)^2 + \frac{c}{2}u^2\right]dx - \int_0^L wf\,dx - w(L)P. \end{aligned} \tag{7.126}$$

The weak form development up to Eq. (7.125) is valid even for the case in which the coefficients a and c are functions of the dependent variable u, making the problem nonlinear. In that case, $B(w,u)$ will be no longer a bilinear form and the functional $I(u)$ may not exist. However, to use the Ritz method one needs only the variational statement (7.125), which always exists.

Model Equation 2 Here we consider a fourth-order differential equation of the form

$$\frac{d^2}{dx^2}\left(b(x)\frac{d^2u}{dx^2}\right) + c(x)u = f, \qquad 0 < x < L, \tag{7.127}$$

We will not be concerned with any specific boundary conditions, as the weak form development naturally leads to the classification of the variables into primary and secondary type, and their specification constitutes, respectively, the essential and the natural type. This equation arises, for example, in connection with the bending of straight beams, where $u(x)$ denotes the transverse deflection, $b(x) = E(x)I(x)$ the bending rigidity, $c(x) = k$ the foundation modulus (if any), and $f(x)$ the distributed transverse load.

The first two steps in the development of the weak form of the equation are summarized below. Note that in this case we must transfer two derivatives to the weight function so that both w and u are required to have the same order of continuity (or differentiability).

Step 1:

$$0 = \int_0^L w(x)\left[\frac{d^2}{dx^2}\left(b\frac{d^2u}{dx^2}\right) + cu - f\right]dx. \tag{7.128}$$

Step 2:

$$0=\int_0^L\left[-\frac{dw}{dx}\frac{d}{dx}\left(b\frac{d^2u}{dx^2}\right)+cwu-wf\right]dx+\left[w\cdot\frac{d}{dx}\left(b\frac{d^2u}{dx^2}\right)\right]_0^L$$

$$= \int_0^L \left(b\frac{d^2w}{dx^2}\frac{d^2u}{dx^2} + cwu - wf \right) dx + \left[w \cdot \frac{d}{dx}\left(b\frac{d^2u}{dx^2} \right) \right]_0^L - \left[\frac{dw}{dx} \cdot b\frac{d^2u}{dx^2} \right]_0^L. \tag{7.129}$$

It is clear from the boundary expressions that the secondary variables are

$$\frac{d}{dx}\left(b\frac{d^2u}{dx^2} \right), \qquad b\frac{d^2u}{dx^2}. \tag{7.130a}$$

In the case of beams they represent the shear force and bending moment in the beam. The primary variables are $[w \to u$ and $(dw/dx) \to (du/dx)]$

$$u, \qquad \frac{du}{dx}. \tag{7.130b}$$

Step 3: Denoting the shear forces and bending moments at the two ends of the beam as (proper signs are inserted to make all of the Q's and M's have the negative sign in the weak form; this also happens to be the correct definition of the bending moments and shear forces on the left and right ends of the beam):

$$\begin{aligned} \left[\frac{d}{dx}\left(b\frac{d^2u}{dx^2} \right) \right]_{x=0} &= Q_0, & \left[b\frac{d^2u}{dx^2} \right]_{x=0} &= M_0, \\ \left[-\frac{d}{dx}\left(b\frac{d^2u}{dx^2} \right) \right]_{x=L} &= Q_L, & \left[-b\frac{d^2u}{dx^2} \right]_{x=0} &= M_L. \end{aligned} \tag{7.131}$$

Then Eq. (7.129) takes the final form

$$\begin{aligned} 0 = \int_0^L \left(b\frac{d^2w}{dx^2}\frac{d^2u}{dx^2} + cwu - wf \right) dx - w(0)Q_0 - w(L)Q_L \\ - \left(-\frac{dw}{dx} \right)\Big|_0 M_0 - \left(-\frac{dw}{dx} \right)\Big|_L M_L. \end{aligned} \tag{7.132}$$

The bilinear form, linear form, and functional for the problem are

$$\begin{aligned} B(w, u) &= \int_0^L \left(b\frac{d^2w}{dx^2}\frac{d^2u}{dx^2} + cwu \right) dx, \\ l(w) &= \int_0^L wf\, dx + w(0)Q_0 + w(L)Q_L + \left(-\frac{dw}{dx} \right)\Big|_0 M_0 + \left(-\frac{dw}{dx} \right)\Big|_L M_L, \\ I(u) &= \int_0^L \left[\frac{b}{2}\left(\frac{d^2u}{dx^2} \right)^2 + \frac{c}{2}u^2 - fu \right] dx \\ &\quad - u(0)Q_0 - u(L)Q_L - \left(-\frac{du}{dx} \right)\Big|_0 M_0 - \left(-\frac{du}{dx} \right)\Big|_L M_L. \end{aligned} \tag{7.133}$$

In the case of beam bending, $I(u)$ is nothing but the total potential energy $\Pi(u)$.

The weight function $w(x)$ is required to be twice differentiable and to vanish at the points where u and to du/dx are specified. Equation (7.132) can be specialized to any beam with specific boundary conditions and loading.

Model Equation 3 Lastly, we consider the problem of determining the solution $u(x, y)$ to the partial differential equation

$$-\frac{\partial}{\partial x}\left(a_1\frac{\partial u}{\partial x}\right)-\frac{\partial}{\partial y}\left(a_2\frac{\partial u}{\partial y}\right)+a_0u=f \qquad \text{in } R \tag{7.134}$$

in a two-dimensional domain R. Here a_0, a_1, a_2, and f are known functions of position (x, y) in R. The function u is required to satisfy, in addition to the differential equation (7.134), certain boundary conditions on the boundary S of R. The variational formulation to be presented tells us the precise form of the essential and natural boundary conditions of the equation. Equation (7.134) arises in many fields of engineering, including in 2D heat transfer, stream function or velocity potential formulation of inviscid flows, transverse deflections of a membrane, and torsion of a cylindrical member.

The three-step procedure applied to Eq. (7.134) results in the following equations:

Step 1:

$$0=\int_R w\left[-\frac{\partial}{\partial x}\left(a_1\frac{\partial u}{\partial x}\right)-\frac{\partial}{\partial y}\left(a_2\frac{\partial u}{\partial y}\right)+a_0u-f\right]dxdy. \tag{7.135}$$

Step 2:

$$\begin{aligned}0=&\int_R\left(a_1\frac{\partial w}{\partial x}\frac{\partial u}{\partial x}+a_2\frac{\partial w}{\partial y}\frac{\partial u}{\partial y}+a_0wu-wf\right)dxdy\\&-\oint_S w\left(a_1\frac{\partial u}{\partial x}n_x+a_2\frac{\partial u}{\partial y}n_y\right)ds,\end{aligned} \tag{7.136}$$

where we used integration by parts [or the Green–Gauss theorem, Eq. (2.89)] to transfer differentiation to w so that both u and w have the same order derivatives. The boundary term shows that u is the primary variable while

$$a_1\frac{\partial u}{\partial x}n_x+a_2\frac{\partial u}{\partial y}n_y$$

is the secondary variable.

Step 3: The last step in the procedure is to impose the specified boundary conditions. Suppose that u is specified on portion S_1 and the natural boundary condition is specified on the remaining portion S_2 of the boundary:

$$u = \hat{u} \quad \text{on } S_1, \qquad a_1 \frac{\partial u}{\partial x} n_x + a_2 \frac{\partial u}{\partial y} n_y = \hat{g} \quad \text{on } S_2. \tag{7.137}$$

Then w is arbitrary on S_2 and equal to zero on S_1. Consequently, Eq. (7.136) simplifies to

$$0 = \int_R \left(a_1 \frac{\partial w}{\partial x} \frac{\partial u}{\partial x} + a_2 \frac{\partial w}{\partial y} \frac{\partial u}{\partial y} + a_0 w u - w f \right) dx dy - \int_{S_2} w \hat{g} \, ds. \tag{7.138}$$

The bilinear form, linear form, and functionals are

$$B(w, u) = \int_R \left(a_1 \frac{\partial w}{\partial x} \frac{\partial u}{\partial x} + a_2 \frac{\partial w}{\partial y} \frac{\partial u}{\partial y} + a_0 w u \right) dx dy, \tag{7.139a}$$

$$l(w) = \int_R w f \, dx dy + \int_{S_2} w \hat{g} \, ds, \tag{7.139b}$$

$$I(u) = \frac{1}{2} \int_R \left[a_1 \left(\frac{\partial u}{\partial x} \right)^2 + a_2 \left(\frac{\partial u}{\partial y} \right)^2 + a_0 u^2 \right] dx dy - \int_R u f \, dx dy - \int_{S_2} \hat{g} u \, ds. \tag{7.139c}$$

Once again, $I(u)$ represents the total potential energy in the case of a membrane problem.

7.4.2 Ritz Approximations

The Ritz method can be applied directly to the weak forms (7.124), (7.132), and (7.138)—or into the variational problem

$$B(w, u) = l(w). \tag{7.140}$$

Here we consider the case in which $B(\cdot, \cdot)$ is a bilinear form. Substituting

$$u \approx U_N = \sum_{j=1}^{N} c_j \phi_j + \phi_0, \qquad w = \phi_i, \tag{7.141}$$

into Eq. (7.140) to obtain the ith equation

$$\sum_{j=1}^{N} a_{ij} c_j = b_i, \qquad j = 1, 2, \ldots, N, \tag{7.142a}$$

where

$$a_{ij} = B(\phi_i, \phi_j), \qquad b_i = l(\phi_i) - B(\phi_0, \phi_i). \tag{7.142b}$$

The specific expressions of A_{ij} and b_i of each model equation can be written using the respective bilinear and linear forms. For example, for the third model equation, we have

$$a_{ij} = \int_R \left(a_1 \frac{\partial \phi_i}{\partial x}\frac{\partial \phi_j}{\partial x} + a_2 \frac{\partial \phi_i}{\partial y}\frac{\partial \phi_j}{\partial y} + a_0 \phi_i \phi_j \right) dxdy,$$

$$b_i = \int_R \left(\phi_i f - a_1 \frac{\partial \phi_i}{\partial x}\frac{\partial \phi_0}{\partial x} - a_2 \frac{\partial \phi_i}{\partial y}\frac{\partial \phi_0}{\partial y} - a_0 \phi_i \phi_0 \right) dxdy + \int_{S_2} \phi_i \hat{g}\, dS. \tag{7.143}$$

Once ϕ_i and ϕ_0 are selected, subject to the conditions stated in Section 7.2, the coefficients of matrix $[A]$ and column vector $\{b\}$ can be computed, and the linear algebraic equations in Eq. (7.142a) can be solved for the Ritz coefficients. We consider specific examples next.

Example 7.9 We wish to solve the partial differential equation (Laplace's equation)

$$-\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) = 0 \qquad \text{in } 0 < (x, y) < 1, \tag{7.144a}$$

subject to the boundary conditions (see Fig. 7.8):

$$\begin{aligned} u &= 0 \qquad \text{on } x = 0, 1 \text{ and } y = 0, \\ u &= \sin \pi x \qquad \text{on } y = 1. \end{aligned} \tag{7.144b}$$

The weak form of the equation can be obtained as a special case from Eq. (7.138) by setting $f = 0$, $a_1 = a_2 = 1$, $a_0 = 0$, and $S_2 = 0$ ($S_1 = S$):

$$0 = \int_0^1 \int_0^1 \left(\frac{\partial \delta u}{\partial x}\frac{\partial u}{\partial x} + \frac{\partial \delta u}{\partial y}\frac{\partial u}{\partial y} \right) dxdy. \tag{7.145}$$

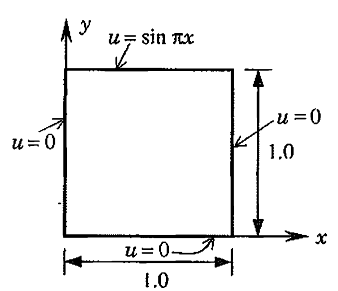

Figure 7.8 Domain and boundary conditions of the problem in Example 7.9.

The Ritz equations are given by Eq. (7.142a) with a_{ij} and b_i given by

$$a_{ij} = \int_0^1 \int_0^1 \left(\frac{\partial \phi_i}{\partial x}\frac{\partial \phi_j}{\partial x} + \frac{\partial \phi_i}{\partial y}\frac{\partial \phi_j}{\partial y} \right) dxdy,$$
$$b_i = -\int_0^1 \int_0^1 \left(\frac{\partial \phi_i}{\partial x}\frac{\partial \phi_0}{\partial x} + \frac{\partial \phi_i}{\partial y}\frac{\partial \phi_0}{\partial y} \right) dxdy. \tag{7.146}$$

Next we select ϕ_0 and ϕ_i for the problem. The function ϕ_0 is required to satisfy the boundary conditions in Eq. (7.144b) because all of them are of the essential type. The following choice for ϕ_0 meets the conditions:

$$\phi_0 = y \sin \pi x. \tag{7.147a}$$

The functions $\phi_i (i = 1, 2, \ldots, n)$ are required to satisfy the homogenous form of Eq. (7.144b) and to be linearly independent. We choose

$$\phi_1 = \sin \pi x \sin \pi y, \qquad \phi_2 = \sin \pi x \sin 2\pi y, \qquad \phi_3 = \sin 2\pi x \sin \pi y, \text{ etc.} \tag{7.147b}$$

From Eq. (7.146) we obtain

$$a_{ij} = \begin{cases} \frac{\pi^2}{2}, & \text{if } i = j, \\ 0, & \text{if } i \neq j \end{cases}, \quad \text{(i.e., [A] is diagonal),}$$
$$b_i = \begin{cases} -\frac{\pi}{2}, & \text{if } i = 1, \\ 0, & \text{if } i \neq 1. \end{cases} \tag{7.147c}$$

Hence, the Ritz solution becomes ($c_1 = -1/\pi$ and all other $c_i = 0$):

$$U_1(x, y) = c_1\phi_1(x, y) + \phi_0(x, y)$$
$$= y \sin \pi x - \frac{1}{\pi} \sin \pi x \sin \pi y = \sin \pi x \left(y - \frac{1}{\pi} \sin \pi y \right). \tag{7.148a}$$

The exact solution of Eq. (7.144a) is given by

$$u(x, y) = \frac{\sin \pi x \sinh \pi y}{\sinh \pi}. \tag{7.148b}$$

Example 7.10 Consider the following pair of coupled differential equations, which arise in connection with the Timoshenko beam theory (see Exercise 6.15):

$$\frac{\partial}{\partial x}\left[S\left(\frac{\partial w_0}{\partial x} + \phi_x \right) \right] + q = I_0 \frac{\partial^2 w_0}{\partial t^2}, \tag{7.149}$$

$$\frac{\partial}{\partial x}\left(D\frac{\partial \phi_x}{\partial x} \right) - S\left(\frac{\partial w_0}{\partial x} + \phi_x \right) = I_2 \frac{\partial^2 \phi_x}{\partial t^2}, \tag{7.150}$$

where S is the shear stiffness ($S = K_s GA$; K_s is the shear correction coefficient, G the shear modulus, and A the area of cross section), D the bending stiffness, w_0 the transverse deflection, ϕ_x the rotation, q the distributed transverse load, and I_0 and I_2 are mass inertias. Assume that D, S, I_0, and I_2 are constants. The "specified" boundary conditions are of the form (as will be clear from the third step of the weak form)

$$-\left(D\frac{\partial \phi_x}{\partial x}\right)_{x=0} = M_1, \qquad \left(D\frac{\partial \phi_x}{\partial x}\right)_{x=L} = M_2, \tag{7.151a}$$

$$-S\left(\frac{\partial w_0}{\partial x} + \phi_x\right)_{x=0} = Q_1, \qquad S\left(\frac{\partial w_0}{\partial x} + \phi_x\right)_{x=L} = Q_2. \tag{7.151b}$$

We wish to derive the Ritz equations of the problem.

First we develop the weak form of the equations using the three-step procedure. Multiply the first equation with weight function v_1 and the second one with weight function v_2 and integrate over the length of the beam to obtain

$$\begin{aligned} 0 &= \int_0^L v_1 \left\{ \frac{\partial}{\partial x}\left[S\left(\frac{\partial w_0}{\partial x} + \phi_x\right)\right] + q - I_0\frac{\partial^2 w_0}{\partial t^2}\right\} dx \\ &= \int_0^L \left\{ -S\frac{\partial v_1}{\partial x}\left(\frac{\partial w_0}{\partial x} + \phi_x\right) + v_1 q - I_0 v_1 \frac{\partial^2 w_0}{\partial t^2}\right\} dx \\ &\quad + \left[S\left(\frac{\partial w_0}{\partial x} + \phi_x\right) v_1\right]_0^L \\ &= \int_0^L \left\{ -S\frac{\partial v_1}{\partial x}\left(\frac{\partial w_0}{\partial x} + \phi_x\right) + v_1 q - I_0 v_1 \frac{\partial^2 w_0}{\partial t^2}\right\} dx \\ &\quad + Q_1 v_1(0) + Q_2 v_1(L), \end{aligned} \tag{7.152a}$$

$$\begin{aligned} 0 &= \int_0^L v_2 \left\{ \frac{\partial}{\partial x}\left(D\frac{\partial \phi_x}{\partial x}\right) - S\left(\frac{\partial w_0}{\partial x} + \phi_x\right) - I_2\frac{\partial^2 \phi_x}{\partial t^2}\right\} dx \\ &= \int_0^L \left\{ -D\frac{\partial v_2}{\partial x}\frac{\partial \phi_x}{\partial x} - S v_2\left(\frac{\partial w_0}{\partial x} + \phi_x\right) - I_2 v_2 \frac{\partial^2 \phi_x}{\partial t^2}\right\} dx \\ &\quad + \left[D\frac{\partial \phi_x}{\partial x} v_2\right]_0^L \\ &= \int_0^L \left\{ -D\frac{\partial v_2}{\partial x}\frac{\partial \phi_x}{\partial x} - S v_2\left(\frac{\partial w_0}{\partial x} + \phi_x\right) - I_2 v_2 \frac{\partial^2 \phi_x}{\partial t^2}\right\} dx \\ &\quad + M_1 v_2(0) + M_2 v_2(L). \end{aligned} \tag{7.152b}$$

Note that integration by parts was used such that the expression $w_{0,x} + \phi_x$ is preserved, as it enters the boundary term representing the shear force. Such considerations can only be used by knowing the mechanics of the problem at hand. Also, note that the pair of weight functions (v_1, v_2) (from a product space of admissible variations) satisfy the homogeneous form of specified essential boundary conditions on the pair (w_0, ϕ_x) (with the correspondence $v_1 \sim w_0$ and $v_2 \sim \phi_x$). Writing the bilinear form for the problem is a little involved; one may treat $u = (w_0, \phi_x)$ and $v = (v_1, v_2)$ as vectors (from a vector space) and then write the bilinear form $B(v, u)$.

We assume Ritz approximations of the form

$$w_0(x,t) \approx \sum_{j=1}^{M} d_j(t)\psi_j(x) + \psi_0(x), \qquad \phi_x(x,t) \approx \sum_{j=1}^{N} c_j(t)\theta_j(x) + \theta_0(x), \tag{7.153}$$

and *derive* the ordinary differential equations involving the time derivatives of d_i and c_i. From weak forms (7.152a,b) it is clear that all of the specified boundary conditions are of the natural type. Hence $\psi_0(x) = 0$ and $\theta_0(x) = 0$.

Next we substitute the approximations into the weak forms for $w_0(x)$ and $\phi_x(x)$, and $v_1(x) = \psi_i(x)$ and $v_2(x) = \theta_i(x)$, and obtain

$$\begin{aligned} 0 &= \int_0^L \left[-S\frac{d\psi_i}{dx}\left(\sum_{j=1}^{m} \frac{d\psi_j}{dx} d_j + \sum_{j=1}^{n} c_j\theta_j \right) + \psi_i q - I_0\psi_i \left(\sum_{j=1}^{m} \frac{d^2 d_j}{dt^2}\psi_j \right) \right] dx \\ &\quad + Q_1\psi_i(0) + Q_2\psi_i(L) \\ &= -\sum_{j=1}^{m} A_{ij}d_j - \sum_{j=1}^{n} B_{ij}c_j - \sum_{j=1}^{m} M_{ij}^1 \frac{d^2 d_j}{dt^2} + F_i^1, \end{aligned} \tag{7.154}$$

$$\begin{aligned} 0 &= \int_0^L \left[-D\frac{d\theta_i}{dx}\left(\sum_{j=1}^{n} \frac{d\theta_j}{dx} c_j \right) - S\theta_i \left(\sum_{j=1}^{m} \frac{d\psi_j}{dx} d_j + \sum_{j=1}^{n} \theta_j c_j \right) \right. \\ &\qquad \left. - I_2\theta_i \left(\sum_{j=1}^{n} \frac{d^2 c_j}{dt^2}\theta_j \right) \right] dx + M_1\theta_i(0) + M_2\theta_i(L) \\ &= -\sum_{j=1}^{m} C_{ij}d_j - \sum_{j=1}^{n} D_{ij}c_j - \sum_{j=1}^{n} M_{ij}^2 \frac{d^2 c_j}{dt^2} + F_i^2, \end{aligned} \tag{7.155}$$

where

$$A_{ij} = \int_0^L S\frac{d\psi_i}{dx}\frac{d\psi_j}{dx}\,dx, \qquad B_{ij} = \int_0^L S\frac{d\psi_i}{dx}\theta_j\,dx,$$

$$M_{ij}^1 = \int_0^L I_0\psi_i\psi_j\,dx, \qquad F_i^1 = \int_0^L \psi_i q\,dx + Q_1\psi_i(0) + Q_2\psi_i(L),$$

$$C_{ij} = \int_0^L S\theta_i \frac{d\psi_j}{dx}\,dx, \qquad D_{ij} = \int_0^L \left(D\frac{d\theta_i}{dx}\frac{d\theta_j}{dx} + S\theta_i\theta_j\right)dx, \tag{7.156}$$

$$M_{ij}^2 = \int_0^L I_2\theta_i\theta_j\,dx, \qquad F_i^2 = M_1\theta_i(0) + M_2\theta_i(L).$$

In matrix form, we can write this as

$$\begin{bmatrix} [A] & [B] \\ [C] & [D] \end{bmatrix} \begin{Bmatrix} \{d\} \\ \{c\} \end{Bmatrix} + \begin{bmatrix} [M^1] & [0] \\ [0] & [M^2] \end{bmatrix} \begin{Bmatrix} \{\ddot{d}\} \\ \{\ddot{c}\} \end{Bmatrix} = \begin{Bmatrix} \{F^1\} \\ \{F^2\} \end{Bmatrix}, \tag{7.157a}$$

or

$$[K]\{\Delta\} + [M]\{\ddot{\Delta}\} = \{F\}. \tag{7.157b}$$

In closing this section, we make a couple of additional comments on the Ritz method. In developing the weak form, one should bear in mind that the boundary terms obtained from the integration by parts should be physically meaningful. The variational form used for the Ritz method does not have to be a quadratic functional, but it should be a form that includes the natural boundary conditions of the problem. Therefore, the Ritz method can be applied even to nonlinear problems. Of course, the resulting simultaneous algebraic equations are nonlinear and there can be more than one solution to the equations (see Example 7.20). The selection of approximation functions becomes increasingly difficult with the dimension and shape of the domain.

7.5 WEIGHTED-RESIDUAL METHODS

7.5.1 Introduction

As discussed earlier, weighted-residual methods are those in which we seek approximate solutions using a weighted-integral statement of the equation(s). To fix the ideas, consider a boundary-value problem described by the operator equation

$$A(u) = f \quad \text{in } \Omega, \tag{7.158a}$$

subjected to boundary conditions

$$B_1(u) = \hat{u} \quad \text{on } \Gamma_1, \qquad B_2(u) = \hat{g} \quad \text{on } \Gamma_2, \tag{7.158b}$$

where A is a linear or nonlinear differential operator, u is the dependent variable, f is a given force term in the domain Ω, B_1 and B_2 are boundary operators associated with essential and natural boundary conditions of the operator A, and $\hat{u}$ and $\hat{g}$ are specified values on the portions Γ_1 and Γ_2 of the boundary Γ of the domain. An example of Eqs. (7.159a) and (7.159b) is given by

$$A(u) = -\frac{d}{dx}\left(a\frac{du}{dx}\right), \qquad B_1(u) = u, \qquad B_2(u) = a\frac{du}{dx},$$

Γ_1 is the point $x = 0$, and Γ_2 is the point $x = L$. The weighted-integral statement of Eq. (7.159a) is

$$0 = \int_\Omega w(x)\,[A(u) - f]\,dx. \tag{7.159}$$

We seek a solution U_N that satisfies the specified boundary conditions (7.158b) such that the above equation is also satisfied. A complete discussion of the procedure will be given shortly.

The weak form of the operator equation (7.158a) can be constructed whenever the operator A permits the use of integration by parts [or gradient/divergence theorems in Eqs. (2.89a,b)] to transfer part of the differentiation from the dependent variable u to the weight function w and incorporate the natural boundary conditions of the problem. In general, the weak form can be constructed if the operator A is expressible as a product of two operators:

$$A = T^*(aT) \tag{7.160a}$$

where operator T^* is called the *adjoint* of T and is related to T by

$$\int_\Omega T(u)v\,d\mathbf{x} = \int_\Omega uT^*(v)\,d\mathbf{x} + \oint_\Gamma C_1(u)C_2(v)\,dS \tag{7.160b}$$

for all u and v. Here Ω is domain with boundary Γ. The boundary operators C_1 and C_2 depend on the operator A. For example, when $A = -(d/dx)[a(du/dx)]$, operators T, T^*, C_1, and C_2 are given by

$$T = \frac{d}{dx}, \qquad T^* = -\frac{d}{dx}, \qquad C_1 = 1, \qquad C_2 = 1.$$

If $A = -\nabla^2 = -\nabla \cdot \nabla$, then we have [$u$ must be a scalar and $\mathbf{v}$ a vector function of position in Eq. (7.160b)]:

$$T = \nabla \text{ (grad)}, \qquad T^* = -\nabla \cdot \text{(div)}, \qquad C_1 = 1, \qquad C_2 = \hat{\mathbf{n}}\cdot .$$

The weak form of Eq. (7.158a,b), with A given by Eq. (7.160a), can be derived as follows:

$$\begin{aligned}
0 &= \int_\Omega w\,[A(u) - f]\,d\mathbf{x} \\
&= \int_\Omega w\left[T^*(aT(u)) - f\right]d\mathbf{x} \\
&= \int_\Omega [T(w)(aT(u)) - wf]\,d\mathbf{x} - \oint_\Gamma B_1(w)B_2(u)\,dS,
\end{aligned}$$

where $C_1 = B_1$ and $B_2 = C_2(aT(u))$. Since $B_1(w) = 0$ on portion Γ_1 (where $B_1(u)$ is specified), and $B_2(u) = \hat{g}$ on Γ_2, with $\Gamma_1 + \Gamma_2 = \Gamma$, we have

$$0 = \int_\Omega [aT(w)T(u) - wf]\, d\mathbf{x} - \int_{\Gamma_2} B_1(w)\hat{g}\, dS,$$

which is the weak form we set to derive.

Returning to the weighted-residual methods, we seek an approximate solution of u in the form (as in the Ritz method)

$$U_N(x) = \sum_{j=1}^{N} c_j \phi_j(x) + \phi_0(x), \tag{7.161}$$

where the parameters c_j are determined by requiring that the residual in the governing equation due to the approximation

$$\mathcal{R}_N = A\left(\sum_{j=1}^{N} c_j \phi_j + \phi_0\right) - f \neq 0 \tag{7.162}$$

be orthogonal to a set of N linearly independent *weight functions* $\psi_i(x)$, which in general are different from the approximation functions ϕ_i:

$$\int_\Omega \psi_i \mathcal{R}_N(\mathbf{x}, \{c\}, \{\phi\}, f)\, d\mathbf{x} = 0, \qquad (i = 1, 2, \ldots, N). \tag{7.163}$$

Equation (7.163) is the same as that obtained by substituting approximation (7.161) into the weighted-residual statement (7.159). It provides N linearly independent equations for the determination of the parameters c_i. If A is a nonlinear operator, the resulting algebraic equations will be nonlinear.

The approximation functions (ϕ_0, ϕ_i) and weight functions ψ_i in a weighted-residual method must satisfy the following conditions:

1. $\phi_j (j = 1, 2, \ldots, N)$ should satisfy three conditions:
   (a) Each ϕ_j is *continuous* as required in the weighted-residual statement; i.e., ϕ_j should be such that U_N yields a nonzero value of $A(U_N)$.
   (b) Each ϕ_j satisfies the *homogeneous form* of *all* specified (i.e., essential as well as natural) boundary conditions.
   (c) The set $\{\phi_j\}$ is *linearly independent* and *complete*.
2. ϕ_0 has the main purpose of satisfying *all* specified boundary conditions associated with the equation. It is necessarily zero when the specified boundary conditions are homogeneous.
3. The set $\{\psi_i\}$ should be linearly independent. (7.164)

There are two main differences between the approximation functions used in the Ritz method and those used in weighted-residual methods:

1. *Continuity.* The approximation functions used in the weighted-residual methods are required to have the same differentiability as in the differential equation, whereas those used in the Ritz method must be differentiable as required by the weak form.
2. *Boundary conditions.* The approximation functions used in the weighted-residual method must satisfy the homogeneous form of both geometric and force boundary conditions, whereas those used in the Ritz method must satisfy the homogeneous form of only the essential boundary conditions, since the natural boundary conditions are already included in the weak form.

Both of these differences require ϕ_i to be of a higher order than those used in the Ritz method.

Various special cases of the weighted-residual method differ from each other due to the choice of the weight function, ψ_i. The most commonly used weight functions are:

$$\begin{aligned} &\textit{The Petrov–Galerkin method:} && \psi_i \neq \phi_i. \\ &\textit{Galerkin's method:} && \psi_i = \phi_i. \\ &\textit{Least-squares method:} && \psi_i = A(\phi_i). \\ &\textit{Collocation method:} && \psi_i = \delta(\mathbf{x} - \mathbf{x}_i). \end{aligned} \tag{7.165}$$

Here $\delta(\cdot)$ denotes the Dirac delta function. Although the least-squares method is listed as a special case of the weighted-residual method here, it is based on the concept of minimizing an integral statement. In general, the least-squares method is *not* a special case of the weighted-residual method. These remarks will be discussed in more detail below. In addition to the methods listed above, there are other variational methods (methods in which the unknown parameters c_i are adjusted such that the governing equations are satisfied in a certain sense). These include the subdomain method and Trefftz method. These methods will also be discussed briefly in this chapter.

7.5.2 Galerkin's Method

The Galerkin method is a special case of the Petrov–Galerkin method in which the approximation functions and the weighted functions are the same ($\phi_i = \psi_i$). Hence, the Galerkin integral is given by

$$\int_\Omega \phi_i \mathcal{R}_N(\mathbf{x}, \{c\}, \{\phi\}, f)\, d\mathbf{x} = 0, \qquad (i = 1, 2, \ldots, N). \tag{7.166}$$

If the Galerkin method is used for second-order or higher-order equations, it would involve the use of higher-order coordinate functions and the solution of nonsymmetric equations.

The Ritz and Galerkin methods yield the same set of algebraic equations for the following two cases:

1. The specified boundary conditions of the problem are all of the essential type, and therefore the requirements on ϕ_i and ϕ_0 in both methods are the same.
2. The problem has both essential and natural boundary conditions, but the coordinate functions used in the Galerkin method are also used in the Ritz method.

7.5.3 Least-Squares Method

The least-squares method is based on the idea of minimizing the integral of the square of the residual:

$$\text{minimize } I(c_1, c_2, \ldots, c_N) = \int_\Omega \mathcal{R}_N^2(\mathbf{x}, \{c\}, \{\phi\}, f)\, d\mathbf{x}. \tag{7.167a}$$

We obtain (from $\delta I = 0$)

$$0 = \int_\Omega \frac{\partial \mathcal{R}_N}{\partial c_i} \mathcal{R}_N(\mathbf{x}, \{c\}, \{\phi\}, f)\, d\mathbf{x}, \tag{7.167b}$$

which, when A is a linear operator, becomes

$$0 = \int_\Omega A(\phi_i) \mathcal{R}_N(\mathbf{x}, \{c\}, \{\phi\}, f)\, d\mathbf{x}. \tag{7.167c}$$

Clearly, Eq. (7.167c) is a special case of Eq. (7.163) with $\psi_i = A(\phi_i)$. The least-squares method is more suitable for first-order equations. For eigenvalue problems and time-dependent problems it is not suitable, as shown below. On the other hand, the least-squares method is the only other method, in addition to the Ritz method, that is based on the minimization of a functional. The least-squares method also results in a positive-definite coefficient matrix.

7.5.4 Collocation Method

In the collocation method we require the residual to vanish at a selected number of points $\mathbf{x}^i$ in the domain:

$$\mathcal{R}_N(\mathbf{x}^i, \{c\}, \{\phi\}, f) = 0, \qquad (i = 1, 2, \ldots, N), \tag{7.168a}$$

which can be written, with the help of the Dirac delta function, as

$$\int_\Omega \delta(\mathbf{x} - \mathbf{x}^i) \mathcal{R}_N(\mathbf{x}, \{c\}, \{\phi\}, f)\, d\mathbf{x} = 0, \qquad (i = 1, 2, \ldots, N). \tag{7.168b}$$

Thus, the collocation method is a special case of the weighted-residual method (7.163) with $\psi_i(\mathbf{x}) = \delta(\mathbf{x} - \mathbf{x}^i)$. In the collocation method, one must choose as many collocation points as there are undetermined parameters. In general, these points should be distributed uniformly in the domain. Otherwise, ill-conditioned equations among c_j may result.

7.5.5 Eigenvalue and Time-Dependent Problems

It should be noted that if the problem at hand is an eigenvalue problem or a time-dependent problem, the operator equation (7.158a) takes the following alternative forms:

Eigenvalue Problem

$$A(u) - \lambda C(u) = 0. \tag{7.169}$$

Time-Dependent Problem

$$A_t(u) + A(u) = f(x, t). \tag{7.170}$$

In Eq. (7.169), λ is the eigenvalue, which is to be determined along with the eigenvector $u(x)$, and A and C are spatial differential operators. An example of the equation is provided by the buckling of a beam column

$$\frac{d^2}{dx^2}\left(EI\frac{d^2u}{dx^2}\right) + P\frac{d^2u}{dx^2} = 0,$$

where u denotes the lateral deflection and P is the axial compressive load. The problem involves determining the value of P and mode shape $u(x)$ such that the governing equation and certain end conditions of the beam are satisfied. The minimum value of P is called the *critical buckling load*. Comparing the above equation with Eq. (7.169), we see that

$$A(u) = \frac{d^2}{dx^2}\left(EI\frac{d^2u}{dx^2}\right), \qquad \lambda = P, \qquad C(u) = -\frac{d^2u}{dx^2}.$$

In Eq. (7.170), A is a spatial differential operator and A_t is a temporal differential operator. Examples of Eq. (7.170) are provided by the equations of heat transfer in a plane wall and axial motion of a bar, respectively:

$$\rho c_v\frac{\partial u}{\partial t} - \frac{\partial}{\partial x}\left(k\frac{\partial u}{\partial x}\right) = f(x, t),$$

$$-\rho\frac{\partial^2 u}{\partial t^2} - \frac{\partial}{\partial x}\left(EA_0\frac{\partial u}{\partial x}\right) = f(x, t).$$

In the first equation, u denotes temperature, ρ the density, c_v specific heat at constant volume, k the conductivity, and f is internal heat generation. Clearly, we have

$$A_t(u) = \rho c_v \frac{\partial u}{\partial t}, \qquad A(u) = -\frac{\partial}{\partial x}\left(k\frac{\partial u}{\partial x}\right).$$

In the second equation, u denotes the axial displacement, ρ the density, E Young's modulus, A_0 area of cross section, and f body force per unit length. In this case, we have

$$A_t(u) = -\rho\frac{\partial^2 u}{\partial t^2}, \qquad A(u) = -\frac{\partial}{\partial x}\left(EA_0\frac{\partial u}{\partial x}\right).$$

Application of the weighted-residual method to Eqs. (7.169) and (7.170) follows the same idea, i.e., Eq. (7.163) holds. However, two comments are in order.

1. In the case of time-dependent problems, the integral in Eq. (7.163) is still over the spatial domain and the weight function ψ is a function of the spatial coordinate only. Thus, Eq. (7.163) leads to a set of ordinary differential equations in time among $c_j(t)$. These need to be further approximated using time-approximation schemes.
2. In the case of the least-squares method, the question arises as to what should be the weight function ψ_i. Let us examine the least-squares method first for the linear eigenvalue problem (i.e., A and C are linear operators). We have

$$\begin{aligned} 0 &= \delta\int_\Omega \mathcal{R}_N^2\, d\mathbf{x}, \qquad \mathcal{R}_N = A(U_N) - \lambda C(U_N) \\ &= 2\int_\Omega [A(\phi_i) - \lambda C(\phi_i)]\,\mathcal{R}_N\, d\mathbf{x}, \qquad (i = 1, 2, \ldots, N) \\ &= 2\sum_{j=1}^{N}\left\{\int_\Omega [A(\phi_i) - \lambda C(\phi_i)]\left[A(\phi_j) - \lambda C(\phi_j)\right] d\mathbf{x}\right\} c_j. \end{aligned}$$

Clearly, the eigenvalue problem becomes quadratic in λ, which is not desirable from a computational viewpoint.

In the case of time-dependent problems, we have

$$\begin{aligned} 0 &= \delta\int_\Omega \mathcal{R}_N^2\, d\mathbf{x}, \qquad \mathcal{R}_N = A(U_N) + A_t(U_N) \\ &= \int_\Omega \left[A(\phi_i) + \sum_{k=1}^{N}\phi_k\frac{\partial}{\partial c_i}A_t(c_k)\right]\mathcal{R}_N\, d\mathbf{x}, \qquad (i = 1, 2, \ldots, N). \end{aligned}$$

which is also complicated. Thus, the least-squares method leads to complicated systems of equations for eigenvalue or time-dependent problems. An alternative is to use $\psi_i = A(\phi_i)$ in all cases. This avoids the problems seen above.

It is possible to develop the so-called space-time approximations, i.e., to use the variational methods, treating time as an additional coordinate. It is found that such approaches are complicated, and they are not used in practice. Therefore, we will not consider the space-time approximations in the present study.

7.5.6 Equations for Undetermined Parameters

Here we develop the discrete equations for the equilibrium, eigenvalue, and time-dependent problems under the assumption that A is a linear operator, as is the case with most problems considered in this study. The equations are valid for all special cases of the weighted-residual method except for the least-squares method. They are also valid for the least-squares method if one accepts the use of $A(\phi_i)$ for ψ_i, which is certainly true for equilibrium problems. For eigenvalue problems we assume that all boundary conditions are homogeneous and therefore $\phi_0 = 0$.

Equilibrium Problems We have

$$A\left(\sum_{j=1}^{N} c_j\phi_j + \phi_0\right) = \sum_{j=1}^{N} c_j A(\phi_j) + A(\phi_0), \tag{7.171}$$

and Eq. (7.163) for the equilibrium equation (7.158a) becomes

$$\sum_{j=1}^{N}\left[\int_\Omega \psi_i A(\phi_j)\,d\mathbf{x}\right]c_j - \int_\Omega \psi_i[f - A(\phi_0)]\,d\mathbf{x} = 0$$

or

$$\sum_{j=1}^{N} a_{ij}c_j - b_i = 0 \quad (i = 1, 2, \ldots, N), \qquad [A]\{c\} = \{b\}, \tag{7.172a}$$

where

$$a_{ij} = \int_\Omega \psi_i A(\phi_j)\,d\mathbf{x}, \qquad b_i = \int_\Omega \psi_i\,[f - A(\phi_0)]\,d\mathbf{x}. \tag{7.172b}$$

Note that a_{ij} is *not symmetric* in general, even when $\psi_i = \phi_i$ (Galerkin's method). It is symmetric in the least-squares method because $\psi_i = A(\phi_i)$. Also, if (a) A is an operator of the form in Eq. (7.160a), (b) $\psi_i = \phi_i$, and (c) ϕ_i satisfy the homogeneous form of specified essential and natural boundary conditions, it can be shown that

$$a_{ij} = \int_\Omega T(\phi_i)aT(\phi_j)\,d\mathbf{x}, \qquad b_i = \int_\Omega \phi_i f\,d\mathbf{x} + \int_{\Gamma_2} \phi_i \hat{g}\,dS. \tag{7.172c}$$

Eigenvalue Problems For linear eigenvalue problem of the form in Eq. (7.169), we have

$$\sum_{j=1}^{N}\left\{\int_{\Omega}\psi_i[A(\phi_j)-\lambda C(\phi_j)]d\mathbf{x}\right\}c_j=0$$

or

$$\sum_{j=1}^{N}(a_{ij}-\lambda g_{ij})c_j=0\ (i=1,2,\ldots,N),\qquad ([A]-\lambda[G])\{c\}=\{0\},\qquad (7.173a)$$

where a_{ij} are defined by Eq. (7.172b) and

$$g_{ij}=\int_{\Omega}\psi_i C(\phi_j)\,d\mathbf{x}.\qquad (7.173b)$$

Note that using Eqs. (7.173a,b) for the least-squares method amounts to using $\psi_i = A(\phi_i)$; a_{ij} is symmetric but g_{ij} is unsymmetric.

Time-Dependent Problems For linear time-dependent problem of the form in Eq. (7.170), we have

$$\sum_{j=1}^{N}\left\{\int_{\Omega}\psi_i[A(\phi_j)c_j+\phi_j A_t(c_j)]d\mathbf{x}\right\}-\int_{\Omega}\psi_i[f-A(\phi_0)]\,d\mathbf{x}=0$$

or

$$\sum_{j=1}^{N}(a_{ij}c_j+m_{ij}A_t(c_j))=0\ \ (i=1,2,\ldots,N),\qquad [A]\{c\}+\lambda[M]\{A_t(c)\}=\{0\},\qquad (7.174a)$$

where a_{ij} are defined by Eq. (7.172b) and

$$m_{ij}A_t(c_j)=\int_{\Omega}\psi_i\phi_j A_t(c_j)\,d\mathbf{x}.\qquad (7.174b)$$

Again recall that using Eqs. (7.174a,b) for the least-squares method amounts to using $\psi_i = A(\phi_i)$.

7.5.7 Examples

A number of examples are considered here to illustrate the use of various weighted-residual methods. Equilibrium, eigenvalue, and time-dependent problems are considered.

Example 7.11 Consider the eigenvalue problem of Example 7.5 in a nondimensional form [see Eqs. (7.77a,b)]:

$$-\frac{d^2u}{dx^2} - \lambda u = 0, \qquad u(0) = 0, \qquad \frac{du}{dx} + u = 0 \text{ at } x = 1.$$

In the weighted-residual method, ϕ_i must satisfy not only the condition $\phi_i(0) = 0$ but also the condition $\phi_i'(1) + \phi_i(1) = 0$. The lowest-order function that satisfies the two conditions is

$$\phi_1(x) = 3x - 2x^2. \tag{7.175}$$

The one-parameter Galerkin's solution for the natural frequency (see Example 7.5) can be computed using

$$0 = c_1 \int_0^1 \phi_1 \left(\frac{d^2\phi_1}{dx^2} + \lambda\phi_1 \right) dx \quad \text{or} \quad \left(-\frac{10}{3} + \frac{4}{5}\lambda \right) c_1 = 0, \tag{a}$$

which gives (for nonzero c_1) $\lambda = 50/12 = 4.167$. If the same function is used for ϕ_1 in the one-parameter Ritz solution, we obtain, as discussed in Example 7.5, the same result as in the one-parameter Galerkin solution.

If we use the one-parameter collocation method with the collocation point at $x = 0.5$, we obtain $[\phi_1(0.5) = 1.0$ and $(d^2\phi_1/dx^2) = -4.0]$:

$$0 = c_1\phi_1(0.5) \left[\left(\frac{d^2\phi_1}{dx^2} \right) \bigg|_{x=0.5} + \lambda\phi_1(0.5) \right] \quad \text{or} \quad (-4 + \lambda)\, c_1 = 0,$$

which gives $\lambda = 4$.

The one-parameter least-squares approximation with $\psi_1 = A(\phi_1)$ gives

$$0 = c_1 \int_0^1 \frac{d^2\phi_1}{dx^2} \left(\frac{d^2\phi_1}{dx^2} + \lambda\phi_1 \right) dx \quad \text{or} \quad \left(-4 + \tfrac{5}{6}\lambda\right) c_1 = 0$$

which gives $\lambda = 4.8$. If we use $\psi_1 = A(\phi_1) - \lambda\phi_1$, we obtain

$$\begin{aligned} 0 &= c_1 \int_0^1 \left(\frac{d^2\phi_1}{dx^2} + \lambda\phi_1 \right) \left(\frac{d^2\phi_1}{dx^2} + \lambda\phi_1 \right) dx \\ &= \left(\tfrac{4}{5}\lambda^2 - \tfrac{20}{3}\lambda + 16\right) c_1, \end{aligned} \tag{b}$$

whose roots are

$$\lambda_{1,2} = \tfrac{25}{6} \pm \tfrac{1}{6}\sqrt{445} \rightarrow \lambda_1 = 7.6825, \qquad \lambda_2 = 0.6508. \tag{c}$$

Neither root is closer to the exact value of 4.116. This indicates that the least-squares method with $\psi_i = A(\phi_i)$ is perhaps more suitable than $\psi_i = A(\phi_i) - \lambda C(\phi_i)$.

Let us consider a two-parameter weighted-residual solution to the problem

$$U_2(x) = c_1\phi_1(x) + c_2\phi_2(x), \tag{d}$$

where $\phi_1(x)$ is given by Eq. (7.175). To determine $\phi_2(x)$, we begin with a polynomial that is one degree higher than that used for ϕ_1:

$$\phi_2(x) = a + bx + cx^2 + dx^3,$$

and obtain

$$\phi_2(0) = 0 \to a = 0; \qquad \phi_2'(1) + \phi_2(1) = 0 \to 2b + 3c + 4d = 0 \text{ or } d = -\tfrac{2}{4}b - \tfrac{3}{4}c.$$

We can arbitrarily pick the values of b and c, except that they are not both equal to zero (for obvious reasons). Thus we have an infinite number of possibilities. If we pick $b = 0$ and $c = 4$, we have $d = -3$, and ϕ_2 becomes

$$\phi_2(x) = a + bx + cx^2 + dx^3 = 4x^2 - 3x^3. \tag{7.176a}$$

On the other hand, if we choose $b = 1$ and $c = 2$, we have $d = -2$, and ϕ_2 becomes

$$\phi_2(x) = a + bx + cx^2 + dx^3 = x + 2x^2 - 2x^3 \equiv \varphi_2(x). \tag{7.176b}$$

The set $\{\phi_1, \phi_2\}$ is equivalent to the set $\{\phi_1, \varphi_2\}$. Note that

$$\begin{aligned}
U_2(x) &= c_1\phi_1(x) + c_2\phi_2(x) \\
&= c_1(3x - 2x^2) + c_2(4x^2 - 3x^3) \\
&= 3c_1 x + (-2c_1 + 4c_2)x^2 - 3c_2 x^3, \\
U_2(x) &\approx c_1\phi_1(x) + c_2\varphi_2(x) \\
&= \bar{c}_1(3x - 2x^2) + \bar{c}_2(x + 2x^2 - 2x^3) \\
&= (3\bar{c}_1 + \bar{c}_2)\,x + (-2\bar{c}_1 + 2\bar{c}_2)\,x^2 - 2\bar{c}_2 x^3.
\end{aligned}$$

Comparing the two relations, we can show that

$$\bar{c}_1 = c_1 - 0.5c_2, \qquad \bar{c}_2 = 1.5c_2.$$

Hence, either set will yield the same final solution for $U_2(x)$ or λ.

Using ϕ_1 and ϕ_2 [from Eq. (7.176a)], we compute the residual of the approximation as

$$\begin{aligned}
R &= -\frac{d^2U_2}{dx^2} - \lambda U_2 = -c_1\frac{d^2\phi_1}{dx^2} - c_2\frac{d^2\phi_2}{dx^2} - \lambda\,(c_1\phi_1 + c_2\phi_2) \\
&= c_1\left(-\frac{d^2\phi_1}{dx^2} - \lambda\phi_1\right) + c_2\left(-\frac{d^2\phi_2}{dx^2} - \lambda\phi_2\right).
\end{aligned} \tag{e}$$

For the Galerkin method, we set the integral of the weighted residual to zero and obtain

$$0 = \int_0^1 \phi_1(x) R\,dx = \int_0^1 \phi_1(x)\left[-c_1\frac{d^2\phi_1}{dx^2} - c_2\frac{d^2\phi_2}{dx^2} - \lambda\,(c_1\phi_1 + c_2\phi_2)\right]dx$$
$$= K_{11}c_1 + K_{12}c_2 - \lambda\,(M_{11}c_1 + M_{12}c_2)\,,$$
$$0 = \int_0^1 \phi_2(x) R\,dx = \int_0^1 \phi_2(x)\left[-c_1\frac{d^2\phi_1}{dx^2} - c_2\frac{d^2\phi_2}{dx^2} - \lambda\,(c_1\phi_1 + c_2\phi_2)\right]dx$$
$$= K_{21}c_1 + K_{22}c_2 - \lambda\,(M_{21}c_1 + M_{22}c_2)\,.$$

In matrix form, we have

$$[K]\{c\} - \lambda[M]\{c\} = \{0\},$$

where

$$K_{ij} = -\int_0^1 \phi_i \frac{d^2\phi_j}{dx^2}\,dx, \qquad M_{ij} = \int_0^1 \phi_i\phi_j\,dx.$$

First, for the choice of functions in Eqs. (7.176a) and (7.176b), we have

$$\frac{d^2\phi_1}{dx^2} = -4, \qquad \frac{d^2\phi_2}{dx^2} = 8 - 18x.$$

Evaluating the integrals, we obtain

$$K_{11} = -\int_0^1 \phi_1\frac{d^2\phi_1}{dx^2}\,dx = \int_0^1 (3x - 2x^2)(4)\,dx = \frac{10}{3},$$
$$K_{12} = -\int_0^1 \phi_1\frac{d^2\phi_2}{dx^2}\,dx = \int_0^1 (3x - 2x^2)(-8 + 18x)\,dx = \frac{7}{3},$$
$$K_{21} = -\int_0^1 \phi_2\frac{d^2\phi_1}{dx^2}\,dx = \int_0^1 (4x^2 - 3x^3)(4)\,dx = \frac{7}{3},$$
$$K_{22} = -\int_0^1 \phi_2\frac{d^2\phi_2}{dx^2}\,dx = \int_0^1 (4x^2 - 3x^3)(-8 + 18x)\,dx = \frac{38}{15},$$
$$M_{11} = \int_0^1 \phi_1\phi_1\,dx = \int_0^1 (3x - 2x^2)(3x - 2x^2)\,dx = \frac{4}{5},$$
$$M_{12} = \int_0^1 \phi_1\phi_2\,dx = \int_0^1 (3x - 2x^2)(4x^2 - 3x^3)\,dx = \frac{3}{5} = M_{21},$$
$$M_{22} = \int_0^1 \phi_2\phi_2\,dx = \int_0^1 (4x^2 - 3x^3)(4x^2 - 3x^3)\,dx = \frac{17}{35},$$

and

$$\left(\frac{1}{15}\begin{bmatrix}50 & 35\\ 35 & 38\end{bmatrix} - \frac{\lambda}{35}\begin{bmatrix}28 & 21\\ 21 & 17\end{bmatrix}\right)\begin{Bmatrix}c_1\\ c_2\end{Bmatrix} = \begin{Bmatrix}0\\ 0\end{Bmatrix}.$$

For nontrivial solution, $c_1 \neq 0$ and $c_2 \neq 0$, we set the determinant of the coefficient matrix to zero to obtain the characteristic polynomial

$$675 - \frac{1332}{7}\lambda + \frac{315}{49}\lambda^2 = 0 \qquad \text{or} \qquad 525 - 148\lambda + 5\lambda^2 = 0, \tag{f}$$

which gives

$$\lambda_1 = 4.121, \qquad \lambda_2 = 25.479. \tag{g}$$

Clearly, the value of λ_1 has improved over that computed using the one-parameter approximation. The exact value of the second eigenvalue is 24.139.

If we were to use the collocation method, we may select $x = 1/3$ and $x = 2/3$ as the collocation points, among other choices. We leave this as an exercise to the reader.

Example 7.12 Next we consider the transient response of the problem discussed in Example 7.6. The governing equations are

$$-\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial t^2} = 1, \qquad u(0,t) = 0, \qquad \frac{\partial u}{\partial x} + u = 0 \ \text{ at } x = L \text{ for all } t > 0,$$

with zero initial conditions. We use the one-parameter approximation $u(x,t) \approx c_1(t)\phi_1(x)$ with $\phi_1(x)$ defined in Eq. (7.175).

For the Galerkin method, we obtain

$$0 = \int_0^1 \phi_1\left(-c_1\frac{d^2\phi_1}{dx^2} + \frac{d^2 c_1}{dt^2}\phi_1 - 1\right)dx \qquad \text{or} \qquad \frac{4}{5}\frac{d^2c_1}{dt^2} + \frac{10}{3}c_1 = \frac{5}{6}, \tag{a}$$

whose (exact) solution is ($\sqrt{50/12} \approx 2.0412$):

$$c_1(t) = A\sin 2.04t + B\cos 2.04t + \tfrac{1}{4}.$$

For zero initial conditions, $u(x,0) = \dot{u}(x,0) = 0$ [or $c_1(0) = 0$ and $\dot{c}_1(0) = 0$], the total solution becomes ($A = 0$ and $B = -1/4$):

$$u_1(x,t) = \tfrac{1}{4}(1 - \cos 2.04t)(3x - 2x^2). \tag{b}$$

For the one-parameter collocation method with the collocation point at $x = 0.5$, we obtain

$$0 = \phi_1(0.5)\left[-c_1\left(\frac{d^2\phi_1}{dx^2}\right)\bigg|_{0.5} + \frac{d^2c_1}{dt^2}\phi_1(0.5) - 1\right] \qquad \text{or} \qquad \frac{d^2c_1}{dt^2} + 4c_1 = 1, \tag{c}$$

so that

$$c_1(t) = A \sin 2t + B \cos 2t + \tfrac{1}{4},$$

and the one-parameter collocation solution becomes

$$u_1(x,t) = \tfrac{1}{4}(1 - \cos 2t)\left(3x - 2x^2\right). \tag{d}$$

Example 7.13 Consider the simply supported beam problem of Example 7.7. Since all specified boundary conditions are homogeneous, again we have $\phi_0 = 0$. In the Galerkin method, ϕ_i must satisfy the homogeneous form of all specified boundary conditions ($w_0 = M_{xx} = 0$ at $x = 0, L$):

$$\phi_i = 0, \qquad \frac{d^2\phi_i}{dx^2} = 0. \tag{a}$$

For the choice of algebraic polynomials, we assume a five-parameter polynomial because there are four conditions in Eq. (a):

$$\phi_1(x) = a + bx + cx^2 + dx^3 + ex^4.$$

Using the boundary conditions, we find that

$$a = c = 0, \qquad bL + dL^3 + eL^4 = 0, \qquad 6dL + 12eL^2 = 0.$$

Thus we have $b = eL^3$ and $d = -2eL$. The function ϕ_1 is given by (taking $eL^4 = 1$)

$$\phi_1 = \frac{x}{L}\left(1 - 2\frac{x^2}{L^2} + \frac{x^3}{L^3}\right). \tag{b}$$

Substituting the one-parameter Galerkin approximation $W_1 = c_1\phi_1$ into the residual,

$$\mathcal{R} = EI\frac{d^4W_1}{dx^4} - q_0 = \frac{24EI}{L^4}c_1 - q_0. \tag{c}$$

Since the residual is already a constant (which implies that the solution is exact), there is no need to integrate the weighted residual over the domain. By setting $\mathcal{R}$ to zero we obtain $c_1 = q_0L^4/(24EI)$, and the solution becomes

$$W_1(x) = \frac{q_0L^4}{24EI}\left[\left(\frac{x}{L}\right) - 2\left(\frac{x}{L}\right)^3 + \left(\frac{x}{L}\right)^4\right], \tag{d}$$

which coincides with the exact solution. Note that the solution obtained is independent of the particular weighted residual method. It can be shown that a one-parameter Ritz solution with ϕ_1 given by Eq. (b) also yields the same exact solution (d).

Note that the solution to this problem is symmetric about $x = L/2$. Hence, we can use a half-beam model to solve the problem. In using a half beam, we must address the boundary conditions at $x = L/2$. The boundary conditions at this point are that the slope is zero, $(dw_0/dx) = 0$, and shear force is zero. Hence, for this case, the approximation functions ϕ_i in the weighted-residual method must satisfy the conditions

$$\text{at } x = 0:\ \phi_i = 0, \quad \frac{d^2\phi_i}{dx^2} = 0, \qquad \text{and at } x = \frac{L}{2}:\ \frac{d\phi_i}{dx} = 0, \quad \frac{d^3\phi_i}{dx^3} = 0.$$

Obviously, the function $\phi_1(x)$ of Eq. (b) satisfies the above conditions. Another choice of ϕ_i is provided by

$$\phi_i(x) = \sin\frac{(2i-1)\pi x}{L}.$$

Example 7.14 Another weighted-residual method that has not been introduced formally in this study is the *subdomain method*. In the subdomain method, we divide the domain of the problem into as many subdomains as there are undetermined parameters, c_i, and then on each subdomain we require the integral of the residual $\mathcal{R}_N$ to be zero:

$$\int_{\Omega_i} \mathcal{R}_N(\mathbf{x}, \{c\}, \{\phi\}, f)\, d\mathbf{x} = 0, \qquad (i = 1, 2, \ldots, N), \tag{7.177a}$$

where Ω_i is the ith subdomain. The method can be viewed as a piecewise application of the weighted residual method with $\psi_i = 1$:

$$\psi_i = \begin{cases} 1, & \text{if } \mathbf{x} \in \Omega_i, \\ 0, & \text{otherwise.} \end{cases} \tag{7.177b}$$

Obviously, in this method, negative errors can cancel positive errors to give zero net error (unless $\mathcal{R}_N$ is a positive function in Ω_i), although the sum of the absolute values of the errors may be very large. The finite volume method that is popular in fluid dynamics is based on ideas similar to the subdomain method.

As an example, we consider the beam problem of Example 7.13 (see Fig. 7.5). Using the half-beam model with the following two-parameter approximation:

$$W_2(x) = c_1 \sin\frac{\pi}{L} + c_2 \sin\frac{3\pi x}{L}, \tag{a}$$

we determine approximate solutions using both the collocation method and the subdomain method.

Collocation Method Using collocation points at $x = L/4$ and $x = L/2$, we obtain

$$EI\left[c_1\left(\frac{\pi}{L}\right)^4 \sin\frac{\pi}{4} + c_2\left(\frac{3\pi}{L}\right)^4 \sin\frac{3\pi}{4}\right] - q_0 = 0,$$

$$EI\left[c_1\left(\frac{\pi}{L}\right)^4 \sin\frac{\pi}{2} + c_2\left(\frac{3\pi}{L}\right)^4 \sin\frac{3\pi}{2}\right] - q_0 = 0,$$

which yield

$$EI\left(\frac{\pi}{L}\right)^4 \begin{bmatrix} 1/\sqrt{2} & 81/\sqrt{2} \\ 1 & -81 \end{bmatrix} \begin{Bmatrix} c_1 \\ c_2 \end{Bmatrix} = q_0 \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}.$$

or

$$c_1 = \frac{\sqrt{2}+1}{2\pi^4}\frac{q_0 L^4}{EI}, \qquad c_2 = \frac{\sqrt{2}-1}{162\pi^4}\frac{q_0 L^4}{EI}$$

The two-parameter collocation solution becomes

$$W_2(x) = \frac{q_0 L^4}{162 EI\pi^4}\left(195.55 \sin\frac{\pi x}{L} + 0.414 \sin\frac{3\pi x}{L}\right). \tag{b}$$

The maximum deflection, $W_2(L/2) = 1.205(q_0 L^4/EI\pi^4) = (q_0 L^4/80.87EI)$, is 5% in error compared to the exact value $(q_0 L^4/76.8EI)$.

Subdomain Method We consider two subdomains, $\Omega_1 = (0, L/4)$ and $\Omega_2 = (L/4, L/2)$, in the first half of the beam. For the same choice of coordinate functions as in the collocation method, we obtain

$$\int_0^{L/4}\left[c_1 EI\left(\frac{\pi^4}{L}\right)^4 \sin\frac{\pi x}{L} + c_2 EI\left(\frac{3\pi}{L}\right)^4 \sin\frac{3\pi x}{L} - q_0\right]dx = 0,$$

$$\int_{L/4}^{L/2}\left[c_1 EI\left(\frac{\pi^4}{L}\right)^4 \sin\frac{\pi x}{L} + c_2 EI\left(\frac{3\pi}{L}\right)^4 \sin\frac{3\pi x}{L} - q_0\right]dx = 0,$$

or

$$EI\left(\frac{\pi}{L}\right)^4 \frac{L}{\pi}\left(1 - \frac{1}{\sqrt{2}}\right)c_1 + EI\left(\frac{3\pi}{L}\right)^4 \frac{L}{3\pi}\left(1 + \frac{1}{\sqrt{2}}\right)c_2 = \frac{q_0 L}{4},$$

$$EI\left(\frac{\pi}{L}\right)^4 \frac{L}{\pi}\frac{1}{\sqrt{2}}c_1 + EI\left(\frac{3\pi}{L}\right)^4 \frac{L}{3\pi}\left(-\frac{1}{\sqrt{2}}\right)c_2 = \frac{q_0 L}{4}.$$

The solution of these equations is

$$c_1 = \frac{\sqrt{2}+1}{4\sqrt{2}\pi^3}\frac{q_0 L^4}{EI}, \qquad c_2 = \frac{\sqrt{2}-1}{108\sqrt{2}\pi^3}\frac{q_0 L^4}{EI},$$

and the solution W_2 becomes

$$W_2(x) = \frac{q_0 L^4}{108\sqrt{2}EI\pi^3}\left(65.184 \sin\frac{\pi x}{L} + 0.414 \sin\frac{3\pi x}{L}\right). \tag{c}$$

The center deflection obtained in the subdomain method, $W_2(L/2) = (q_0 L^4/73.12EI)$, is −5% in error compared to the exact value.

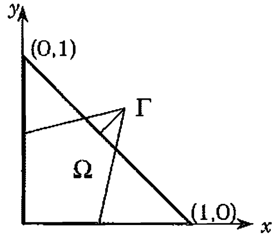

Figure 7.9 A triangular membrane.

Example 7.15 Consider the equation

$$A(u) \equiv \nabla^2 u + \lambda u = 0 \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \Gamma, \tag{a}$$

where Ω is the triangular domain shown in Fig. 7.9 and Γ is its boundary. Equation (a) describes a nondimensional form of the equation governing the natural vibration of a triangular membrane of side a, mass density ρ, and tension T ($\lambda = \rho a^2\omega^2/T$, ω the frequency of vibration). We wish to determine the fundamental frequency (i.e., determine λ) of vibration by using a one-parameter Galerkin approximation of the problem.

The Galerkin method is based on the weighted-integral statement

$$0 = \int_\Omega c_1 \left(\frac{\partial^2 \phi_1}{\partial x^2} + \frac{\partial^2 \phi_1}{\partial y^2} + \lambda \phi_1 \right) \phi_1 \, dxdy. \tag{b}$$

The function $\phi_1(x, y)$ must vanish on the boundary Γ. Thus, we have

$$\phi_1(x, 0) = 0, \qquad \phi_1(0, y) = 0, \qquad \phi_1(x, y) = 0 \text{ on the line } x + y - 1 = 0. \tag{c}$$

Hence the choice for $\phi_1(x, y)$ is

$$\phi_1(x, y) = (x - 0)(y - 0)(x + y - 1), \qquad \frac{\partial^2 \phi_1}{\partial x^2} + \frac{\partial^2 \phi_1}{\partial y^2} = 2(x + y). \tag{d}$$

Hence we have

$$0 = c_1 \int_0^1 \int_0^{1-y} [2(x + y) + \lambda xy(x + y - 1)]\, xy(x + y - 1)\, dxdy$$

or

$$\lambda = -\frac{\int_0^1 \int_0^{1-y} 2(x + y)xy(x + y - 1)\, dxdy}{\int_0^1 \int_0^{1-y} x^2y^2(x + y - 1)^2\, dxdy} = 56. \tag{e}$$

Example 7.16 Consider the differential equation (Poisson's equation)

$$A(u) \equiv -\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) = f_0 \quad \text{in} \quad -a \leq x,\ y \leq a$$
$$u = 0 \quad \text{on the boundary} \tag{7.178}$$

in a square region. The origin of the coordinate system is taken at the center of the domain, as shown in Fig. 7.10. The equation arises, among others, in connection with (1) the transverse deflection of a membrane fixed on all sides and subjected to uniform pressure f_0, (2) the study of the torsion of a square cross-section prismatic bar with $f_0 = 2G\theta$, where G denotes the shear modulus and θ is the angle of twist per unit length, and (3) conduction heat transfer in a square region with internal heat generation of f_0 unit area. The function u denotes the deflection u in the case of a membrane, the Prandtl stress function Φ in the case of the torsion problem, and the temperature T in the case of conduction heat transfer.

The quadratic functional associated with Eq. (7.178) is given by

$$I(u) = \int_{-a}^{a}\int_{-a}^{a} \frac{1}{2}\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 - 2f_0 u\right] dx\,dy, \tag{7.179}$$

which represents the total potential energy functional for the membrane problem and the total complementary energy functional for the torsion problem. For the heat conduction problem, $I(u)$ does not have a physical meaning. The functional $I(u)$ can be derived by the weak form procedure outlined in Section 7.4.1. The boundary condition in Eq. (7.178) is of the essential type. The natural boundary condition for the equation involves the specification of $\partial u/\partial n$, which is not the specified boundary condition of the problem, unless a quadrant is used.

For the Galerkin method, we assume ($\phi_0 = 0$)

$$U_N(x, y) = \sum_{j=1}^{N} c_j \phi_j(x, y) \tag{7.180}$$

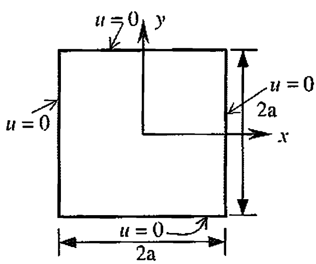

Figure 7.10 The square domain of Example 7.16.

where

$$\phi_1 = (a^2 - x^2)(a^2 - y^2), \qquad \phi_2 = (x^2 + y^2)\phi_1, \ldots. \tag{7.181}$$

Clearly, ϕ_i are twice differentiable with respect to x and y and satisfy the boundary conditions. From Eq. (7.172b), we have

$$a_{ij} = \int_{-a}^{a}\int_{-a}^{a} \phi_i A(\phi_j)\,dx\,dy, \qquad b_i = \int_{-a}^{a}\int_{-a}^{a} \phi_i f_0\,dx\,dy. \tag{7.182}$$

For $N = 1$, we have

$$\frac{256}{45}c_1 = \frac{16}{9}\frac{f_0}{a^2}$$

and the one-parameter solution is given by

$$U_1(x, y) = \frac{5f_0 a^2}{16}\left(1 - \frac{x^2}{a^2}\right)\left(1 - \frac{y^2}{a^2}\right). \tag{7.183a}$$

For the two-parameter approximation, we have

$$a^8 \begin{bmatrix} \frac{256}{45} & \frac{1024}{525}a^2 \\ \frac{1024}{525}a^2 & \frac{11264}{4725}a^4 \end{bmatrix} \begin{Bmatrix} c_1 \\ c_2 \end{Bmatrix} = f_0 a^6 \begin{Bmatrix} \frac{16}{9} \\ \frac{32}{45}a^2 \end{Bmatrix}, \tag{a}$$

whose solution yields

$$c_1 = \frac{1295 f_0}{4432 a^2}, \qquad c_2 = \frac{525 f_0}{8864 a^4}. \tag{b}$$

The two-parameter Galerkin solution of Eq. (7.178) is given by

$$U_2(x, y) = \frac{f_0 a^2}{8864}[2590 + 525(\bar{x}^2 + \bar{y}^2)](1 - \bar{x}^2)(1 - \bar{y}^2), \tag{7.183b}$$

where $\bar{x} = x/a$ and $\bar{y} = y/a$.

For the Ritz method, the same approximation functions as used in the Galerkin method must be used (why?), and we obtain

$$\begin{aligned} a_{ij} &= \int_{-a}^{a}\int_{-a}^{a}\left(\frac{\partial\phi_i}{\partial x}\frac{\partial\phi_j}{\partial x} + \frac{\partial\phi_i}{\partial y}\frac{\partial\phi_j}{\partial y}\right)dx\,dy \\ b_i &= \int_{-a}^{a}\int_{-a}^{a} f_0\phi_i\,dx\,dy. \end{aligned} \tag{7.184}$$

Calculations will show that a_{ij} and b_i are the same as those in Eq. (a). Therefore, the Ritz and Galerkin solutions coincide.

The exact solution to Eq. (7.178) can be obtained using the separation of variables method, and it is given by

$$u(x, y) = \frac{16 f_0 a^2}{\pi^3} \sum_{n=1,3,5,\ldots}^{\infty} \frac{1}{n^3}(-1)^{(n-1)/2}\left[1 - \frac{\cosh(n\pi y/2a)}{\cosh(n\pi/2)}\right]\cos\frac{n\pi x}{2a}. \tag{7.185}$$

The exact solution at the center of the region is

$$u_0(0, 0) = 0.2942 f_0 a^2,$$

whereas the two-parameter Galerkin/Ritz solution is $0.2922 f_0 a^2$, which is only 0.68% in error.

Examples 7.17 and 7.18 presented below illustrate the use of variational methods for the solution of problems in two dimensions and time-dependent problems. The approximation is selected such that the parameters c_i are functions of one coordinate (say, time) and ϕ_i are functions of the remaining (say, spatial) coordinates. The procedure used in Examples 7.17 and 7.18 to obtain ordinary differential equations from partial differential equation(s) is termed the *semidiscretization method* or *Kantorovich method*. The Lévy method of solution to be discussed in Chapter 8 is also similar to these methods, and all of them are based on the separation of variables concept.

Example 7.17 Consider the following nondimensionalized partial differential equation (such as the one arising in transient heat transfer):

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = 0, \qquad 0 < x < 2,$$

$$u(0, t) = u(2, t) = 0 \qquad \text{for } t > 0,$$

$$u(x, 0) = 1.0 \qquad \text{for } 0 < x < 2.$$

Owing to the symmetry about $x = 1$, we solve the following equivalent problem:

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = 0, \qquad 0 < x < 1, \tag{7.186a}$$

$$u(0, t) = 0, \qquad \frac{\partial u}{\partial x}(1, t) = 0 \qquad \text{for } t > 0, \tag{7.186b}$$

$$u(x, 0) = 1.0 \qquad \text{for } 0 < x < 1. \tag{7.187}$$

Let us consider the following form of a one-parameter Galerkin approximation:

$$U_1(x, t) = c_1(t)\phi_1(x) = c_1(t)(2x - x^2). \tag{7.188}$$

The function $\phi_1(x)$, which is a function of x only, satisfies the boundary conditions in Eq. (7.186b). We should determine $c_1(t)$ such that the initial condition (7.187) is satisfied. This requires that

$$c_1(0) = 1. \tag{7.189}$$

Substituting Eq. (7.188) into Eq. (7.186a) and setting up the Galerkin integral, we obtain

$$\begin{aligned} 0 &= \int_0^1 \left(\frac{\partial u_1}{\partial t} - \frac{\partial^2 u_1}{\partial x^2} \right) \phi_1 \, dx \\ &= \int_0^1 \left[\frac{dc_1}{dt}(2x - x^2) - c_1(-2) \right] (2x - x^2) \, dx \\ &= \frac{8}{15}\frac{dc_1}{dt} + \frac{4}{3}c_1. \end{aligned} \tag{7.190}$$

The solution of Eq. (7.190) is given by

$$c_1(t) = Ae^{-(5/2)t}, \qquad c_1(0) = 1 \;\rightarrow\; A = 1,$$

and the one-parameter Galerkin solution becomes

$$U_1(x, t) = e^{-2.5t}(2x - x^2). \tag{7.191}$$

The exact solution of Eqs. (7.186)–(7.187) is given by

$$u(x, t) = 2\sum_{n=0}^{\infty} \frac{e^{-\lambda_n^2 t} \sin \lambda_n x}{\lambda_n}, \qquad \lambda_n = \frac{(2n+1)\pi}{2}. \tag{7.192}$$

For a two-parameter Ritz approximation, we seek the solution in the form

$$u(x, t) \approx U_2(x, t) = c_1(t)\phi_1(x) + c_2(t)\phi_2(x), \tag{7.193}$$

where ϕ_i satisfy only the essential boundary condition, $\phi_i(0) = 0$. Thus the functions

$$\phi_1(x) = x, \qquad \phi_2(x) = x^2 \tag{7.194}$$

are admissible.

We use the weak form (for the Ritz approximation)

$$0 = \int_0^1 \phi_i \left(\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} \right) dx = \int_0^1 \left(\phi_i \frac{\partial u}{\partial t} + \frac{\partial \phi_i}{\partial x}\frac{\partial u}{\partial x} \right) dx. \tag{7.195}$$

Substituting Eqs. (7.193) and (7.194) into Eq. (7.195) and carrying out the integration, we obtain

$$\frac{1}{3}\frac{dc_1}{dt} + \frac{1}{4}\frac{dc_2}{dt} + c_1 + c_2 = 0, \tag{7.196a}$$

$$\frac{1}{4}\frac{dc_1}{dt} + \frac{1}{5}\frac{dc_2}{dt} + c_1 + \frac{4}{3}c_2 = 0, \tag{7.196b}$$

which must be solved subject to the initial condition in Eq. (7.187). It is clear that there is no way in which the initial condition will be satisfied exactly by the selected approximation (7.193). Therefore, we satisfy the initial condition in the Galerkin-integral sense:

$$\int_0^1 [u(x,0) - 1]\phi_i \, dx = 0, \qquad i = 1, 2, \tag{7.197a}$$

which gives

$$\tfrac{1}{3}c_1(0) + \tfrac{1}{4}c_2(0) = \tfrac{1}{2}, \qquad \tfrac{1}{4}c_1(0) + \tfrac{1}{5}c_2(0) = \tfrac{1}{3}. \tag{7.197b}$$

Now we use the Laplace transform method to solve Eqs. (7.196a,b) and (7.197b). Let $\bar{c}_i(s)$ denote the Laplace transform of $c_i(t)$:

$$\bar{c}_i = L[c_i(t)] \equiv \int_{-\infty}^{\infty} c_i(t)e^{-st}\, dt. \tag{7.198}$$

The Laplace transform of the derivative of $c_i(t)$ is given by

$$L\left[\frac{dc_i}{dt}\right] = s\bar{c}_i(s) - c_i(0). \tag{7.199}$$

Using Eq. (7.199), Eqs. (7.196a,b) can be transformed to

$$\left(\tfrac{1}{3}s + 1\right)\bar{c}_1 + \left(\tfrac{1}{4}s + 1\right)\bar{c}_2 = \tfrac{1}{3}c_1(0) + \tfrac{1}{4}c_2(0), \tag{7.200a}$$

$$\left(\tfrac{1}{4}s + 1\right)\bar{c}_1 + \left(\tfrac{1}{5}s + \tfrac{4}{3}\right)\bar{c}_2 = \tfrac{1}{4}c_1(0) + \tfrac{1}{5}c_2(0). \tag{7.200b}$$

The right-hand side can be substituted from Eq. (7.197b), and the resulting equations can be solved for $\bar{c}_1(s)$ and $\bar{c}_2(s)$:

$$\bar{c}_1(s) = \frac{\frac{1}{2}\left(\frac{1}{5}s + \frac{4}{3}\right) - \frac{1}{3}\left(\frac{1}{4}s + 1\right)}{\left(\frac{1}{5}s + \frac{4}{3}\right)\left(\frac{1}{3}s + 1\right) - \left(\frac{1}{4}s + 1\right)^2} = \frac{12s + 240}{3s^2 + 104s + 240}, \tag{7.201a}$$

$$\bar{c}_2(s) = \frac{\frac{1}{3}\left(\frac{1}{3}s + 1\right) - \frac{1}{2}\left(\frac{1}{4}s + 1\right)}{\left(\frac{1}{5}s + \frac{4}{3}\right)\left(\frac{1}{3}s + 1\right) - \left(\frac{1}{4}s + 1\right)^2} = -\frac{10s + 120}{3s^2 + 104s + 240}. \tag{7.201b}$$

The inverse transform of Eqs. (7.201a,b) can be obtained by using the identity

$$L^{-1}\left[\frac{s + a_1}{(s + a)(s + b)}\right] = \frac{a_1 - a}{b - a}e^{-at} + \frac{a_1 - b}{a - b}e^{-bt}. \tag{7.202}$$

Hence we have

$$c_1(t) = 1.6408e^{-32.1807t} + 2.3592e^{-2.486t}, \tag{7.203a}$$

$$c_2(t) = -\left(2.265e^{-32.1807t} + 1.068e^{-2.486t}\right). \tag{7.203b}$$

Equations (7.193) and (7.203a,b) together give the two-parameter Ritz solution.

The two variational solutions in Eqs. (7.191) and (7.193) are compared, for various values of (x, t), with the series solution (7.192) in Table 7.3. The two-parameter Ritz

Table 7.3 Comparison of the variational solutions with the series solution of a parabolic equation with initial condition, $u(x, 0) = 1$ [see Eqs. (7.186)–(7.187)]

		Series Solution	Variational Solution	
t	x	Eq. (7.192)	Eq. (7.191)	Eq. (7.193)
0.05	0.2	0.4727	0.3177	0.4265
	0.4	0.7938	0.5648	0.7413
	0.6	0.9418	0.7413	0.9443
	0.8	0.9880	0.8472	1.0357
	1.0	0.9965	0.8825	1.0154
0.25	0.2	0.2135	0.1927	0.2306
	0.4	0.4052	0.3426	0.4152
	0.6	0.5560	0.4496	0.5538
	0.8	0.6520	0.5139	0.6466
	1.0	0.6848	0.5353	0.6934
0.50	0.2	0.1145	0.1031	0.1238
	0.4	0.2177	0.1834	0.2230
	0.6	0.2996	0.2407	0.2975
	0.8	0.3522	0.2751	0.3473
	1.0	0.3703	0.2865	0.3725
0.75	0.2	0.0617	0.0552	0.0665
	0.4	0.1174	0.0981	0.1198
	0.6	0.1616	0.1288	0.1600
	0.8	0.1900	0.1472	0.1866
	1.0	0.1997	0.1534	0.2001
1.0	0.2	0.0333	0.0296	0.0357
	0.4	0.0633	0.0525	0.0643
	0.6	0.0872	0.0690	0.0858
	0.8	0.1025	0.0788	0.1002
	1.0	0.1077	0.0821	0.1075
1.50	0.2	0.0097	0.0085	0.0103
	0.4	0.0184	0.0151	0.0186
	0.6	0.0254	0.0198	0.0248
	0.8	0.0298	0.0226	0.0289
	1.0	0.0313	0.0235	0.0310

solution is more accurate than the one-parameter Galerkin solution and agrees more closely with the series solution for large values of time.

Example 7.18 Consider the membrane/torsion problem considered in Example 7.16. Here we seek a one-parameter approximation of the form

$$U_1(x, y) = c_1(x)\phi_1(y) = c_1(x)(y^2 - a^2). \tag{7.204a}$$

Since $u = 0$ on $x = \pm a$ and on $y = \pm a$, it follows that we must determine $c_1(x)$ such that it satisfies the conditions

$$c_1(-a) = c_1(a) = 0. \tag{7.204b}$$

Substituting Eq. (7.204a) into the Galerkin integral ($u \approx U_1$),

$$0 = \int_{-a}^{a}\int_{-a}^{a}\left[-\left(\frac{\partial^2 U_1}{\partial x^2} + \frac{\partial^2 U_1}{\partial y^2}\right) - f_0\right]\phi_1\, dx dy, \tag{7.205a}$$

we obtain

$$\begin{aligned} 0 &= \int_{-a}^{a}\int_{-a}^{a}\left(-\frac{d^2c_1}{dx^2}\phi_1 - c_1\frac{d^2\phi_1}{dy^2} - f_0\right)\phi_1\, dx dy \\ &= -\int_{-a}^{a}\left\{\int_{-a}^{a}\left[(y^2 - a^2)\frac{d^2c_1}{dx^2} + 2c_1 + f_0\right](y^2 - a^2)\, dy\right\} dx. \end{aligned} \tag{7.205b}$$

Performing the integration with respect to y and dividing throughout by the coefficient of d^2c_1/dx^2, we obtain

$$0 = \int_{-a}^{a}\left(\frac{d^2c_1}{dx^2} - \frac{5}{2a^2}c_1 - \frac{5f_0}{4a^2}\right) dx. \tag{7.206a}$$

An examination of the integrand in (7.206a) shows that $c_1(x)$ is *not* a periodic function. Hence, the integral vanishes only if the integrand is identically zero:

$$\frac{d^2c_1}{dx^2} - \frac{5}{2a^2}c_1 - \frac{5f_0}{4a^2} = 0. \tag{7.206b}$$

This completes the application of the Galerkin method. We can solve the ordinary differential equation (7.206) either exactly or by an approximate method, such as the Ritz or Galerkin method. We consider the exact solution of Eq. (7.206) subjected to the boundary conditions in Eq. (7.204b).

The exact solution of Eq. (7.206) is given by

$$c_1(x) = A\cosh kx + B\sinh kx - \frac{f_0}{2}, \qquad k = \sqrt{\frac{5}{2a^2}}. \tag{7.207}$$

Using the conditions in Eq. (7.204b), the constants of integration, A and B, are evaluated to be

$$A = \frac{f_0}{2\cosh ka}, \qquad B = 0.$$

The solution in Eq. (7.204a) becomes

$$U_1(x, y) = \frac{f_0 a^2}{2}\left(1 - \frac{y^2}{a^2}\right)\left[1 - \frac{\cosh kx}{\cosh ka}\right]. \tag{7.208}$$

A two-parameter approximation of the form

$$U_2(x, y) = (y^2 - a^2)[c_1(x) + c_2(x)y^2] \tag{7.209}$$

gives the differential equations

$$\begin{aligned}\frac{8}{15}a^2\frac{d^2c_1}{dx^2} + \frac{8}{105}a^4\frac{d^2c_2}{dx^2} - \left(\frac{4}{3}c_1 + \frac{4}{15}a^2c_2\right) &= \frac{2}{3}f_0\\ \frac{8}{105}a^4\frac{d^2c_1}{dx^2} + \frac{8}{315}a^6\frac{d^2c_2}{dx^2} - \left(\frac{4a^2}{15}c_1 + \frac{44}{105}a^4c_2\right) &= \frac{2}{15}f_0a^2,\end{aligned} \tag{7.210}$$

with the boundary conditions

$$c_1(-a) = c_1(a) = c_2(-a) = c_2(a) = 0. \tag{7.211}$$

The simultaneous differential equations in Eq. (7.210) can be solved as follows: let $D = d/dx$, $D^2 = d^2/dx^2$, and so on. Then we can write Eq. (7.210) in the operator form

$$\begin{bmatrix}\frac{8a^2}{15}D^2 - \frac{4}{3} & \frac{8a^4}{105}D^2 - \frac{4a^2}{15}\\ \frac{8a^4}{105}D^2 - \frac{4a^2}{15} & \frac{8a^6}{315}D^2 - \frac{44a^4}{105}\end{bmatrix}\begin{Bmatrix}c_1\\ c_2\end{Bmatrix} = \frac{2}{15}f_0\begin{Bmatrix}5\\ a^2\end{Bmatrix}. \tag{7.212}$$

Using Cramer's rule, but keeping in mind that D's operate on the quantities in front of them, we obtain

$$L(c_1) = \begin{vmatrix}\frac{2}{3}f_0 & \frac{8a^4}{105}D^2 - \frac{4a^2}{15}\\ \frac{2}{15}f_0a^2 & \frac{8a^6}{315}D^2 - \frac{44a^4}{105}\end{vmatrix}$$

$$= \left(\frac{8}{315}a^6D^2 - \frac{44}{105}a^4\right)\frac{2}{3}f_0 - \left(\frac{8}{105}a^4D^2 - \frac{4a^2}{15}\right)\frac{2}{15}f_0a^2,$$

$$= -\frac{384}{1575}f_0a^4, \tag{7.213a}$$

$$L(c_2) = \begin{vmatrix} \frac{8a^2}{15}D^2 - \frac{4}{3} & \frac{2}{3}f_0 \\ \frac{8}{105}a^4D^2 - \frac{4a^2}{15} & \frac{2}{15}f_0a^2 \end{vmatrix} = 0, \tag{7.213b}$$

where $L(\cdot)$ is the determinant of the operator matrix in Eq. (7.212),

$$L = \frac{256a^8}{33075}\left(D^4 - \frac{28}{a^2}D^2 + \frac{63}{a^4}\right). \tag{7.214}$$

The general solutions of Eqs. (7.213a,b) are

$$c_1(x) = -\frac{f_0}{2} + A_1 \cosh k_1x + A_2 \sinh k_1x + A_3 \cosh k_2x + A_4 \sinh k_2x,$$

$$c_2(x) = B_1 \cosh k_1x + B_2 \sinh k_1x + B_3 \cosh k_2x + B_4 \sinh k_2x, \tag{7.215}$$

where A_i and $B_i\,(i = 1, 2, 3, 4)$ are constants to be determined, and k_1^2 and k_2^2 are the roots of the quadratic equation

$$k^4 - \frac{28}{a^2}k^2 + \frac{63}{a^4} = 0,$$

$$k_1^2 = 14 - \sqrt{133}, \qquad k_2^2 = 14 + \sqrt{133}. \tag{7.216}$$

Substituting Eq. (7.215) into the first equation in (7.213a), we obtain the relationships

$$\left(\frac{8a^2}{15}k_1^2 - \frac{4}{3}\right)A_i = \left(\frac{4a^2}{15} - \frac{8}{105}a^4k_1^2\right)B_i, \qquad i = 1, 2,$$

$$\left(\frac{8a^2}{15}k_2^2 - \frac{4}{3}\right)A_i = \left(\frac{4a^2}{15} - \frac{8}{105}a^4k_2^2\right)B_i, \qquad i = 3, 4. \tag{7.217}$$

Use of the boundary conditions in Eq. (7.211) gives $A_2 = A_4 = B_2 = B_4 = 0$, and

$$A_1 \cosh k_1a + A_3 \cosh k_2a = 0.5f_0,$$

$$B_1 \cosh k_1a + B_3 \cosh k_2a = 0, \tag{7.218}$$

which can be solved along with Eq. (7.217) for A_1, B_1, A_3, and B_3, and we have

$$c_1(x) = 0.5f_0 + 0.516f_0\frac{\cosh k_1x}{\cosh k_1a} - 0.0156f_0\frac{\cosh k_2x}{\cosh k_2a},$$

$$c_2(x) = -0.1138f_0\frac{\cosh k_1x}{\cosh k_1a} + 0.1138f_0\frac{\cosh k_2x}{\cosh k_2a}. \tag{7.219}$$

Table 7.4 Comparison of the solution $\bar{u}(x, 0)$ of Eq. (7.178) obtained by the Galerkin/Ritz and semidiscretization methods with the series solution[a]

x/a ($y = 0$)	Series Solution Eq. (7.185)	Ritz–Galerkin Solution[b] Eqs. (7.183a,b) One Parameter	Ritz–Galerkin Solution[b] Eqs. (7.183a,b) Two Parameter	Semidiscretization Eq. (7.209) One Parameter	Semidiscretization Eq. (7.209) Two Parameter
0.0	0.29445	0.31250	0.29219	0.30261	0.29473
0.1	0.29194	0.30937	0.28986	0.30014	0.29222
0.2	0.28437	0.30000	0.28278	0.29266	0.28462
0.3	0.27158	0.28437	0.27075	0.27999	0.27177
0.4	0.25328	0.26250	0.25340	0.26180	0.25339
0.5	0.22909	0.23437	0.23025	0.23765	0.22909
0.6	0.19854	0.20000	0.20065	0.20693	0.19839
0.7	0.16104	0.15937	0.16382	0.16886	0.16072
0.8	0.11591	0.11250	0.11884	0.12250	0.11546
0.9	0.06239	0.05937	0.06463	0.06667	0.06203
1.0	−0.00038	0.00000	0.00000	−0.00447	0.00000
		Shear stress, $\bar{\sigma}_{yz}(a, 0)$			
1.0	0.67500	0.62500	0.70284	0.72636	0.66416

[a] $\bar{u} = u/q_0 a^2$; $\bar{\sigma}_{yz} = \sigma_{yz}/f_0 a$.
[b] The Ritz and Galerkin solutions are the same for the same coordinate functions.

Equations (7.219) and (7.208a) together define the two-parameter solution of the torsion problem.

A comparison of the solutions obtained using the Galerkin method and the semi-discretization method (with one and two parameters) with the series solution of the torsion problem is presented in Table 7.4. The table also includes the shearing stress, $\sigma_{yz} = -(\partial u/\partial x)$, at $x = a$ and $y = 0$. It is clear from the results that the two-parameter solutions are more accurate than the one-parameter solutions.

Example 7.19 *(The Trefftz method)* In all the variational methods discussed in this chapter, the coordinate functions were selected such that they satisfied the boundary conditions of the problem, and the unknown parameters were determined using a variational procedure, such as the minimization of a quadratic functional or setting the weighted residual to zero. An alternative approach that is complementary to the above approach is to select the approximation functions to satisfy the governing differential equation and determine the unknown parameters such that the boundary conditions are satisfied in a variational/integral sense. This method is known as the Trefftz method, and its application is illustrated via the torsion problem of Example 7.16.

The torsion of cylindrical members can be formulated alternatively in terms of the conjugate function Ψ, which is related to the Prandtl stress function Φ:

$$\Phi = \Psi - \frac{f_0}{4}(x^2 + y^2) \qquad \text{in } \Omega, \tag{7.220}$$

where $f_0 = 2G\theta$, as in Example 7.16. Consequently, Ψ is governed by the Laplace equation

$$\nabla^2\Psi = 0 \qquad \text{in } \Omega = \{(x, y)\colon -a < x, y < a\}, \tag{7.221}$$

subjected to the boundary condition

$$\Psi = \frac{f_0}{4}(x^2 + y^2) \qquad \text{on } \Gamma. \tag{7.222}$$

Note that a nonhomogeneous equation (i.e., Poisson's equation) with homogeneous boundary conditions is transformed to a homogeneous equation (i.e., Laplace's equation) with nonhomogeneous boundary conditions. For the square domain of Example 7.16, we wish to determine an approximate solution of Eqs. (7.221) and (7.222) using the Trefftz method.

We select an N-parameter approximation of the form

$$\Psi \approx \Psi_N = \sum_{j=1}^{N} c_j\phi_j(x, y), \tag{7.223}$$

where ϕ_j are selected such that Ψ_N satisfies the governing equation $\nabla^2\Psi_N = 0$ (i.e., ϕ_j should be harmonic functions). The parameters c_j are determined by the requirement that the boundary condition (7.222) is satisfied in an integral sense:

$$\begin{aligned} 0 &= \oint_\Gamma \left[\Psi_N - \frac{f_0}{4}(x^2 + y^2)\right]\frac{\partial\phi_i}{\partial n}\,ds \\ &= \oint_\Gamma \left[\sum_{j=1}^{N} c_j\phi_j - \frac{f_0}{4}(x^2 + y^2)\right]\frac{\partial\phi_i}{\partial n}\,ds \\ &\equiv \sum_{j=1}^{N} t_{ij}c_j - b_i, \end{aligned} \tag{7.224}$$

where

$$t_{ij} = \oint_\Gamma \frac{\partial\phi_i}{\partial n}\phi_j\,ds, \qquad b_i = \oint_\Gamma \frac{f_0}{4}(x^2 + y^2)\frac{\partial\phi_i}{\partial n}\,ds. \tag{7.225}$$

As a specific example, we choose a one-parameter approximation with

$$\phi_1 = x^4 - 6x^2y^2 + y^4, \tag{7.226}$$

which satisfies the equation $\nabla^2\phi_1 = 0$. We have

$$t_{11} = 4\int_{-a}^{a} (4a^3 - 12ax^2)(a^4 - 6a^2x^2 + x^4)\,dx = \frac{1536}{35}a^8,$$
$$b_1 = -\int_{-a}^{a} f_0(-12x^2a + 4a^3)(x^2 + a^2)\,dx = -\frac{64}{30}a^6 f_0, \qquad (7.227)$$

and the solution is given by

$$\Psi_1 = -\frac{7f_0}{144a^2}(x^4 - 6x^2y^2 + y^4). \qquad (7.228)$$

The Prandtl stress function Φ becomes

$$\Phi = \Psi - \frac{f_0}{4}(x^2 + y^2) = -\frac{7f}{144a^2}(x^4 - 6x^2y^2 + y^4) - \frac{f}{4}(x^2 + y^2). \qquad (7.229)$$

The maximum shear stress σ_{yz} is

$$\sigma_{yz}(a, 0) = -\left(\frac{\partial \Phi_1}{\partial x}\right)_{(a,0)} = \frac{5}{6} f_0 a = 0.833 f_0 a.$$

The maximum shear stress is about 23.4% greater than the exact solution.

The Trefftz method has limited use, because it can only be utilized for Dirichlet-type boundary-value problems (i.e., boundary-value problems in which the function is specified on the boundary). Furthermore, it is not easy to find approximation functions that satisfy the governing differential equations. Some of the hybrid finite element models are based on ideas similar to the Trefftz method.

Example 7.20 Consider the nonlinear differential equation

$$-\frac{d}{dx}\left(u\frac{du}{dx}\right) + 1 = 0, \qquad 0 < x < 1, \qquad (7.230a)$$

subject to the boundary conditions

$$u(1) = \sqrt{2}, \qquad \left(\frac{du}{dx}\right)\bigg|_{x=0} = 0. \qquad (7.230b)$$

We wish to determine a one-parameter Ritz solution to the problem.

The weak form of Eq. (7.230a) is given by

$$0 = \int_0^1 \left[u\frac{dw}{dx}\frac{du}{dx} + w \cdot 1\right] dx.$$

The boundary term is zero because $w(1) = 0$ and $(du/dx)(0) = 0$.

Let

$$u \approx U_1 = c_1\phi_1 + \phi_0, \qquad \text{with } \phi_0 = \sqrt{2}, \qquad \phi_1 = 1 - x.$$

Substituting into the weak form, we obtain

$$0 = \int_0^1 \left\{\left[c_1(1-x)+\sqrt{2}\right](-1)(-c_1)+(1-x)\right\} dx$$
$$= c_1\left[c_1\cdot\tfrac{1}{2}+\sqrt{2}\right]+\tfrac{1}{2}$$

or

$$c_1^2+2\sqrt{2}c_1+1=0, \qquad (c_1)_{1,2}=-\sqrt{2}\pm\sqrt{2-1}=-\sqrt{2}\pm 1.$$

Thus, there are two approximate solutions to the nonlinear problem. We must choose one value of c_i using some meaningful criterion. We shall take the value of c_1 that yields the smallest integral of the residual in the differential equation:

$$\int_0^1 \left[-\frac{d}{dx}\left(u\frac{du}{dx}\right)+1\right]dx=\left(c_1^2+1\right).$$

Clearly, the smaller (in absolute value) root gives the smaller value of the residual. Hence, we choose $(c_1)_1=-\sqrt{2}+1$. The solution becomes

$$U_1=\left(1-\sqrt{2}\right)(1-x)+\sqrt{2}=1+\left(\sqrt{2}-1\right)x.$$

The exact solution is $u(x)=\sqrt{1+x^2}$. At $x=0$ the approximate solution matches with the exact.

7.6 SUMMARY

In this chapter, the Ritz and weighted-residual (e.g., Galerkin, least-squares, and collocation) methods were presented and their application to simple problems was illustrated. The Ritz method makes use of the weak form provided by the principle of virtual displacements, the principles of minimum total potential energy, or the one developed from the governing equations of the problem as discussed in Sections 7.4.1 and 7.5.1. The key feature of the weak form is that it includes the governing equation(s) as well as the natural boundary condition(s) of the problem. Hence, the Ritz approximation is not required to satisfy the natural boundary conditions. All weighted-residual methods, except the least-squares method, are based on an weighted-integral statement of the governing equation(s), whereas the least-squares method is based on the minimization of the square of the governing equation(s). Thus, the integral statements used in all weighted-residual methods do not include any boundary conditions as a part of the statements. Hence, the approximations chosen for the weighted-residual methods are required to satisfy *all* (both natural and essential) specified boundary conditions of the problem. Consequently, the approximation functions are of higher order than those used in the Ritz method. The Ritz as well as the weighted-residual methods may be used for linear and nonlinear differential equations. However, in the case of the least-squares method, there are some limitations, as discussed in Section 7.5.5. Although the least-squares method can be interpreted as a special case of the weighted-residual methods for linear static problems, it is based on an integral statement whose Euler equations, if derived, are *not* the same as the governing equations. Thus, the least-squares method is quite different from the other weighted-residual methods. Overall, the Ritz method is the most efficient method, especially for solid and structural mechanics problems.

The single most difficult step in using the variational methods presented in this chapter is the selection of the approximation functions. The requirements (7.62) and (7.164) on the approximation functions merely provide the guidelines for their selection. The selection of the approximation functions becomes even more difficult for problems with irregular domains (i.e., noncircular and nonrectangular) or discontinuous data (loading as well as geometry). Further, the generation of coefficient matrices for the resulting algebraic equations cannot be automated for a *class* of problems that differ from each other only in the geometry of the domain, boundary conditions, or loading. These limitations of the classical variational methods can be overcome by representing a given domain as a collection of geometrically simple subdomains for which we can systematically generate the approximation functions (see Example 7.8). One such technique, namely, the finite element method, is discussed in Chapter 9. The finite element method is based on ideas similar to the classical variational methods, especially in developing the system of algebraic equations for the unknown coefficients, but the method views a given domain as a collection of conveniently chosen subdomains that allow a systematic generation of the approximation functions.

EXERCISES

7.1 Let $\mathcal{P}$ be the vector space of all polynomials with real coefficients. Determine which of the following subsets of $\mathcal{P}$ are subspaces:

(a) $S_1 = \{p(x): \ p(1) = 0\}$.
(b) $S_2 = \{p(x): \ \text{degree of } p(x) = 3\}$.
(c) $S_3 = \{p(x): \ \text{degree of } p(x) \leq 3\}$.
(d) $S_4 = \{p(x): \ \text{constant term is zero}\}$.

7.2 Determine which of the following sets of vectors in $\Re^3$ are linearly independent over the real number field $\Re$:

(a) $\{(-1, 1, 0), (-1, 1, 1), (-2, -1, 1), (1, 1, 1)\}$.
(b) $\{(1, 0, 0), (1, 1, 0), (1, 1, 1)\}$.
(c) $\{(1, 1, 1), (1, 2, 3), (2, -1, 1)\}$.

7.3 Determine if the following sets of functions are linearly independent.

(a) $\{\sin(n\pi x/L)\}_{n=1}^{3}$, $0 \leq x \leq L$.
(b) $\{x^n(1-x)\}_{n=0}^{3}$, $0 \leq x \leq 1$.
(c) $\{1 + x + x^2, 1 + 2x + 3x^2, 2 - 3x + x^3\}$, $0 \leq x \leq 1$.

7.4 Compute the L_2 norm and the sup-norm of the following functions in the interval indicated:

(a) $u(x) = \sin \pi x - x$, on $0 \leq x \leq 1$.

(b) $u(x) = x^{1/3}$, on $0 \leq x \leq 1$.

(c) $u(x) = \cos \pi x + 2x - 1$, on $0 \leq x \leq 1$.

(d) $u(x) = \sqrt{1 + x^2}$, on $0 \leq x \leq 1$.

(e) $u(x) = \begin{cases} 100 \sin 100\pi x, & 0 \leq x \leq 0.01 \\ 0, & 0.01 \leq x \leq 1. \end{cases}$

7.5 Prove the following relations in a real inner product space:

(a) Parallelogram law: $\|u + v\|^2 + \|u - v\|^2 = 2(\|u\|^2 + \|v\|^2)$.

(b) $(u, v) = \frac{1}{4}[\|u + v\|^2 - \|u - v\|^2]$.

(c) $|\|u\| - \|v\|| \leq \|u - v\|$.

7.6 Let $\mathbf{u} = \mathbf{u}(x)$ and $\mathbf{v} = \mathbf{v}(x)$ be two vector functions of x. Which one of the following products qualifies as an inner product?

(a) $(\mathbf{u}, \mathbf{v}) \equiv \int_0^L \mathbf{u} \cdot \mathbf{v}\, dx$.

(b) $(\mathbf{u}, \mathbf{v}) \equiv \int_0^L \mathbf{u}(L - x) \cdot \mathbf{v}(x)\, dx$.

7.7 Compute the inner product of the following pairs of functions on the interval indicated. Use the L_2-inner product and the H^1-inner product.

(a) $u = x - x^2$, $v = \sin \pi x$, on $0 \leq x \leq 1$.

(b) $u = (1 + x)$, $v = 3x^2 - 1$, on $-1 \leq x \leq 1$.

(c) $u = \sin \pi x$, $v = \cos \pi x$, on $0 \leq x \leq 1$.

(d) $u = \sin \pi x$, $v = a + bx + cx^2$, on $0 \leq x \leq 1$.

(e) $u = \sin \pi x \sin \pi y$, $v = (1 - x^2 - y^2)$, on $0 \leq x, y \leq 1$.

(f) $u = (x^2 - a^2)(y^2 - b^2)$, $0 \leq x \leq a$ and $0 \leq y \leq b$.

7.8 Find the distance between the following pair of functions using the inner product in Eq. (7.23a): $u = x^3 - 3x + 2$ and $v = (x - 1)^2$, $0 \leq x \leq 1$.

7.9 Check whether the following pair of functions are orthogonal in the $L_2(0, 1)$-space: $u(x) = 2 + 3x^2 - x$ and $v(x) = \frac{1}{3} + 3x - 5x^2$.

7.10 Check whether the pair of functions in Exercise 7.9 are orthogonal in the $H^1(0, 1)$-space.

7.11 Determine the constants a and b such that $w(x) = a + bx + 3x^2$ is orthogonal in $L_2(0, 1)$ to both $u(x)$ and $v(x)$ of Exercise 7.9.

7.12 Determine C if the indicated pairs of vectors are orthogonal in $\Re^4$:

(a) $(1, C - 1, 2, 1 + C)$, $(2C, 4, C, 1)$.

(b) $(2, 1 - C, 4, 3C)$, $(C, 1 + C, 3C, -7)$.

7.13 Determine a vector in the space $\mathcal{P}$ of polynomials of degree 2 such that the vector is orthogonal to the polynomials $p_1 = 1 + x - 2x^2$ and $p_2 = -2 + 4x + x^2$ in $L_2\,(1{,}0)$.

7.14 Which of the following operators qualify as linear operators?

(a) $A(u) = -\dfrac{d}{dx}\left(a\dfrac{du}{dx}\right)$.

(b) $T(u) = \nabla^2 u + 1$.

(c) $I(u) = \int_0^a K(x)\dfrac{du}{dx}\,dx - u(0)$.

7.15 If T_1 and T_2 are linear operators defined on $\Re^3$ by $T_1(\mathbf{x}) = (0, x_1 - x_3, x_2 + x_3)$, $T_2(\mathbf{x}) = (x_2 - x_3, 2x_1 - x_2, x_1)$, determine:

(a) $T_1 + T_2$.

(b) $T_1 T_2$.

(c) $T_2 T_1$.

7.16 Find the matrix representing the linear transformation from $\Re^4$ into $\Re^3$ with respect to the standard bases of $\Re^4$ and $\Re^3$:

$$T(x_1, x_2, x_3, x_4) = (x_1 + 2x_2, 2x_2 + x_3, x_3 + x_1).$$

7.17 Identify the linear and bilinear functionals:

(a) $I(u, v) = \int_\Omega \nabla u \cdot \nabla v\, dxdy$.

(b) $I(u, v) = \displaystyle\int_0^a \left(b\frac{d^2u}{dx^2}\frac{d^2v}{dx^2} + fv\right) dx$.

7.18 Determine which of the following operators represent bilinear forms:

(a) $B\colon\ \Re^2 \times \Re^2 \to \Re,\ \ B(\mathbf{x}, \mathbf{y}) = (x_1 + y_1)^2 - (x_1 - y_1)^2$.

(b) $B\colon\ \Re^2 \times \Re^2 \to \Re,\ \ B(\mathbf{x}, \mathbf{y}) = x_1 y_2 - x_2 y_1$.

7.19 Consider the functional

$$B(u, v) = \tfrac{1}{4}(\|u + v\|^2 - \|u - v\|^2)$$

on an inner product space. Show that:

(a) $B(\cdot, \cdot)$ is bilinear in u and v.

(b) For fixed u, $f_u(v) = B(u, v)$ is a (bounded) linear functional.

(c) $B(u, v)$ satisfies the axioms of an inner product.

7.20 If $I(u)$ is a quadratic functional, show that (a) its first variation is a bilinear functional of u and δu, and (b) its first variation is a linear functional of δu.

Give admissible approximation functions, either algebraic or trigonometric, for a two-parameter Ritz approximation of problems in Exercises 7.21–7.27. Assume that the total potential energy principle or weak form is used to construct the Ritz approximation.

7.21 A cable suspended between points $A : (0, 0)$ and $B : (L, h)$ and subjected to uniformly distributed transverse load of intensity f_0.

7.22 A cantilever beam subjected to uniformly distributed load of intensity q_0.

7.23 The symmetric half of the simply supported beam problem considered in Example 7.7 $(0 \leq x \leq L/2)$.

7.24 A beam clamped at the left end and simply supported at the right end, and subjected to point load F_0 at $x = L/2$.

7.25 A simply supported beam with a spring support at $x = L/2$, and subjected to uniformly distributed load of intensity q_0.

7.26 A square elastic membrane fixed on all its sides and subjected to a uniformly distributed load of intensity f_0.

7.27 A quadrant model (because of the biaxial symmetry) of the membrane problem of Exercise 7.26.

7.28–7.34 Find the two-parameter Ritz approximation using algebraic polynomials of the problems in Exercises 7.21–7.27 and compare the solutions with the exact solutions when possible.

7.35 Find the first two natural frequencies of a cantilever beam. Take $EI = (a + bx)^{-1}$ where a and b are constants.

7.36 Find a two-parameter Ritz approximation of the transverse deflection of a simply supported beam on an elastic foundation that is subjected to uniformly distributed load. Use (a) algebraic polynomials and (b) trigonometric polynomials.

7.37 Derive the matrix equations corresponding to the N-parameter Ritz approximation

$$W_N = c_1 x^2 + c_2 x^3 + \cdots + c_N x^{N+1}$$

of a cantilever beam with a uniformly distributed load, q_0. Compute a_{ij} and b_j in explicit form in terms of i, j, L, EI, and q_0.

7.38 Use a two-parameter Ritz approximation with trigonometric functions to determine the critical buckling load P of a simply supported beam.

7.39 Consider the buckling of a uniform beam according to the Timoshenko beam theory. The total potential energy functional for the problem can be written as

$$\Pi(w_0, \phi_x) = \frac{1}{2}\int_0^L \left[D\left(\frac{d\phi_x}{dx}\right)^2 + S\left(\frac{dw_0}{dx} + \phi_x\right)^2 - N\left(\frac{dw_0}{dx}\right)^2 \right] dx,$$

where $w_0(x)$ is the transverse deflection, ϕ_x is the rotation, D is the flexural stiffness, S is the shear stiffness, and N is the axial compressive load. We wish to determine the critical buckling load N_{cr} of a simply supported beam using the Ritz method. Assume a one-parameter approximation of w_0 and ϕ_x and determine the critical buckling load.

7.40 Determine the N-parameter Ritz solution for the transient response of a simply supported beam under step loading $q(x,t) = q_0 H(t-t_0)$, where $H(t)$ denotes the Heaviside step function. Use trigonometric functions for $\phi_i(x)$.

7.41 Show that the two-parameter Ritz solution for the transient response of the bar considered in Example 7.6 yields the equations

$$\begin{bmatrix} \frac{1}{3} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{5} \end{bmatrix} \begin{Bmatrix} \ddot{c}_1 \\ \ddot{c}_2 \end{Bmatrix} + \begin{bmatrix} 2 & 2 \\ 2 & \frac{7}{3} \end{bmatrix} \begin{Bmatrix} c_1 \\ c_2 \end{Bmatrix} = \begin{Bmatrix} \frac{1}{2} \\ \frac{1}{3} \end{Bmatrix}. \quad \text{(a)}$$

Use the Laplace transform method to determine the solution of these equations.

7.42 Derive the weak forms of the following nonlinear equations:

$$-\frac{dN_{xx}}{dx} = f(x), \qquad 0 \le x \le L, \quad \text{(a)}$$

$$-\frac{d}{dx}\left(\frac{dw_0}{dx}N_{xx}\right) - \frac{d^2 M_{xx}}{dx^2} = q(x), \qquad 0 \le x \le L, \quad \text{(b)}$$

where

$$N_{xx} = EA\left[\frac{du_0}{dx} + \frac{1}{2}\left(\frac{dw_0}{dx}\right)^2\right], \qquad M_{xx} = -EI\frac{d^2 w_0}{dx^2}.$$

7.43 Consider a uniform beam fixed at one end and supported by an elastic spring (spring constant k) in the vertical direction. Assume that the beam is loaded by a uniformly distributed load q_0. Determine the one-parameter Ritz solution using algebraic functions.

7.44 Consider the problem of finding the fundamental frequency of a circular membrane of radius a, fixed at its edge. The governing equation for axisymmetric vibration is

$$-\frac{1}{r}\frac{d}{dr}\left(r\frac{du}{dr}\right) - \lambda u = 0, \qquad 0 < r < a,$$

where λ is the frequency parameter and u is the deflection of the membrane. (a) Construct the weak form, (b) use one-parameter Ritz approximation to determine λ, and (c) use two-parameter Ritz approximation to determine λ. Select trigonometric functions for $\phi_i(r)$.

7.45 Consider the problem of finding the solution of the equation

$$\frac{d^2u}{dx^2} + u + x = 0, \qquad 0 < x < 1; \qquad u(0) = u(1) = 0.$$

(a) Develop the weak form, (b) assume N-parameter Ritz approximation of the form

$$U_N(x) = x(1-x)\left(c_1 + c_2 x + \cdots + c_N x^{N-1}\right)$$

and obtain the Ritz equations for the unknown coefficients, and (c) determine the two-parameter solution and compare it with the exact solution

$$u(x) = \frac{\sin x}{\sin 1} - x.$$

7.46 Consider the Bessel equation

$$x^2 \frac{d^2u}{dx^2} + x\frac{du}{dx} + (x^2 - 1)u = 0, \qquad 1 < x < 2$$

subject to the boundary conditions $u(1) = 1$ and $u(2) = 2$. Assume $u = w + x$ and reduce the equation to

$$-\frac{d}{dx}\left(x\frac{dw}{dx}\right) - \frac{x^2-1}{x}w = x^2, \qquad 1 < x < 2,$$

subject to the boundary conditions $w(1) = w(2) = 0$. Determine a one-parameter Ritz approximation of the problem and compare it with the analytical solution

$$u(x) = 3.6072 J_1(x) + 0.75195 Y_1(x),$$

where J_1 and Y_1 are Bessel functions of the first and second kind, respectively.

7.47 Derive the weak form and obtain a one-parameter Ritz solution of the problem

$$-\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) = 1 \text{ in a unit square,}$$

$$u(1, y) = u(x, 1) = 0,$$

$$\frac{\partial u}{\partial x}(0, y) = \frac{\partial u}{\partial y}(x, 0) = 0.$$

The origin of the coordinate system is taken at the lower left corner of the unit square.

7.48 Determine the two-parameter Petrov–Galerkin solution (with $\psi_1 = 1$ and $\psi_2 = x$) of the following differential equation and compare, if possible, with the exact solution:

$$-\frac{d^2u}{dx^2} = \frac{1}{1+x}, \qquad 0 < x < 1; \qquad u(0) = u'(1) = 0.$$

7.49 Determine the two-parameter Petrov–Galerkin solution (with $\psi_1 = 1$ and $\psi_2 = x$) of the following differential equation:

$$-\frac{d}{dx}\left[(1+x^2)\frac{du}{dx}\right]+u=\sin\pi x+3x-1, \quad 0<x<1; \quad u(0)=u'(1)=0.$$

7.50 Determine the one-parameter Galerkin solution of the equation

$$\frac{d^2}{dx^2}\left[\left(2+\frac{x}{L}\right)\frac{d^2w_0}{dx^2}\right]+kw_0=q_0\frac{x}{L}$$

that governs a cantilever beam on an elastic foundation and subjected to linearly varying load (from zero at the free end to q_0 at the fixed end). Take $k = L = 1$ and $q_0 = 3$, and use algebraic polynomials.

7.51 Determine a two-parameter Petrov–Galerkin solution (with $\psi_1 = 1$ and $\psi_2 = xy$) of the problem

$$-\nabla^2 u = 0 \qquad \text{in } 0 < (x, y) < 1,$$
$$u = \sin\pi x \qquad \text{on } y = 0,$$
$$u = 0 \qquad \text{on all other sides.}$$

7.52 Solve the nonlinear differential equation

$$-\frac{d}{dx}\left(u\frac{du}{dx}\right)+1=0, \qquad 0<x<1,$$
$$u(1)=\sqrt{2}, \qquad u'(0)=0.$$

Use a one-parameter Petrov–Galerkin approximation with (a) $\psi_1 = 1$ and (b) $\psi_1 = x$. Compare the results with the exact solution, $u(x) = \sqrt{1+x^2}$. See Example 7.20.

7.53 Find the first two eigenvalues associated with the differential equation

$$-\frac{d^2u}{dx^2}=\lambda u, \qquad 0<x<1,$$
$$u(0)=0, \qquad u(1)+u'(1)=0.$$

Use the least-squares method. Use the operator definition $A = -(d^2/dx^2)$ to avoid increasing the degree of the characteristic polynomial for λ.

7.54 Solve the Poisson equation

$$-\nabla^2 u = f_0 \qquad \text{in a unit square,}$$
$$u = 0 \qquad \text{on the boundary.}$$

using the following N-parameter Galerkin approximation:

$$U_N = \sum_{i,j=1}^{N} c_{ij} \sin i\pi x \sin j\pi y.$$

7.55 Solve the nonlinear equation in Exercise 7.52 by the Galerkin method.

7.56 Find a two-parameter Galerkin solution of a clamped (at both ends) beam under uniformly distributed load.

7.57 Solve the equation in Exercise 7.49 using the least-squares method.

7.58 Solve the problem of Exercise 7.52 using the least-squares method.

7.59 Solve the equation in Exercise 7.53 using the least-squares method.

7.60 Solve the equation in Exercise 7.54 using the least-squares method.

7.61 Consider a cantilever beam of variable flexural rigidity, $EI = a_0[2-(x/L)^2]$, and carrying a distributed load, $q = q_0[1 - (x/L)]$. Find a three-parameter solution using the collocation method.

7.62 Repeat Exercise 7.52 using the collocation method.

7.63 Solve the differential equation in Exercise 7.53 by the collocation method.

7.64 Solve the problem in Exercise 7.54 using the one-point collocation method.

7.65 Consider the Laplace equation

$$-\nabla^2 u = 0, \qquad 0 < x < 1, \qquad 0 < y < \infty,$$
$$u(0, y) = u(1, y) = 0 \quad \text{for} \quad y > 0,$$
$$u(x, 0) = x(1 - x), \qquad u(x, \infty) = 0, \qquad 0 \le x \le 1.$$

Assuming an approximation of the form

$$u(x, y) = c_1(y)x(1 - x),$$

find the differential equation for $c_1(y)$ and solve it exactly.

7.66 Use the semidiscretization method to find a two-parameter approximation of the form

$$u(x, y) = c_1(y) \cos \frac{\pi x}{2a} + c_2(y) \cos \frac{3\pi x}{2a},$$

to determine an approximate solution of the torsion problem in Example 7.16.

7.67 Use the semidiscretization method to find a two-parameter approximation of the form

$$w(x, t) = c_1(t)(1 - \cos 2\pi x),$$

to determine an approximate solution of the equation

$$\frac{\partial^4 w}{\partial x^4} + \frac{\partial^2 w}{\partial t^2} = 0, \qquad 0 < x < 1, \qquad t > 0,$$

$$w = \frac{\partial w}{\partial x} = 0 \qquad \text{at } x = 0, 1 \text{ and } t > 0,$$

$$w = \sin \pi x - \pi x(1 - x), \qquad \frac{\partial w}{\partial t} = 0 \text{ at } t = 0, \quad 0 < x < 1.$$

Use the Galerkin method to satisfy the initial conditions.

7.68 Consider the problem of finding the eigenvalues associated with the equation

$$\nabla^2 u + \lambda u = 0 \quad \text{in } \Omega; \qquad u = 0 \quad \text{on } \Gamma,$$

where Ω is a rectangle, $-a < x < a$ and $-b < y < b$, and Γ is its boundary (see Example 7.18). Assuming approximation of the form

$$U_1 = c_1(x)\phi_1(y) = c_1(x)(y^2 - b^2),$$

determine the first eigenvalue λ_1.

7.69 Repeat Exercise 7.68 with a two-parameter approximation of the form

$$U_2 = c_1(x)\phi_1(y) + c_2(x)\phi_2(y) = [c_1(x) + c_2(x)y^2](y^2 - b^2).$$

7.70 Solve the problem in Example 7.19 using a two-parameter approximation of the form (which is more complete than the one-parameter approximation used in Example 7.19, although it has no effect on the derivative of the solution)

$$\Psi_2 = c_1 + c_2(x^4 - 6x^2y^2 + y^4).$$

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8

THEORY AND ANALYSIS OF PLATES

8.1 INTRODUCTION

8.1.1 General Comments

In this chapter we study bending, buckling, and natural vibration of plate structures. A *plate* is a flat structural element with planform dimensions much larger than its thickness and is subjected to loads that cause bending deformation in addition to stretching (see Fig. 8.1). Street manhole covers, table tops, side panels and roofs of buildings and transportation vehicles, glass window panels, turbine disks, bulkheads, and tank bottoms provide familiar examples of plate structures. A *shell* is initially a curved structural element with thickness much smaller than the other dimensions. Like a plate, a shell is subjected to loads that cause stretching and bending deformations. Examples of shell structures are provided by pressure vessels, pipes, curved panels of a variety of structures including automobiles and aerospace vehicles, tires, and roof domes and sheds.

In most cases, the thickness of plate and shell structures is about one-tenth or less of the smallest in-plane (i.e., within the plane) dimension. When the thickness is one-twentieth of an in-plane dimension or less, they are termed *thin*; otherwise they are said to be *thick*. Because of the smallness of the thickness dimension, it is often not necessary to model plate and shell structures using 3D elasticity equations. Simple 2D theories that are based on some kinematic assumptions can be developed to study deformation, stresses, natural frequencies, and global buckling loads of plate and shell structures. In the present study, the governing equations of the 2D theories are derived using the principle of virtual displacements for circular plates in cylindrical coordinates and for noncircular plates in rectangular Cartesian coordinates. Bending, buckling, and vibration solutions are obtained for particular cases of

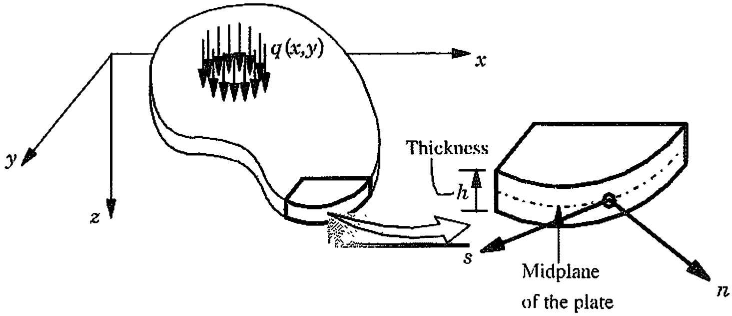


Figure 8.1 Geometry of a plate with curved boundary.

geometry, boundary conditions, and loads using the energy and variational methods developed in the earlier chapters.

The two-dimensional theories (in terms of the planform coordinates x and y) of plates and shells are developed from an assumed displacement or stress field. They are often taken as a finite linear combination of unknown functions and the thickness coordinate z. For example, the ith displacement component u_i is expanded as

$$u_i(x,y,z,t)=\varphi_i^{(0)}(x,y,t)+z\varphi_i^{(1)}+\cdots+(z)^N\varphi_i^{(N)}=\sum_{j=0}^{N}(z)^j\varphi_i^{(j)}(x,y,t), \quad (8.1)$$

where (x, y) are the coordinates in the plane of the plate, z the thickness coordinate, t the time, and $\varphi_i^{(j)}$ are functions that describe the deformation and are to be determined.

The equations governing $\varphi_i^{(j)}$ $(j = 0, 1, 2, \ldots, N)$ are determined by the principle of virtual displacements:

$$0 = \int_0^T (\delta U + \delta V - \delta K)\,dt, \quad (8.2)$$

where δU, δV, and δK denote the virtual strain energy, virtual potential energy due to external applied forces, and virtual kinetic energy, respectively. These quantities are determined in terms of actual stresses and virtual strains, which depend on the assumed displacement expansion (8.1). The integration over the domain of the plate structure is represented as the tensor product of the integration over the midplane (mid surface) of the plate (shell) and integration over the thickness. This is possibly due to the explicit nature of the assumed displacement field in the thickness coordinate. Thus, we can write

$$\int_{Vol.} (\cdot)\,dV = \int_{-h/2}^{h/2}\int_{\Omega_0} (\cdot)\,d\Omega dz, \quad (8.3)$$

where h denotes the total thickness of the plate and Ω_0 denotes the undeformed midplane of the plate, which is assumed to coincide with the xy-plane. Since all undetermined variables $\varphi_i^{(j)}$ are explicit functions of the thickness coordinate z, the integration over plate thickness is carried out explicitly, reducing the problem to a two-dimensional one. Consequently, the Euler–Lagrange equations resulting from Eq. (8.2) consist of differential equations involving the dependent variables $\varphi_i^{(j)}(x, y, t)$ and thickness-averaged stress resultants, $R_{ij}^{(m)}$:

$$R_{ij}^{(m)} = \int_{-h/2}^{h/2} (z)^m \sigma_{ij}\, dz. \tag{8.4}$$

The stress resultants $R_{ij}^{(m)}$ can be written in terms of $\varphi_i^{(j)}$ with the help of the assumed constitutive equations (stress–strain relations) and strain–displacement relations. More complete development of this procedure is forthcoming in this chapter.

The same approach is used when u_i denote stress components, except that the basis of the derivation of the governing equations is the principle of virtual forces. In the present book, the stress-based theories will receive very little attention.

8.1.2 An Overview of Plate/Shell Theories

There exist a number of plate and shell theories, and they differ from each other in two principal ways: (1) the choice of the assumed field (i.e., displacement or stress), and (2) the choice of terms in the expansion [i.e., number of terms and powers of the thickness coordinate z retained in Eq. (8.1)]. The choice of the field is often restricted to either displacements or stresses, although a mixed approach is possible. The number of terms retained in the displacement or stress expansions is often limited, at most, to cubic in thickness coordinate in the interest of keeping the theory simple. Thus, a plate or shell theory can be developed for any combination of the field variables and number of terms in the expansion of the variables. The number of theories further multiply if different order expansions are used for different components of a field variable and types of nonlinear terms included in the strain–displacement relations.

A general but brief review of various theories of plates is presented before closing this section. The ideas are also applicable to shells. No review of plate/shell theories will be complete, as there are thousands of papers dealing with one aspect or the other of the many combinations mentioned above.

Kirchhoff Plate Theory The simplest plate theory is known as the *Kirchhoff plate theory* or simply the *classical plate theory* (CPT), which is an extension of the Euler–Bernoulli beam theory (see Example 4.3) to two dimensions. The theory is based on the following assumptions, known as the *Kirchhoff hypotheses* [1]:

1. Straight lines perpendicular to the midplane (i.e., transverse normals) before deformation remain straight after deformation.
2. The transverse normals do not experience elongation (i.e., they are inextensible).

3. The transverse normals rotate such that they remain perpendicular to the mid surface after deformation.

These assumptions allow us to describe the plate deformation in terms of certain displacement quantities. Assumption (1) requires that the displacement field be a linear function of the thickness coordinate z:

$$\begin{aligned} u(x, y, z, t) &= u_0 + zF_1(x, y, t), \\ v(x, y, z, t) &= v_0 + zF_2(x, y, t), \\ w(x, y, z, t) &= w_0 + zF_3(x, y, t), \end{aligned} \tag{8.5}$$

where $(u_0, v_0, w_0, F_1, F_2, F_3)$ are functions to be determined such that the remaining two assumptions of the Kirchhoff hypothesis are satisfied. The inextensibility assumption (2) requires that

$$\frac{\partial w}{\partial z} = 0 \rightarrow F_3 = 0 \qquad \text{for all } x, y, \text{ and } t.$$

Thus, w is independent of z, i.e., $w = w_0(x, y, t)$. Assumption (3) requires that

$$\begin{aligned} \frac{\partial u}{\partial z} &= -\frac{\partial w}{\partial x} \rightarrow F_1 = -\frac{\partial w_0}{\partial x}, \\ \frac{\partial v}{\partial z} &= -\frac{\partial w}{\partial y} \rightarrow F_2 = -\frac{\partial w_0}{\partial y}. \end{aligned} \tag{8.6}$$

Hence, the displacement field (8.5) takes the form

$$\begin{aligned} u(x, y, z, t) &= u_0(x, y, t) - z\frac{\partial w_0}{\partial x}, \\ v(x, y, z, t) &= v_0(x, y, t) - z\frac{\partial w_0}{\partial y}, \\ w(x, y, z, t) &= w_0(x, y, t). \end{aligned} \tag{8.7}$$

Thus, the Kirchhoff plate deformation is completely determined by the functions (u_0, v_0, w_0), which denote the displacements of a point on the midplane along the three coordinate directions. Note that the displacement field (8.7) will result, as will be seen shortly, in the neglect of all transverse strains, i.e., $\varepsilon_{zz} = \varepsilon_{xz} = \varepsilon_{yz} = 0$.

Displacement-Based Shear Deformation Theories The simplest displacement-based plate theory that accounts for transverse shear strains is one based on the displacement field [2–8]

$$\begin{aligned} u(x, y, z) &= u_0 + z\phi_x(x, y), \\ v(x, y, z) &= v_0 + z\phi_y(x, y), \\ w(x, y, z) &= w_0(x, y), \end{aligned} \tag{8.8}$$

where ϕ_x and $-\phi_y$ denote rotations about the y and x axes, respectively:

$$\phi_x(x, y) = \left(\frac{\partial u}{\partial z}\right)_{z=0}, \qquad \phi_y(x, y) = \left(\frac{\partial v}{\partial z}\right)_{z=0}. \tag{8.9}$$

In this theory, the normality restriction is removed by allowing for independent and arbitrary rotations (ϕ_x, ϕ_y) of a transverse normal. The theory is known in the literature as the *Mindlin plate theory*, although the use of the displacement field (8.8) and associated plate theory were developed much earlier by Bassett [2], Hildebrand et al. [3], and Hencky [4]. Mindlin [5] extended the theory developed by Hencky [4] to the vibration of crystal plates. Hence, it would be incorrect to attribute the theory to Mindlin alone. The theory is now being referred to as the first-order shear deformation plate theory (FSDT); see Reddy [6–8].

Note that the classical plate theory (CPT) is also a first-order theory in the sense that the first-order terms in the thickness coordinate are included; however, it is not a shear deformation theory. As will be seen later in this chapter, the transverse shear strains in FSDT are represented as constants through the plate thickness. Therefore, the transverse shear stresses are also constant through the plate thickness, whereas the stress equilibrium equations predict them to be quadratic. To make the shear forces (Q_x, Q_y) computed in the FSDT to be equal to those obtained using the transverse shear stresses from the stress–equilibrium equations, shear correction factors were introduced. In both CPT and FSDT, the plane-stress state assumption is used and the plane-stress reduced form of the constitutive law is used. In both theories, the inextensibility (i.e., $\varepsilon_{zz} = 0$) and/or straightness of transverse normals can be removed. Such extensions lead to second- and higher-order theories of plates.

Second- and higher-order plate theories[1] use higher-order polynomials [i.e., $n > 1$ in Eq. (8.1)] in the expansion of the displacement components through the thickness of the plate. The higher-order theories introduce additional unknowns that are often difficult to interpret in physical terms.

A third-order plate theory with transverse inextensibility is based on the displacement field [9–16]:

$$\begin{aligned} u(x, y, z, t) &= u_0(x, y, t) + f(z)\varphi_x(x, y, t), \\ v(x, y, z, t) &= v_0(x, y, t) + f(z)\varphi_y(x, y, t), \\ w(x, y, z, t) &= w_0(x, y, t), \end{aligned} \tag{8.10}$$

where $f(z)$ is a cubic function of z. There are number of people who have used a displacement field of the form (8.10), but they differ in actual form because of the choice of variables; consequently, the resulting governing equations have different looks. However, it can be shown that all third-order theories are special cases

[1]The order referred here is to the degree n of the thickness coordinate in the displacement expansion and not to the order of the governing differential equations.

of that derived by Reddy [9–11] for laminated plates and shells using the displacement field in Eq. (8.10) and the principle of virtual displacements. The displacement field (8.10) accommodates quadratic variation of transverse shear strains (and hence stresses) and vanishing of transverse shear stresses on the top and bottom of a plate. Thus, there is no need to use shear correction factors in a third-order plate theory.

The third-order plate theory of Levinson [16] is based on the same displacement field as in Eq. (8.10), but he used the equilibrium equations of the FSDT, i.e., he did not use the principle of virtual displacements to derive the equilibrium equations. The Levinson theory results in much simpler equations and does not involve the higher-order stress resultants. However, the theory leads to an unsymmetric stiffness matrix even for linear problems.

Stress-Based Theories The plate theories based on the expansion of the stress field are due to Reissner [17–19] and Kromm [20,21], and the book by Panc [22] contains chapters devoted to these theories and their extensions. In the *Reissner plate theory*, the distribution of the stress components through the plate thickness, for the static case, is assumed to be of the form

$$\sigma_{xx} = z\frac{12M_{xx}}{h^3}, \qquad \sigma_{yy} = z\frac{12M_{yy}}{h^3}, \qquad \sigma_{xy} = z\frac{12M_{xy}}{h^3}, \tag{8.11a}$$

$$\sigma_{xz} = \frac{3Q_x}{2h}\left[1-\left(\frac{2z}{h}\right)^2\right], \qquad \sigma_{yz} = \frac{3Q_y}{2h}\left[1-\left(\frac{2z}{h}\right)^2\right], \tag{8.11b}$$

$$\sigma_{zz} = -\frac{q}{4}\left[2-3\left(\frac{2z}{h}\right)+\left(\frac{2z}{h}\right)^3\right], \tag{8.11c}$$

where (M_{xx}, M_{yy}, M_{xy}) are the bending moments per unit length, (Q_x, Q_y) the shear forces per unit length, q is the distributed transverse load per unit area, and h is the plate thickness. The stress field in Eq. (8.11a) is the same as that of the classical plate theory (to be shown shortly); the transverse shear stress field in Eqs. (8.11b,c) is the same as that obtained from 3D stress–equilibrium equations after using the in-plane stress field from (8.11a). Thus, the in-plane stresses are linear functions of z, the transverse shear stresses are quadratic functions of z, and transverse normal stress is cubic in z. Obviously, these stress components satisfy the stress–equilibrium equations of 3D elasticity, Eqs. (3.13). The transverse displacement w of Reissner's theory is a function of (x, y, z), and this complicates its determination. To make the theory tractable, Reissner introduced the thickness-integrated transverse displacement (a "mean deflection with respect to the plate thickness"):

$$w(x, y) = \frac{3}{2h}\int_{-h/2}^{h/2} w\left[1-\left(\frac{2z}{h}\right)^2\right]dz. \tag{8.12}$$

The refined theory of Kromm [20,21] is based on a more general stress distribution, especially for the transverse shear stress components across the thickness of the plate. For a complete description of theory and its governing equations, the reader may consult the book by Panc [22].

The stress-based theories have not received as much attention as the displacement-based plate theories. This might be due to the fact that the stress-based theories are relatively more complicated, and inconsistencies between the actual (i.e., consistent with the assumed stress distributions) and adopted displacements may exist.

In this chapter we consider only small strain and small displacement bending deformation of plates; i.e., the stretching deformation is omitted in the derivation of the governing equations, as the equations governing stretching deformation are not coupled to bending deformation. Therefore, in the absence of loads that cause stretching, the extensional displacement field (u_0, v_0) is identically zero. We begin with an assumed displacement field, compute strains, and then use Hamilton's principle (or the dynamic version of the principle of virtual displacements) to derive the governing equations of motion.

8.2 CLASSICAL PLATE THEORY

8.2.1 Governing Equations of Circular Plates

Consider a circular plate on a linear elastic foundation, with foundation modulus k. Let r denote the radial coordinate outward from the center of the plate, z the thickness coordinate along the height of the plate, and θ the coordinate along a circumference

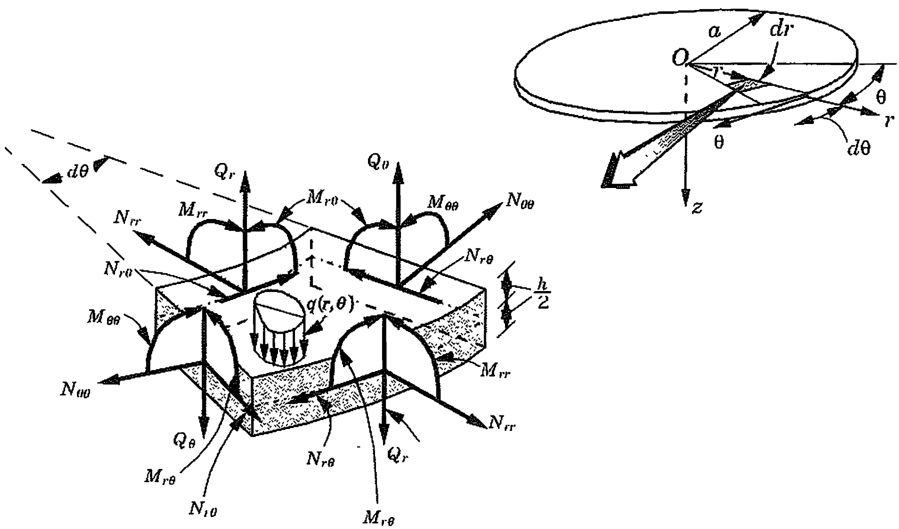


Figure 8.2 A circular plate with stress resultants.

of the plate, as shown in Fig. 8.2. In a general case where applied loads and boundary conditions are not axisymmetric, the displacements (u_r, u_θ, u_z) along the coordinates (r, θ, z) are functions of r, θ, and z coordinates.

We begin with the following displacement field implied by the Kirchhoff assumptions:

$$\begin{aligned} u_r(r,\theta,z,t) &= -z\frac{\partial w_0}{\partial r}, \\ u_\theta(r,\theta,z,t) &= -z\left(\frac{1}{r}\frac{\partial w_0}{\partial \theta}\right), \\ u_z(r,\theta,z,t) &= w_0(r,\theta,t), \end{aligned} \tag{8.13}$$

where w_0 is the transverse displacement at point (x, y) in the plate. The linear strain components referred to the cylindrical coordinate system are given by (see Exercise 3.34)

$$\begin{aligned} \varepsilon_{rr} &= \frac{\partial u_r}{\partial r}, & \varepsilon_{\theta\theta} &= \frac{u_r}{r} + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta}, \\ \varepsilon_{zz} &= \frac{\partial u_z}{\partial z}, & \varepsilon_{r\theta} &= \frac{1}{2}\left(\frac{1}{r}\frac{\partial u_r}{\partial \theta} + \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r}\right), \\ \varepsilon_{z\theta} &= \frac{1}{2}\left(\frac{\partial u_\theta}{\partial z} + \frac{1}{r}\frac{\partial u_z}{\partial \theta}\right), & \varepsilon_{rz} &= \frac{1}{2}\left(\frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r}\right). \end{aligned} \tag{8.14}$$

For the choice of the displacement field in Eq. (8.13), the only nonzero strains are

$$\varepsilon_{rr} = z\varepsilon_{rr}^{(1)}, \qquad \varepsilon_{\theta\theta} = z\varepsilon_{\theta\theta}^{(1)}, \qquad 2\varepsilon_{r\theta} = z\gamma_{r\theta}^{(1)}, \tag{8.15}$$

where

$$\begin{aligned} \varepsilon_{rr}^{(1)} &= -\frac{\partial^2 w_0}{\partial r^2}, \\ \varepsilon_{\theta\theta}^{(1)} &= -\frac{1}{r}\left(\frac{\partial w_0}{\partial r} + \frac{1}{r}\frac{\partial^2 w_0}{\partial \theta^2}\right), \\ \gamma_{r\theta}^{(1)} &= -\frac{2}{r}\left(\frac{\partial^2 w_0}{\partial r \partial \theta} - \frac{1}{r}\frac{\partial w_0}{\partial \theta}\right). \end{aligned} \tag{8.16}$$

Next we use Hamilton's principle

$$\int_{t_1}^{t_2} (\delta U + \delta V - \delta K)\, dt = 0, \tag{8.17}$$

to derive the Euler–Lagrange equations. The virtual strain energy δU is given by

$$\delta U = \int_{\Omega_0} \int_{-h/2}^{h/2} (\sigma_{rr}\delta\varepsilon_{rr} + \sigma_{\theta\theta}\delta\varepsilon_{\theta\theta} + \sigma_{r\theta}\delta\gamma_{r\theta})\, dz\, r dr d\theta$$
$$= \int_{\Omega_0} \left[-M_{rr}\frac{\partial^2 \delta w_0}{\partial r^2} - \frac{1}{r}M_{\theta\theta}\left(\frac{\partial \delta w_0}{\partial r} + \frac{1}{r}\frac{\partial^2 \delta w_0}{\partial \theta^2}\right) - \frac{2}{r}M_{r\theta}\left(\frac{\partial^2 \delta w_0}{\partial r \partial \theta} - \frac{1}{r}\frac{\partial \delta w_0}{\partial \theta}\right)\right] r\, dr d\theta, \tag{8.18}$$

where h is the plate thickness and $(M_{rr}, M_{r\theta}, M_{\theta\theta})$ are the moments per unit length:

$$M_{rr} = \int_{-h/2}^{h/2} \sigma_{rr} z\, dz, \qquad M_{\theta\theta} = \int_{-h/2}^{h/2} \sigma_{\theta\theta} z\, dz, \qquad M_{r\theta} = \int_{-h/2}^{h/2} \sigma_{r\theta} z\, dz. \tag{8.19}$$

The virtual total potential energy due to applied loads is calculated as follows. Suppose that $q = q(r, \theta)$ is the distributed transverse load, F_s is the reaction force of a linear elastic foundation, and $(\hat{N}_{rr}, \hat{N}_{\theta\theta}, \hat{N}_{r\theta})$ are the compressive and shear forces per unit length applied in the midplane of the plate. We have

$$\delta V = -\int_{\Omega_0} (q + F_s)\delta w_0\, r\, dr d\theta - \int_{\Omega_0} \left[\hat{N}_{rr}\frac{\partial w_0}{\partial r}\frac{\partial \delta w_0}{\partial r} + \hat{N}_{\theta\theta}\frac{1}{r}\frac{\partial w_0}{\partial \theta}\frac{1}{r}\frac{\partial \delta w_0}{\partial \theta} + \hat{N}_{r\theta}\frac{1}{r}\left(\frac{\partial w_0}{\partial \theta}\frac{\partial \delta w_0}{\partial r} + \frac{\partial w_0}{\partial r}\frac{\partial \delta w_0}{\partial \theta}\right)\right] r\, dr d\theta, \tag{8.20}$$

where the foundation force is given by $F_s = -kw_0$, and k is the modulus of the elastic foundation.

The virtual kinetic energy δK is

$$\delta K = \int_{\Omega_0} \int_{-h/2}^{h/2} \rho \left[z^2 \frac{\partial \dot{w}_0}{\partial r}\frac{\partial \delta \dot{w}_0}{\partial r} + \frac{z^2}{r}\frac{\partial \dot{w}_0}{\partial \theta}\frac{1}{r}\frac{\partial \delta \dot{w}_0}{\partial \theta} + \dot{w}_0 \delta \dot{w}_0 \right] r\, dr d\theta$$
$$= \int_{\Omega_0} \left[I_2 \left(\frac{\partial \dot{w}_0}{\partial r}\frac{\partial \delta \dot{w}_0}{\partial r} + \frac{1}{r^2}\frac{\partial \dot{w}_0}{\partial \theta}\frac{\partial \delta \dot{w}_0}{\partial \theta}\right) + I_0 \dot{w}_0 \delta \dot{w}_0 \right] r\, dr d\theta, \tag{8.21}$$

where I_0 is the principal mass inertia, and I_2 is the rotatory inertia:

$$I_0 = \int_{-h/2}^{h/2} \rho\, dz = \rho h, \qquad I_2 = \int_{-h/2}^{h/2} z^2 \rho\, dz = \frac{\rho h^3}{12}, \tag{8.22}$$

and the superposed dot on w_0 denotes differentiation with respect to t.

Next, substitute the expressions for δU, δV, and δK into Eq. (8.17), and carry out integration by parts with respect to r, θ, and t to relieve δw_0 of any derivatives. Note that $\delta w_0(r, \theta, t_1) = 0$ and $\delta w_0(r, \theta, t_2) = 0$. The mixed derivative term $\partial^2 \delta w_0 / \partial r \partial \theta$ in Eq. (8.18) requires special attention. Instead of integrating by parts with respect to r and then with respect to θ, or vice versa, it should be split into two terms so that the integration by parts is carried out in a symmetric way:

$$\int_{\Omega_0} \frac{2}{r} M_{r\theta} \frac{\partial^2 \delta w_0}{\partial r \partial \theta}\, r\, dr d\theta = \int_{\Omega_0} M_{r\theta} \left(\frac{\partial^2 \delta w_0}{\partial r \partial \theta} + \frac{\partial^2 \delta w_0}{\partial \theta \partial r} \right) dr d\theta.$$

We obtain

$$\begin{aligned}
0 = \int_{t_1}^{t_2} \int_{\Omega_0} \Bigg\{ & -\frac{1}{r} \left[\frac{\partial^2}{\partial r^2}(r M_{rr}) - \frac{\partial M_{\theta\theta}}{\partial r} + \frac{1}{r} \frac{\partial^2 M_{\theta\theta}}{\partial \theta^2} + 2 \frac{\partial^2 M_{r\theta}}{\partial r \partial \theta} + \frac{2}{r} \frac{\partial M_{r\theta}}{\partial \theta} \right] \\
& + \frac{1}{r} \left[\frac{\partial}{\partial r} \left(r \hat{N}_{rr} \frac{\partial w_0}{\partial r} \right) + \frac{\partial}{\partial \theta} \left(\hat{N}_{\theta\theta} \frac{1}{r} \frac{\partial w_0}{\partial \theta} \right) \right. \\
& \left. + \frac{\partial}{\partial r} \left(\hat{N}_{r\theta} \frac{\partial w_0}{\partial \theta} \right) + \frac{\partial}{\partial \theta} \left(\hat{N}_{r\theta} \frac{\partial w_0}{\partial r} \right) \right] - q + k w_0 \\
& + \frac{1}{r} \left[-I_2 \frac{\partial}{\partial r} \left(r \frac{\partial \ddot{w}_0}{\partial r} \right) - I_2 \frac{1}{r^2} \frac{\partial^2 \ddot{w}_0}{\partial \theta^2} + I_0 \ddot{w}_0 \right] \Bigg\} \delta w_0 r\, dr d\theta dt \\
- \int_{t_1}^{t_2} \oint_{\Gamma} \Bigg\{ & (r M_{rr} n_r + M_{r\theta} n_\theta) \frac{\partial \delta w_0}{\partial r} + \frac{1}{r} (r M_{r\theta} n_r + M_{\theta\theta} n_\theta) \frac{\partial \delta w_0}{\partial \theta} \\
& + \left[\frac{\partial}{\partial r}(r M_{rr}) + \frac{\partial M_{r\theta}}{\partial \theta} - M_{\theta\theta} \right] n_r + \left[\frac{1}{r} \frac{\partial M_{\theta\theta}}{\partial \theta} + \frac{\partial M_{r\theta}}{\partial r} + \frac{2}{r} M_{r\theta} \right] n_\theta \\
& - \frac{1}{r} \left(r \hat{N}_{rr} \frac{\partial w_0}{\partial r} + \hat{N}_{r\theta} \frac{\partial w_0}{\partial \theta} \right) n_r - \frac{1}{r} \left(\hat{N}_{\theta\theta} \frac{1}{r} \frac{\partial w_0}{\partial \theta} + \hat{N}_{r\theta} \frac{\partial w_0}{\partial r} \right) n_\theta \\
& + I_2 \left(\frac{\partial \ddot{w}_0}{\partial r} n_r + \frac{1}{r^2} \frac{\partial \ddot{w}_0}{\partial \theta} n_\theta \right) \Bigg\} \delta w_0\, r\, dr d\theta dt,
\end{aligned} \tag{8.23}$$

where (n_r, n_θ) are the direction cosines of unit normal $\hat{\mathbf{n}} = n_r \hat{\mathbf{e}}_r + n_\theta \hat{\mathbf{e}}_\theta$ to the boundary Γ of the domain (midplane) Ω_0 of the plate. Setting the coefficient of δw_0 to zero inside Ω_0, we obtain the Euler–Lagrange equation:

$$\begin{aligned}
& -\frac{1}{r} \left[\frac{\partial^2}{\partial r^2}(r M_{rr}) - \frac{\partial M_{\theta\theta}}{\partial r} + \frac{1}{r} \frac{\partial^2 M_{\theta\theta}}{\partial \theta^2} + 2 \frac{\partial^2 M_{r\theta}}{\partial r \partial \theta} + \frac{2}{r} \frac{\partial M_{r\theta}}{\partial \theta} \right] \\
& + \frac{1}{r} \left[\frac{\partial}{\partial r} \left(r \hat{N}_{rr} \frac{\partial w_0}{\partial r} \right) + \frac{\partial}{\partial \theta} \left(\hat{N}_{\theta\theta} \frac{1}{r} \frac{\partial w_0}{\partial \theta} \right) \right. \\
& \left. + \frac{\partial}{\partial r} \left(\hat{N}_{r\theta} \frac{\partial w_0}{\partial \theta} \right) + \frac{\partial}{\partial \theta} \left(\hat{N}_{r\theta} \frac{\partial w_0}{\partial r} \right) \right] - q + k w_0 \\
& + \frac{1}{r} \left[-I_2 \frac{\partial}{\partial r} \left(r \frac{\partial \ddot{w}_0}{\partial r} \right) - I_2 \frac{1}{r^2} \frac{\partial^2 \ddot{w}_0}{\partial \theta^2} + I_0 \ddot{w}_0 \right] = 0.
\end{aligned} \tag{8.24}$$

The natural boundary conditions follow from the boundary expressions in Eq. (8.23). Assuming that δw_0, $\partial \delta w_0/\partial r$, and $\partial \delta w_0/\partial \theta$ are arbitrary, which means that w_0, $\partial w_0/\partial r$, and $\partial w_0/\partial \theta$ are not specified on the boundary, we obtain

$$\delta w_0: \qquad r V_r n_r + V_\theta n_\theta = 0, \tag{8.25a}$$

$$\frac{\partial \delta w_0}{\partial r}: \qquad r M_{rr} n_r + M_{r\theta} n_\theta = 0, \tag{8.25b}$$

$$\frac{\partial \delta w_0}{\partial \theta}: \qquad r M_{r\theta} n_r + M_{\theta\theta} n_\theta = 0, \tag{8.25c}$$

where V_r and V_θ are the effective shear forces:

$$V_r = Q_r - \frac{1}{r}\left(r\hat{N}_{rr}\frac{\partial w_0}{\partial r} + \hat{N}_{r\theta}\frac{\partial w_0}{\partial \theta}\right) + I_2\frac{\partial \ddot{w}_0}{\partial r}, \tag{8.26a}$$

$$V_\theta = Q_\theta - \frac{1}{r}\left(\hat{N}_{\theta\theta}\frac{1}{r}\frac{\partial w_0}{\partial \theta} + \hat{N}_{r\theta}\frac{\partial w_0}{\partial r}\right) + I_2\frac{1}{r^2}\frac{\partial \ddot{w}_0}{\partial \theta}, \tag{8.26b}$$

$$Q_r \equiv \frac{1}{r}\left[\frac{\partial}{\partial r}(r M_{rr}) + \frac{\partial M_{r\theta}}{\partial \theta} - M_{\theta\theta}\right], \tag{8.26c}$$

$$Q_\theta \equiv \frac{1}{r}\left[\frac{\partial}{\partial r}(r M_{r\theta}) + \frac{\partial M_{\theta\theta}}{\partial \theta} + M_{r\theta}\right]. \tag{8.26d}$$

The equation of motion, Eq. (8.24), can be expressed, in view of the definitions (8.26c,d), as

$$-\frac{1}{r}\left[\frac{\partial}{\partial r}(r Q_r) + \frac{\partial Q_\theta}{\partial \theta}\right] + k w_0 + \frac{1}{r}\left[-I_2\frac{\partial}{\partial r}\left(r\frac{\partial \ddot{w}_0}{\partial r}\right) - I_2\frac{1}{r^2}\frac{\partial^2 \ddot{w}_0}{\partial \theta^2} + I_0\ddot{w}_0\right] = q$$
$$-\frac{1}{r}\left[\frac{\partial}{\partial r}\left(r\hat{N}_{rr}\frac{\partial w_0}{\partial r}\right) + \frac{\partial}{\partial \theta}\left(\hat{N}_{\theta\theta}\frac{1}{r}\frac{\partial w_0}{\partial \theta}\right) + \frac{\partial}{\partial r}\left(\hat{N}_{r\theta}\frac{\partial w_0}{\partial \theta}\right) + \frac{\partial}{\partial \theta}\left(\hat{N}_{r\theta}\frac{\partial w_0}{\partial r}\right)\right]. \tag{8.27}$$

For the axisymmetric case (i.e., when the material properties, load, and boundary conditions are independent of θ), Eq. (8.27) becomes

$$-\frac{1}{r}\frac{\partial}{\partial r}(r Q_r) + k w_0 + \frac{1}{r}\frac{\partial}{\partial r}\left(r\hat{N}_{rr}\frac{\partial w_0}{\partial r}\right) + \frac{1}{r}\left[-I_2\frac{\partial}{\partial r}\left(r\frac{\partial \ddot{w}_0}{\partial r}\right) + I_0\ddot{w}_0\right] = q, \tag{8.28}$$

for $b < r < a$, where a and b are the outer and inner radii, respectively, of an annular plate. For the solid circular plate, we set $b = 0$.

This completes the derivation of the governing equations of a circular plate. The equations are valid for any suitable constitutive behavior of the plate material.

In order to express the governing equations in terms of the displacement w_0, the bending moments in Eq. (8.19) must be expressed in terms of the displacement w_0 via stress–strain relations and strain–displacement relations. For an isotropic linear elastic material, assuming that the elastic stiffnesses are independent of the temperature, the stress–strain relations are [see Eqs. (3.45) and (3.41)]:

$$\begin{Bmatrix} \sigma_{rr} \\ \sigma_{\theta\theta} \\ \sigma_{r\theta} \end{Bmatrix} = \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix} \begin{Bmatrix} \varepsilon_{rr} - \alpha\,\Delta T \\ \varepsilon_{\theta\theta} - \alpha\,\Delta T \\ 2\varepsilon_{r\theta} \end{Bmatrix}, \tag{8.29}$$

where $\Delta T(r, \theta, z)$ is the temperature rise from a stress-free (or undeformed) state, α is the coefficient of thermal expansion, E is Young's modulus, and ν is Poisson's ratio. Then we find that

$$\begin{aligned} M_{rr} &= \int_{-h/2}^{h/2} \sigma_{rr} z\, dz = \frac{Eh^3}{12(1-\nu^2)} \left(\varepsilon_{rr}^{(1)} + \nu\varepsilon_{\theta\theta}^{(1)}\right) - M^T, \\ M_{\theta\theta} &= \int_{-h/2}^{h/2} \sigma_{\theta\theta} z\, dz = \frac{Eh^3}{12(1-\nu^2)} \left(\nu\varepsilon_{rr}^{(1)} + \varepsilon_{\theta\theta}^{(1)}\right) - M^T, \\ M_{r\theta} &= \int_{-h/2}^{h/2} \sigma_{r\theta} z\, dz = \frac{Gh^3}{12} \gamma_{r\theta}^{(1)}, \end{aligned} \tag{8.30}$$

where M^T is the thermal moment:

$$M^T = \frac{E\alpha}{1-\nu} \int_{-h/2}^{h/2} \Delta T z\, dz. \tag{8.31}$$

Now, the stress resultants can be expressed in terms of the deflection w_0 using the strain–displacement relations (8.16) as

$$M_{rr} = -D\left[\frac{\partial^2 w_0}{\partial r^2} + \frac{\nu}{r}\left(\frac{\partial w_0}{\partial r} + \frac{1}{r}\frac{\partial^2 w_0}{\partial \theta^2}\right)\right] - M^T, \tag{8.32a}$$

$$M_{\theta\theta} = -D\left[\nu\frac{\partial^2 w_0}{\partial r^2} + \frac{1}{r}\left(\frac{\partial w_0}{\partial r} + \frac{1}{r}\frac{\partial^2 w_0}{\partial \theta^2}\right)\right] - M^T, \tag{8.32b}$$

$$M_{r\theta} = -(1-\nu)D\frac{1}{r}\left(\frac{\partial^2 w_0}{\partial r \partial \theta} - \frac{1}{r}\frac{\partial w_0}{\partial \theta}\right). \tag{8.32c}$$

The bending stiffness D is given by

$$D = \frac{Eh^3}{12(1-\nu^2)}. \tag{8.33}$$

The equation of equilibrium (8.27) can now be written in terms of the deflection w_0 as

$$D\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right]\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial w_0}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 w_0}{\partial\theta^2}\right]+kw_0$$
$$+\frac{1}{r}\left[\frac{\partial}{\partial r}\left(r\hat{N}_{rr}\frac{\partial w_0}{\partial r}\right)+\frac{\partial}{\partial\theta}\left(\hat{N}_{\theta\theta}\frac{1}{r}\frac{\partial w_0}{\partial\theta}\right)+\frac{\partial}{\partial r}\left(\hat{N}_{r\theta}\frac{\partial w_0}{\partial\theta}\right)+\frac{\partial}{\partial\theta}\left(\hat{N}_{r\theta}\frac{\partial w_0}{\partial r}\right)\right]$$
$$+\frac{1}{r}\left[-I_2\frac{\partial}{\partial r}\left(r\frac{\partial\ddot{w}_0}{\partial r}\right)-I_2\frac{1}{r^2}\frac{\partial^2\ddot{w}_0}{\partial\theta^2}+I_0\ddot{w}_0\right]$$
$$=q-\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial M^T}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 M^T}{\partial\theta^2}\right]. \tag{8.34}$$

For the axisymmetric case, Eq. (8.34) simplifies to

$$\frac{D}{r}\frac{\partial}{\partial r}\left\{r\frac{\partial}{\partial r}\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial w_0}{\partial r}\right)\right]\right\}+kw_0+\frac{1}{r}\frac{\partial}{\partial r}\left(r\hat{N}_{rr}\frac{\partial w_0}{\partial r}\right)$$
$$+\frac{1}{r}\left[-I_2\frac{\partial}{\partial r}\left(r\frac{\partial\ddot{w}_0}{\partial r}\right)+I_0\ddot{w}_0\right]=q-\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial M^T}{\partial r}\right). \tag{8.35a}$$

For the static case, Eq. (8.35a) becomes

$$\frac{D}{r}\frac{d}{dr}\left\{r\frac{d}{dr}\left[\frac{1}{r}\frac{d}{dr}\left(r\frac{dw_0}{dr}\right)\right]\right\}+kw_0+\frac{1}{r}\frac{d}{dr}\left(r\hat{N}_{rr}\frac{dw_0}{dr}\right)=q-\frac{1}{r}\frac{d}{dr}\left(r\frac{dM^T}{dr}\right). \tag{8.35b}$$

The standard boundary conditions for axisymmetric bending of solid circular plates and annular plates (see Fig. 8.3) are listed in Table 8.1.

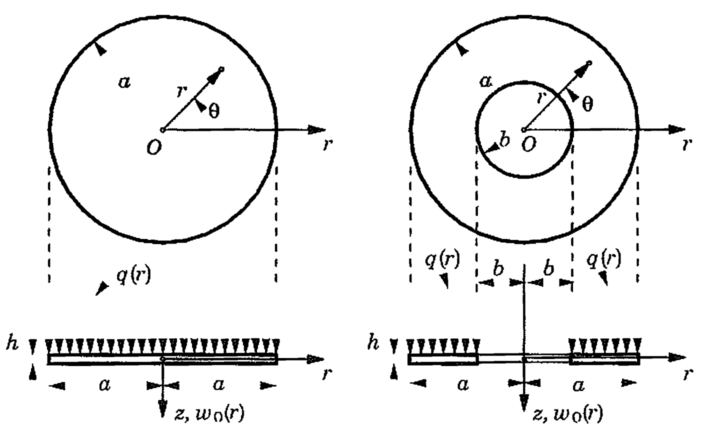


Figure 8.3 Axisymmetric circular and annular plates.

Table 8.1 Typical boundary conditions for solid circular and annular plates

Plate Type/Edge	Free	Hinged	Clamped
		Circular Plate	
$r = 0$ (for all cases)	$\frac{dw_0}{dr} = 0$	$2\pi\, r Q_r = -Q_0$	
$r = a$	$Q_r = Q_a$	$w_0 = 0$	$w_0 = 0$
	$M_{rr} = M_a$	$M_{rr} = M_a$	$\frac{dw_0}{dr} = 0$
		Annular Plate	
$r = b$	$Q_r = Q_b$	$w_0 = 0$	$w_0 = 0$
	$M_{rr} = M_b$	$M_{rr} = M_b$	$\frac{dw_0}{dr} = 0$
$r = a$	$Q_r = Q_a$	$w_0 = 0$	$w_0 = 0$
	$M_{rr} = M_a$	$M_{rr} = M_a$	$\frac{dw_0}{dr} = 0$

Q_a, Q_b, M_a, and M_b are distributed edge forces and moments and Q_0 is a concentrated force.

8.2.2 Analysis of Circular Plates

Analytical Solutions: Bending Here, we develop analytical solutions of isotropic circular plates for axisymmetric boundary conditions and loads (see Reddy [8] for additional details). For axisymmetric circular plates with $k = 0$, $M^T = 0$, and $\hat{N}_{rr} = 0$, Eq. (8.35b) simplifies to

$$\frac{D}{r}\frac{\partial}{\partial r}\left\{r\frac{\partial}{\partial r}\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial w_0}{\partial r}\right)\right]\right\} + \frac{1}{r}\left[-I_2\frac{\partial}{\partial r}\left(r\frac{\partial \ddot{w}_0}{\partial r}\right) + I_0\ddot{w}_0\right] = q. \qquad (8.36a)$$

For the static case, this further simplifies to

$$\frac{D}{r}\frac{d}{dr}\left\{r\frac{d}{dr}\left[\frac{1}{r}\frac{d}{dr}\left(r\frac{dw_0}{dr}\right)\right]\right\} = q, \qquad (8.36b)$$

and the stresses, bending moments, and shear force are related to the deflection by

$$\sigma_{rr} = -\frac{Ez}{(1-\nu^2)}\left(\frac{d^2 w_0}{dr^2} + \frac{\nu}{r}\frac{dw_0}{dr}\right), \qquad (8.37a)$$

$$\sigma_{\theta\theta} = -\frac{Ez}{(1-\nu^2)}\left(\nu\frac{d^2 w_0}{dr^2} + \frac{1}{r}\frac{dw_0}{dr}\right), \qquad (8.37b)$$

$$M_{rr} = -D\left(\frac{d^2 w_0}{dr^2} + \frac{\nu}{r}\frac{dw_0}{dr}\right), \qquad (8.38a)$$

$$M_{\theta\theta} = -D\left(\nu\frac{d^2 w_0}{dr^2} + \frac{1}{r}\frac{dw_0}{dr}\right), \qquad (8.38b)$$

$$Q_r = -D\frac{d}{dr}\left[\frac{1}{r}\frac{d}{dr}\left(r\frac{dw_0}{dr}\right)\right]. \qquad (8.38c)$$

The general solution of Eq. (8.36b) can be obtained by successive integrations [see Eq. (8.28)]:

$$-rQ(r) = \int rq(r)\,dr + c_1, \tag{8.39}$$

$$D\frac{d}{dr}\left(r\frac{dw_0}{dr}\right) = r\int\frac{1}{r}\int rq(r)\,dr dr + c_1 r\log r + c_2 r, \tag{8.40}$$

$$D\frac{dw_0}{dr} = F'(r) + \frac{r}{4}(2\log r - 1)c_1 + \frac{r}{2}c_2 + \frac{1}{r}c_3, \tag{8.41}$$

$$Dw_0(r) = F(r) + \frac{r^2}{4}(\log r - 1)c_1 + \frac{r^2}{4}c_2 + c_3\log r + c_4, \tag{8.42}$$

where

$$F(r) = \int\frac{1}{r}\int r\int\frac{1}{r}\int rq(r)\,dr dr dr dr, \qquad F'(r) = \frac{1}{r}\int r\int\frac{1}{r}\int rq(r)\,dr dr dr, \tag{8.43}$$

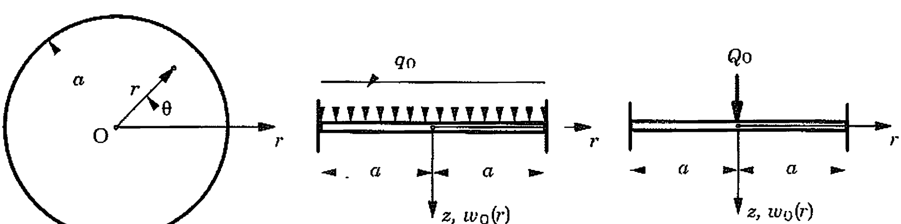
and c_i $(i = 1, 2, 3, 4)$ are constants of integration that will be evaluated using boundary conditions. Note that

$$D\frac{d^2w_0}{dr^2} = F''(r) + \frac{1}{4}(2\log r + 1)c_1 + \frac{1}{2}c_2 - \frac{1}{r^2}c_3. \tag{8.44}$$

For solid circular plates, the requirement that the slope be zero due to symmetry at $r = 0$ yields $c_3 = 0$ [see Eq. (8.41)]. Further, if there is no point load at $r = 0$, the shear force is zero there, giving $c_1 = 0$.

Example 8.1 Consider a clamped circular plate under distributed load. For this case, $c_1 = c_3 = 0$. The boundary conditions associated with the clamped outer edge are $w_0 = 0$ and $dw_0/dr = 0$ at $r = a$ (see Fig. 8.4). These conditions yield

$$c_2 = -\frac{2F'(a)}{a}, \qquad c_4 = -F(a) + \frac{F'(a)a}{2}.$$



(a) (b)

Figure 8.4 A clamped circular plate under (a) a uniformly distributed load and (b) a point load.

For uniformly distributed load q_0, wc have

$$c_2 = -\frac{q_0 a^2}{8}, \qquad c_4 = \frac{q_0 a^4}{64}.$$

Hence the deflection becomes

$$w_0(r) = \frac{q_0 a^4}{64D}\left(1 - \frac{r^2}{a^2}\right)^2, \tag{8.45}$$

and the maximum deflection occurs at the center of the plate ($r = 0$):

$$w_{max} = \frac{q_0 a^4}{64D}. \tag{8.46}$$

Expressions for the bending moments and stresses become

$$M_{rr}(r) = \frac{q_0 a^2}{16}\left[(1+\nu) - (3+\nu)\frac{r^2}{a^2}\right], \tag{8.47a}$$

$$M_{\theta\theta}(r) = \frac{q_0 a^2}{16}\left[(1+\nu) - (1+3\nu)\frac{r^2}{a^2}\right], \tag{8.47b}$$

$$\sigma_{rr}(r, z) = \frac{3q_0 a^2 z}{4h^3}\left[(1+\nu) - (3+\nu)\frac{r^2}{a^2}\right], \tag{8.47c}$$

$$\sigma_{\theta\theta}(r, z) = \frac{3q_0 a^2 z}{4h^3}\left[(1+\nu) - (1+3\nu)\frac{r^2}{a^2}\right]. \tag{8.47d}$$

Thc maximum (in magnitude) values of the bending moments are found to be at the fixed edge:

$$M_{rr}(a) = -\frac{q_0 a^2}{8}, \qquad M_{\theta\theta}(a) = -\frac{\nu q_0 a^2}{8}. \tag{8.48}$$

Hence, the maximum stress is given by

$$\sigma_{rr}\left(a, -\frac{h}{2}\right) = -\frac{6M_{rr}(a)}{h^2} = \frac{3q_0}{4}\left(\frac{a}{h}\right)^2. \tag{8.49}$$

For a solid circular plate with a point load Q_0 at the center, we require $2\pi(rQ_r)_0 = -Q_0$. From Eq. (8.39) we obtain

$$c_1 = \frac{Q_0}{2\pi}.$$

The boundary conditions of the clamped edgc at $r = a$ give

$$c_2 = -\frac{Q_0}{4\pi}(2\log a - 1), \qquad c_4 = \frac{Q_0 a^2}{16\pi}.$$

Hence, the solution becomes

$$w_0(r) = \frac{Q_0 a^2}{16\pi D}\left[1 - \frac{r^2}{a^2} + 2\frac{r^2}{a^2}\log\left(\frac{r}{a}\right)\right], \tag{8.50}$$

$$M_{rr}(r) = -\frac{Q_0}{4\pi}\left[1 + (1+\nu)\log\left(\frac{r}{a}\right)\right], \tag{8.51a}$$

$$M_{\theta\theta}(r) = -\frac{Q_0}{4\pi}\left[\nu + (1+\nu)\log\left(\frac{r}{a}\right)\right]. \tag{8.51b}$$

The maximum deflection occurs at $r = 0$ and is given by

$$w_{max} = \frac{Q_0 a^2}{16\pi D}. \tag{8.52}$$

Example 8.2 Consider a simply supported annular plate under uniformly distributed load of intensity q_0. The boundary conditions are

$$\text{At } r = b: \quad M_{rr} = 0, \quad (rQ_r) = 0, \tag{8.53a}$$

$$\text{At } r = a: \quad w_0 = 0, \quad M_{rr} = 0. \tag{8.53b}$$

These conditions give $c_1 = -q_0 b^2/2$ and

$$c_2 = -\left(\frac{1+3\nu}{1+\nu}\right)\frac{q_0 b^2}{8} - \left(\frac{3+\nu}{1+\nu}\right)\frac{q_0 a^2}{8} + \frac{q_0 b^2}{2}\log a - \frac{q_0 b^4}{2(a^2-b^2)}\log\beta,$$

$$c_3 = -\left(\frac{3+\nu}{1-\nu}\right)\frac{q_0 b^2 a^2}{16} - \left(\frac{1+\nu}{1-\nu}\right)\frac{q_0 a^2 b^4}{4(a^2-b^2)}\log\beta,$$

$$c_4 = -\frac{q_0 a^4}{64} + \left(\frac{3+\nu}{1+\nu}\right)\frac{q_0 a^2(a^2-b^2)}{32} + \left(\frac{3+\nu}{1-\nu}\right)\frac{q_0 b^2 a^2}{16}\log a$$
$$+ \frac{q_0 b^4 a^2}{8(a^2-b^2)}\log\beta + \left(\frac{1+\nu}{1-\nu}\right)\frac{q_0 b^4 a^2}{4(a^2-b^2)}\log a \log\beta.$$

The deflection and bending moments become

$$w_0 = \frac{q_0 a^4}{64D}\left\{-1 + \left(\frac{r}{a}\right)^4 + \frac{2\alpha_1}{1+\nu}\left[1 - \left(\frac{r}{a}\right)^2\right] - \frac{4\alpha_2\beta^2}{1-\nu}\log\left(\frac{r}{a}\right)\right\}, \tag{8.54}$$

$$M_{rr} = \frac{q_0 a^2}{16}\left\{(3+\nu)\left[1 - \left(\frac{r}{a}\right)^2\right] - \beta^2(3+\nu)\left[1 + \left(\frac{r}{a}\right)^2\right]\right.$$
$$\left. + 4(1+\nu)\beta^2\kappa\left[1 - \left(\frac{r}{a}\right)^2\right] + 4(1+\nu)\beta^2\log\left(\frac{r}{a}\right)\right\}, \tag{8.55a}$$

$$M_{\theta\theta} = \frac{q_0 a^2}{16}\left\{(3+\nu)\left[1-\left(\frac{r}{a}\right)^2\right] + \beta^2\left[(5\nu - 1) + (3+\nu)\left(\frac{r}{a}\right)^2\right] + 4(1+\nu)\beta^2\kappa\left[1+\left(\frac{r}{a}\right)^2\right]\right\}, \tag{8.55b}$$

where

$$\alpha_1 = (3+\nu)(1-\beta^2) - 4(1+\nu)\beta^2\kappa, \qquad \alpha_2 = (3+\nu) + 4(1+\nu)\kappa,$$

$$\kappa = \frac{\beta^2}{1-\beta^2}\log\beta, \qquad \beta = \frac{b}{a}.$$

For a simply supported solid circular plate under uniform load, we set $b = 0$ or $\beta = 0$ (hence, $c_1 = c_3 = 0$) in Eqs. (8.54)–(8.55a,b) and obtain

$$w_0 = \frac{q_0 a^4}{64D}\left[\left(\frac{r}{a}\right)^4 - 2\frac{3+\nu}{1+\nu}\left(\frac{r}{a}\right)^2 + \frac{5+\nu}{1+\nu}\right], \tag{8.56}$$

$$M_{rr} = \frac{q_0 a^2}{16}(3+\nu)\left[1-\left(\frac{r}{a}\right)^2\right], \tag{8.57a}$$

$$M_{\theta\theta} = \frac{q_0 a^2}{16}\left[(3+\nu) - (1+3\nu)\left(\frac{r}{a}\right)^2\right]. \tag{8.57b}$$

The maximum deflection, bending moments, and stresses occur at $r = 0$, and they are

$$w_{max} = \left(\frac{5+\nu}{1+\nu}\right)\frac{q_0 a^4}{64D}, \qquad M_{max} = (3+\nu)\frac{q_0 a^2}{16}, \qquad \sigma_{max} = 3(3+\nu)\frac{q_0 a^2}{8h^2}. \tag{8.58}$$

Finally, for a simply supported solid circular plate with a central point load Q_0, we have $c_1 = Q_0/2\pi$ and $c_3 = 0$. Using the boundary conditions $w_0(a) = 0$ and $M_{rr}(a) = 0$ gives

$$c_2 = -\frac{Q_0}{2\pi}\left[\log a + \frac{1}{2}\left(\frac{1-\nu}{1+\nu}\right)\right], \qquad c_4 = \left(\frac{3+\nu}{1+\nu}\right)\frac{Q_0 a^2}{16\pi}.$$

The solution for any $r \neq 0$ is given by

$$w_0(r) = \frac{Q_0 a^2}{16\pi D}\left[\left(\frac{3+\nu}{1+\nu}\right)\left(1-\frac{r^2}{a^2}\right) + 2\left(\frac{r}{a}\right)^2\log\left(\frac{r}{a}\right)\right], \tag{8.59}$$

$$\sigma_{rr}(r) = -\frac{3zQ_0(1+\nu)}{h^3\pi}\log\left(\frac{r}{a}\right), \tag{8.60a}$$

$$\sigma_{\theta\theta}(r) = -\frac{3zQ_0}{h^3\pi}\left[(1+\nu)\log\left(\frac{r}{a}\right) - (1-\nu)\right], \tag{8.60b}$$

$$M_{rr}(r) = -\frac{Q_0(1+\nu)}{4\pi}\log\left(\frac{r}{a}\right), \tag{8.61a}$$

$$M_{\theta\theta}(r) = -\frac{Q_0}{4\pi}\left[(1+\nu)\log\left(\frac{r}{a}\right) - (1-\nu)\right]. \tag{8.61b}$$

The maximum deflection is given by

$$w_{max} = w_0(0) = \frac{Q_0 a^2}{16\pi D}\left(\frac{3+\nu}{1+\nu}\right). \tag{8.62}$$

The stresses and bending moments cannot be calculated at $r = 0$ due to the logarithmic singularity. The maximum finite stresses produced by load Q_0 on a very small circular area of radius r_c can be calculated using the so-called equivalent radius r_e in Eqs. (8.60a,b) and (8.61a,b) (see Roark and Young [23]):

$$r_e = \sqrt{1.6r_c^2 + h^2} - 0.675h \qquad \text{when } r_c < 1.7h, \tag{8.63a}$$

$$r_e = r_c \qquad \text{when } r_c \geq 1.7h. \tag{8.63b}$$

Analytical Solutions: Buckling Here we consider exact buckling solutions of circular plates. Consider the shear force–deflection relation (8.38c):

$$Q_r = -D\left[\frac{1}{r}\frac{d}{dr}\left(r\frac{d^2w_0}{dr^2}\right) - \frac{1}{r^2}\frac{dw_0}{dr}\right]. \tag{8.64a}$$

We write the equation in terms of $\phi = (dw_0/dr)$, which represents the angle between the central axis of the plate and the normal to the deflected surface at any point (see Fig. 8.5), as

$$Q_r = -D\left[\frac{1}{r}\frac{d}{dr}\left(r\frac{d\phi}{dr}\right) - \frac{\phi}{r^2}\right]. \tag{8.64b}$$

For a circular plate under the action of uniform radial compressive force $\hat{N}_{rr} = N_0$ per unit length, the shear force at any point is given by [an integration of Eq. (8.28), for the static case with $q = k = 0$, gives the result, because $Q_r = 0$ at $r = 0$ for axisymmetric mode]:

$$Q_r = \hat{N}_{rr}\frac{dw_0}{dr} = \hat{N}_{rr}\phi. \tag{8.64c}$$

From Eqs. (8.64b,c) we obtain

$$-D\left[\frac{1}{r}\frac{d}{dr}\left(r\frac{d\phi}{dr}\right) - \frac{1}{r^2}\phi\right] = \hat{N}_{rr}\phi$$

or

$$r\frac{d}{dr}\left(r\frac{d\phi}{dr}\right) + \left(\frac{\hat{N}_{rr}}{D}r^2 - 1\right)\phi = 0. \tag{8.65}$$

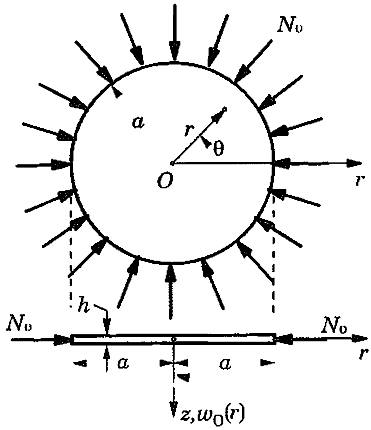


Figure 8.5 Buckling of a circular plate under uniform radial compressive load.

Equation (8.65) can be recast in an alternative form by invoking the transformation

$$\bar{r} = r\alpha, \qquad \alpha^2 = \frac{\hat{N}_{rr}}{D}, \tag{8.66}$$

and the alternative form of Eq. (8.65) is (with $n = 1$)

$$\bar{r}\frac{d}{d\bar{r}}\left(\bar{r}\frac{d\phi}{d\bar{r}}\right) + (\bar{r}^2 - n^2)\phi = 0, \tag{8.67}$$

which is recognized as the *Bessel differential equation of order n*.

The general solution of Eq. (8.67) is

$$\phi(\bar{r}) = AJ_n(\bar{r}) + BY_n(\bar{r}), \tag{8.68}$$

where J_n is the Bessel function of the first kind of order n, Y_n is the Bessel function of the second kind of order n, and A and B are constants to be determined using the boundary conditions. In the case of buckling, we do not actually find these constants but determine the stability criterion.

Example 8.3

Clamped Plate For a clamped plate, the boundary conditions are (dw_0/dr is zero at $r = 0, a$)

$$\phi(0) = 0, \qquad \phi(a) = 0. \tag{8.69}$$

Using the general solution (8.68) for an isotropic plate, we obtain

$$AJ_1(0) + BY_1(0) = 0, \qquad AJ_1(\alpha a) + BY_1(\alpha a) = 0, \qquad \alpha^2 = \frac{\hat{N}_{rr}}{D}.$$

Since $J_1(0) = 0$ and $Y_1(0)$ is unbounded, we must have $B = 0$. The fact that $B = 0$ reduces the second equation (since $A \neq 0$ for a nontrivial solution) to

$$J_1(\alpha a) = 0, \tag{8.70}$$

which is the stability criterion. The smallest root of Eq. (8.70) is $\alpha a = 3.8317$. Thus we have

$$\hat{N}_{\text{cr}} = 14.682\frac{D}{a^2}. \tag{8.71}$$

Simply Supported Plate For a simply supported plate, the boundary conditions are ($dw_0/dr = 0$ at $r = 0$ and $M_{rr} = 0$ at $r = a$)

$$\phi(0) = 0, \qquad \left[\frac{d\phi}{dr} + \nu\frac{1}{r}\phi\right]_{r=a} = 0. \tag{8.72a}$$

The second boundary condition can be written in terms of $\bar{r}$ as

$$\left[\frac{d\phi}{d\bar{r}} + \nu\frac{1}{\bar{r}}\phi\right]_{\bar{r}=\alpha a} = 0. \tag{8.72b}$$

Using the general solution (8.68), we obtain

$$AJ_1(0) + BY_1(0) = 0, \qquad AJ_1'(\alpha a) + BY_1'(\alpha a) + \frac{\nu}{\alpha a}\left[AJ_1(\alpha a) + BY_1(\alpha a)\right] = 0.$$

The first equation gives $B = 0$, and the second equation, in view of $B = 0$ and the identity

$$\frac{dJ_n}{d\bar{r}} = J_{n-1}(\bar{r}) - \frac{1}{\bar{r}}J_n(\bar{r}), \tag{8.73}$$

gives

$$\alpha a J_0'(\alpha a) - (1 - \nu)J_1(\alpha a) = 0, \tag{8.74}$$

which is the stability condition for the simply supported plate. For $\nu = 0.3$, the smallest root of the transcendental equation (8.74) is $\alpha a = 2.05$. Hence the buckling load becomes

$$\hat{N}_{cr} = 4.198\frac{D}{a^2}. \tag{8.75}$$

Simply Supported Plate with Rotational Restraint For a simply supported plate with rotational restraint (see Fig. 8.6), the boundary conditions are

$$\phi(0) = 0, \qquad D\left[r\frac{d\phi}{dr} + \nu\phi\right]_{r=a} + k_R a\phi(a) = 0, \tag{8.76}$$

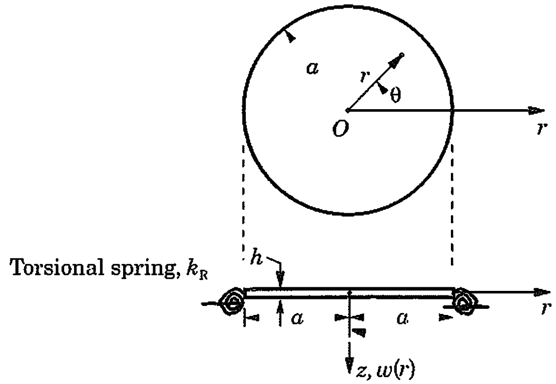


Figure 8.6 Buckling of a rotationally restrained circular plate under uniform radial compressive load.

Table 8.2 Buckling load factors $\bar{N} = \hat{N}_{cr}(a^2/D)$ for rotationally restrained circular plates

$\beta \rightarrow$	0	0.1	0.5	1	5	10	100	∞
$\bar{N} \rightarrow$	4.198	4.449	5.369	6.353	10.462	12.173	14.392	14.682

where k_R denotes the rotational spring constant. We obtain

$$\alpha a J_0'(\alpha a) - (1 - \nu - \beta) J_1(\alpha a) = 0, \qquad \beta = \frac{a k_R}{D}, \tag{8.77}$$

as the stability condition. When $\beta = 0$, we obtain Eq. (8.74), and when $\beta = \infty$, we obtain Eq. (8.70) as special cases. Table 8.2 contains numerical values of the buckling load factor $\bar{N} = \hat{N}_{cr}(a^2/D)$ for various values of the parameter, β (for $\nu = 0.3$).

Variational Solutions: Bending Next, we consider the Ritz and Galerkin solutions for axisymmetric bending of circular plates, possibly on an elastic foundation. Toward this end, we first write the variational statement (i.e., the weak form) needed for the Ritz method. From the virtual work statement in Eq. (8.17), with $(\delta U, \delta V)$ from Eqs. (8.18) and (8.20), $\delta K = 0$, and omitting terms involving derivatives with respect to θ, we have

$$0 = -2\pi \int_b^a \left(M_{rr} \frac{d^2 \delta w_0}{dr^2} + M_{\theta\theta} \frac{1}{r} \frac{d\delta w_0}{dr} - k w_0 \delta w_0 + q \delta w_0 - \hat{N}_{rr} \frac{dw_0}{dr} \frac{d\delta w_0}{dr} \right) r\, dr$$
$$+ 2\pi \left[a \bar{Q}_a \delta w_0(a) - b \bar{Q}_b \delta w_0(b) - a M_a \left(\frac{d\delta w_0}{dr} \right)_a + b M_b \left(\frac{d\delta w_0}{dr} \right)_b \right]$$

$$= 2\pi \int_b^a \left\{ D \left[\left(\frac{d^2 w_0}{dr^2} + \frac{\nu}{r} \frac{dw_0}{dr} \right) \frac{d^2 \delta w_0}{dr^2} + \left(\nu \frac{d^2 w_0}{dr^2} + \frac{1}{r} \frac{dw_0}{dr} \right) \frac{1}{r} \frac{d\delta w_0}{dr} \right] \right.$$
$$\left. + k w_0 \delta w_0 - q \delta w_0 - \hat{N}_{rr} \frac{dw_0}{dr} \frac{d\delta w_0}{dr} \right\} r\, dr$$
$$+ 2\pi \left[a \bar{Q}_a \delta w_0(a) - b \bar{Q}_b \delta w_0(b) \right]$$
$$+ 2\pi \left[-a M_a \left(\frac{d\delta w_0}{dr} \right)_a + b M_b \left(\frac{d\delta w_0}{dr} \right)_b \right], \tag{8.78a}$$

where a and b denote the outer and inner radii of an annular plate, k is the spring constant of the linear elastic foundation, D the bending stiffness, q the distributed load, $\bar{Q}_a$ and $\bar{Q}_b$ the intensities of effective line loads (including in-plane compressive force) at the outer and inner edges, respectively:

$$\bar{Q}_a = \left[Q - \hat{N}_{rr} \frac{dw_0}{dr} \right]_{r=a}, \qquad \bar{Q}_b = \left[Q - \hat{N}_{rr} \frac{dw_0}{dr} \right]_{r=b}, \tag{8.78b}$$

and M_a and M_b the distributed edge moments at the outer and inner edges, respectively. When $b = 0$ (for a solid circular plate), we have $M_b = 0$ and $2\pi b Q_b = Q_0$, Q_0 being the applied point load at the center of the plate. Equation (8.78a) is the weak form used in the Ritz method.

The weak form (8.78a) should be modified if the edge $r = a$ of the plate is elastically restrained (see Fig. 8.7):

$$\left(r \bar{Q}_r \right)_{r=a} + k_E w_0(a) = 0, \qquad (-r M_{rr})_{r=a} + k_R \left(\frac{dw_0}{dr} \right)_{r=a} = 0, \tag{8.79}$$

where k_E and k_R are spring constants associated with extensional and rotational springs, respectively. One can simulate the simply supported boundary condition ($k_E = \infty$ and $k_R = 0$), the clamped boundary condition ($k_E = \infty$ and $k_R = \infty$),

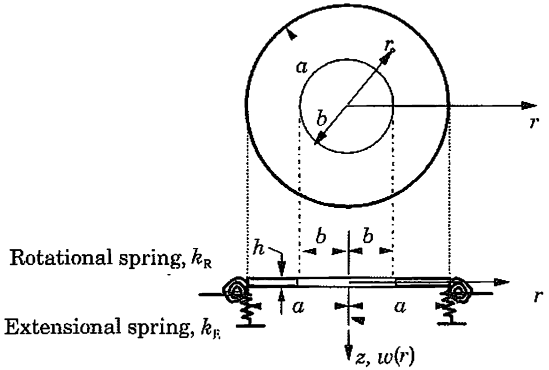


Figure 8.7 An elastically restrained annular plate.

and free edge condition ($k_E = 0$ and $k_R = 0$). Of course, similar expressions can be written for the edge at $r = a$. The weak form (8.78a) takes the form

$$0 = 2\pi \int_b^a \left\{ D \left[\left(\frac{d^2 w_0}{dr^2} + \frac{\nu}{r}\frac{dw_0}{dr} \right) \frac{d^2 \delta w_0}{dr^2} + \left(\nu \frac{d^2 w_0}{dr^2} + \frac{1}{r}\frac{dw_0}{dr} \right) \frac{1}{r}\frac{d\delta w_0}{dr} \right] \right.$$
$$\left. + k w_0 \delta w_0 - q \delta w_0 - \hat{N}_{rr} \frac{dw_0}{dr}\frac{d\delta w_0}{dr} \right\} r\, dr$$
$$+ 2\pi \left[k_E \delta w_0(a) w_0(a) - b \bar{Q}_b \delta w_0(b) \right]$$
$$+ 2\pi \left[k_R \left(\frac{d\delta w_0}{dr}\frac{dw_0}{dr} \right)_a + b M_b \left(\frac{d\delta w_0}{dr} \right)_b \right], \tag{8.80}$$

Assume an N-parameter solution of the form (assuming that all specified geometric boundary conditions are homogeneous so that we can take $\phi_0(r) = 0$):

$$w_0(r) \approx W_N(r) = \sum_{j=1}^{N} c_j \phi_j(r). \tag{8.81}$$

The approximation functions ϕ_i for the Ritz method must be continuous as required by the weak form, linearly independent, and satisfy the homogeneous form of specified essential boundary conditions.

Substituting (8.81) into Eq. (8.80), we obtain

$$\sum_{j=1}^{N} a_{ij} c_j - b_i = 0 \; (i = 1, 2, \ldots, N); \qquad \text{or} \qquad [A]\{c\} = \{b\}, \tag{8.82a}$$

where

$$a_{ij} = 2\pi D \int_b^a \left[\frac{d^2\phi_i}{dr^2}\frac{d^2\phi_j}{dr^2} + \frac{\nu}{r}\left(\frac{d\phi_i}{dr}\frac{d^2\phi_j}{dr^2} + \frac{d^2\phi_i}{dr^2}\frac{d\phi_j}{dr} \right) + \frac{1}{r^2}\frac{d\phi_i}{dr}\frac{d\phi_j}{dr} \right] r\, dr$$
$$+ 2k\pi \int_b^a \phi_i \phi_j \, r\, dr - 2\pi \int_b^a \hat{N}_{rr} \frac{d\phi_i}{dr}\frac{d\phi_j}{dr} \, r\, dr, \tag{8.82b}$$
$$b_i = 2\pi \int_b^a q\phi_i \, r\, dr - 2\pi \left[a\bar{Q}_a \phi_i(a) - b\bar{Q}_b \phi_i(b) - aM_a \left(\frac{d\phi_i}{dr} \right)_a + bM_b \left(\frac{d\phi_i}{dr} \right)_b \right]. \tag{8.82c}$$

Similar expressions can be obtained using the variational statement (8.80). Once ϕ_i are selected, a_{ij} and b_i can be computed by evaluating the integrals in Eqs. (8.82b,c), and Eq. (8.82a) can be solved for c_i ($i = 1, 2, \ldots, N$). Then, the solution is given by Eq. (8.81).

For a weighted-residual method, we use the weighted-integral statement

$$0 = 2\pi \int_b^a \psi_i(r) \left[\frac{D}{r} \frac{d}{dr} \left\{ r \frac{d}{dr} \left[\frac{1}{r} \frac{d}{dr} \left(r \frac{dw_0}{dr} \right) \right] \right\} + kw_0 \right.$$
$$\left. + \frac{1}{r} \frac{d}{dr} \left(r \hat{N}_{rr} \frac{dw_0}{dr} \right) - q \right] r\, dr, \tag{8.83}$$

where $\psi_i(r)$ is the weight function that takes different forms depending on the method used. Substituting the N-parameter approximation of the form

$$w_0(r) \approx W_N(r) = \sum_{j=1}^{N} c_j \phi_j(r) + \phi_0(r) \tag{8.84}$$

into Eq. (8.83), we obtain Eq. (8.82a) with

$$a_{ij} = 2\pi D \int_b^a \psi_i \left[\frac{D}{r} \frac{d}{dr} \left\{ r \frac{d}{dr} \left[\frac{1}{r} \frac{d}{dr} \left(r \frac{d\phi_j}{dr} \right) \right] \right\} + k\phi_j \right.$$
$$\left. + \frac{1}{r} \frac{d}{dr} \left(r \hat{N}_{rr} \frac{d\phi_j}{dr} \right) \right] r\, dr, \tag{8.85a}$$

$$b_i = 2\pi \int_b^a q \psi_i\, r\, dr - 2\pi D \int_b^a \psi_i \left[\frac{D}{r} \frac{d}{dr} \left\{ r \frac{d}{dr} \left[\frac{1}{r} \frac{d}{dr} \left(r \frac{d\phi_0}{dr} \right) \right] \right\} \right.$$
$$\left. + \frac{1}{r} \frac{d}{dr} \left(r \hat{N}_{rr} \frac{d\phi_0}{dr} \right) + k\phi_0 \right] r\, dr. \tag{8.85b}$$

The approximation functions ϕ_i must be continuous as required by the weighted-integral statement (8.83), linearly independent, and satisfy the homogeneous form of all specified boundary conditions, while ϕ_0 satisfy all specified boundary conditions.

Example 8.4 As a specific example, we consider a simply supported solid ($b = 0$) circular plate under uniformly distributed load of intensity q_0, as shown in Fig. 8.8. Note that $\hat{N}_{rr} = 0$ here.

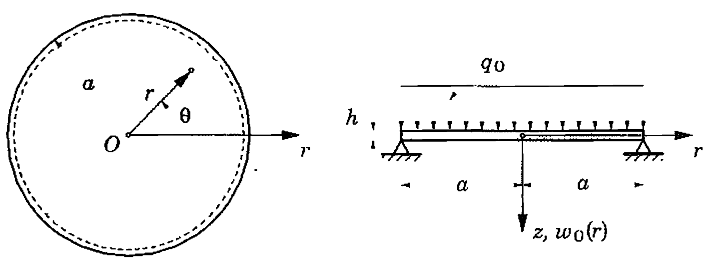


Figure 8.8 A simply supported solid circular plate under uniformly distributed load.

Recall that for this problem the essential boundary conditions are

$$\frac{dw_0}{dr}(0) = 0, \qquad w_0(a) = 0,$$

and the natural boundary conditions are given by

$$M_{rr} = 0 \text{ at } r = a, \qquad Q_r = \frac{1}{r}\left[\frac{d}{dr}(rM_{rr}) - M_{\theta\theta}\right] = 0 \text{ at } r = 0.$$

The natural boundary conditions will have no bearing on the selection of ϕ_i in the Ritz method. Each ϕ_i must satisfy the homogeneous form of the geometric boundary conditions

$$\frac{d\phi_i}{dr}(0) = 0, \qquad \phi_i(a) = 0.$$

For the choice of algebraic polynomials, we assume

$$\phi_1(r) = \alpha_1 + \alpha_2 r + \alpha_3 r^2,$$

and determine, using the above conditions, that $\alpha_2 = 0$ and $\alpha_1 + \alpha_3 a^2 = 0$. Thus, we have (for the choice of $\alpha_1 = 1$)

$$\phi_1 = 1 - \frac{r^2}{a^2}.$$

The procedure can be used to obtain a linearly independent and complete set of functions:

$$\phi_1 = 1 - \frac{r^2}{a^2}, \qquad \phi_2 = 1 - \frac{r^3}{a^3}, \ldots, \qquad \phi_j = 1 - \left(\frac{r}{a}\right)^{j+1}. \tag{8.86}$$

Substituting for ϕ_j from the above approximation into Eqs. (7.82b,c), and noting that $k = 0$, we obtain

$$\begin{aligned} a_{ij} &= \frac{2\pi D}{a^2}\left(\frac{ij+1}{i+j} + \nu\right)(i+1)(j+1), \\ b_i &= 2\pi q_0 a^2 \frac{(i+1)}{2(i+3)}. \end{aligned} \tag{8.87}$$

For $N = 3$ we have the algebraic equations

$$\frac{D}{a^2}\begin{bmatrix} 4(1+\nu) & 6(1+\nu) & 8(1+\nu) \\ 6(1+\nu) & 9(1.25+\nu) & 12(1.4+\nu) \\ 8(1+\nu) & 12(1.4+\nu) & 16(\frac{5}{3}+\nu) \end{bmatrix}\begin{Bmatrix} c_1 \\ c_2 \\ c_3 \end{Bmatrix} = \frac{q_0 a^2}{60}\begin{Bmatrix} 15 \\ 18 \\ 20 \end{Bmatrix}. \tag{8.88}$$

The one-, two- and three-parameter Ritz solutions are

$$W_1(r) = \frac{q_0 a^4}{16D(1+\nu)}\left(1 - \frac{r^2}{a^2}\right),$$

$$W_2(r) = \frac{q_0 a^4}{80D}\left(\frac{9+4\nu}{1+\nu}\right)\left(1 - \frac{r^2}{a^2}\right) - \frac{q_0 a^4}{30D}\left(1 - \frac{r^3}{a^3}\right),$$

$$W_3(r) = \frac{q_0 a^4}{64D}\left(\frac{6+2\nu}{1+\nu}\right)\left(1 - \frac{r^2}{a^2}\right) - \frac{q_0 a^4}{64D}\left(1 - \frac{r^4}{a^4}\right)$$

$$= \frac{q_0 a^4}{64D}\left[\frac{5+\nu}{1+\nu} - 2\left(\frac{3+\nu}{1+\nu}\right)\left(\frac{r}{a}\right)^2 + \left(\frac{r}{a}\right)^4\right]. \tag{8.89}$$

The three-parameter solution (8.89) coincides with the exact solution in Eq. (8.56):

$$w_0(r) = \frac{q_0 a^4}{64D}\left[\left(\frac{r}{a}\right)^4 - 2\left(\frac{3+\nu}{1+\nu}\right)\left(\frac{r}{a}\right)^2 + \left(\frac{5+\nu}{1+\nu}\right)\right].$$

Example 8.5 Consider axisymmetric bending of a clamped (at $r = a$) solid circular plate under uniformly distributed transverse load of intensity q_0 (acting downward), and with $\hat{N}_{rr} = 0$. The boundary conditions are

$$w_0(a) = 0, \qquad \frac{dw_0}{dr} = 0 \quad \text{at } r = 0, a, \tag{8.90a}$$

$$Q_r(0) \equiv -D\left\{\frac{d}{dr}\left[\frac{1}{r}\frac{d}{dr}\left(r\frac{dw_0}{dr}\right)\right]\right\}_{r=0} = 0. \tag{8.90b}$$

The three boundary conditions in Eq. (8.90a) are of the essential type while that in (8.90b) is of a natural type.

Approximation Functions First, we discuss the selection of the approximation functions. Obviously, ϕ_0 is zero since all specified boundary conditions are homogeneous. The choice of ϕ_i depends on the method we use. We derive them for both the Ritz and weighted-residual methods.

In the Ritz method the approximation functions are required to satisfy the homogeneous form of only the geometric boundary conditions. Since there are *three* essential boundary conditions, we begin with a four-parameter polynomial,

$$\phi_1(r) = c_0 + c_1 r + c_2 r^2 + c_3 r^3,$$

and determine the constants (one of them is arbitrary but nonzero) such that the three conditions

$$\phi_i'(0) = 0, \qquad \phi_i(a) = \phi_i'(a) = 0$$

are satisfied. We obtain ($c_0 = a^3c_3/2$, $c_1 = 0$, $c_2 = -3ac_3/2$):

$$\phi_1 = 1 - 3\left(\frac{r}{a}\right)^2 + 2\left(\frac{r}{a}\right)^3. \tag{8.91a}$$

Similarly, we pick the five-term polynomial for ϕ_2 and determine the constants (two of them are arbitrary but the coefficient of the highest-order term, c_4, should never be taken as zero):

$$\phi_2(r) = c_0 + c_1 r + c_2 r^2 + c_3 r^3 + c_4 r^4.$$

We obtain

$$c_0 = \tfrac{3}{2}a^3 c_3 + c_4 a^4, \qquad c_1 = 0, \qquad c_2 = -\tfrac{3}{2}ac_3 - 2a^2 c_4.$$

For simplicity, we take $c_3 = 0$ and obtain

$$\phi_2 = \left[1 - \left(\frac{r}{a}\right)^2\right]^2. \tag{8.91b}$$

In the weighted-residual method, the approximation functions are required to satisfy the homogeneous form of all specified boundary conditions. Since there are *four* specified boundary conditions, we begin with a five-parameter polynomial,

$$\phi_1(r) = c_0 + c_1 r + c_2 r^2 + c_3 r^3 + c_4 r^4,$$

and determine the constants such that the four conditions

$$\phi_i'(0) = 0, \qquad \phi_i(a) = \phi_i'(a) = 0, \qquad \left\{\frac{d}{dr}\left[\frac{1}{r}\frac{d}{dr}\left(r\frac{d\phi_i}{dr}\right)\right]\right\}_{r=0} = 0$$

are satisfied. We obtain ($c_0 = a^4 c_4$, $c_1 = 0$, $c_2 = -2a^2 c_4$, $c_3 = 0$):

$$\phi_1(r) = \left[1 - \left(\frac{r}{a}\right)^2\right]^2. \tag{8.92a}$$

Next, we pick the six-term polynomial for ϕ_2 and determine the constants (two of them are arbitrary but the coefficient of the highest-order term, c_5, should not be taken as zero):

$$\phi_2(r) = c_0 + c_1 r + c_2 r^2 + c_3 r^3 + c_4 r^4 + c_5 r^5.$$

We obtain

$$c_0 = a^4 c_4 + \tfrac{3}{2}a^5 c_5, \qquad c_1 = 0, \qquad c_2 = -2a^2 c_4 - \tfrac{5}{2}a^3 c_5, \qquad c_3 = 0,$$

and

$$\phi_2(r) = \left[1 - \left(\frac{r}{a}\right)^2\right]^2 a^4 c_4 + \frac{1}{2}a^5\left[3 - 5\left(\frac{r}{a}\right)^2 + \left(\frac{r}{a}\right)^5\right]c_5.$$

Since the first part is already represented by ϕ_1, we set $c_4 = 0$ and obtain

$$\phi_2(r) = 3 - 5\left(\frac{r}{a}\right)^2 + \left(\frac{r}{a}\right)^5. \tag{8.92b}$$

Solutions For the two-parameter Ritz approximation, we use the approximation functions ϕ_1 and ϕ_2 from Eqs. (8.91a,b) and compute a_{ij} and b_i from (8.82b,c) (note that k, Q_a, Q_b, M_a, and M_b are zero):

$$\frac{1}{r}\frac{d\phi_1}{dr} = -\frac{6}{a^2}\left(1 - \frac{r}{a}\right), \qquad \frac{d^2\phi_1}{dr^2} = -\frac{6}{a^2}\left[1 - 2\left(\frac{r}{a}\right)\right],$$

$$\frac{1}{r}\frac{d\phi_2}{dr} = -\frac{4}{a^2}\left[1 - \left(\frac{r}{a}\right)^2\right], \qquad \frac{d^2\phi_2}{dr^2} = -\frac{4}{a^2}\left[1 - 3\left(\frac{r}{a}\right)^2\right],$$

$$a_{11} = 2\pi D\left(\frac{32}{3a^2}\right), \qquad a_{12} = a_{21} = 2\pi D\left(\frac{48}{5a^2}\right), \qquad a_{22} = 2\pi D\left(\frac{32}{3a^2}\right),$$

$$b_1 = 2\pi\left(\frac{3q_0a^2}{20}\right), \qquad b_2 = 2\pi\left(\frac{q_0a^2}{6}\right), \qquad c_1 = 0, \qquad c_2 = \frac{q_0a^4}{64D}.$$

Thus, the two-parameter Ritz solution coincides with the exact solution in Eq. (8.45):

$$w_0(r) = \frac{q_0a^4}{64D}\left[1 - \left(\frac{r}{a}\right)^2\right]^2.$$

Note that the one-parameter solution is given by ($c_1 = b_1/a_{11}$):

$$W_1(r) = c_1\phi_1(r) = \frac{9q_0a^4}{640D}\left[1 - 3\left(\frac{r}{a}\right)^2 + 2\left(\frac{r}{a}\right)^3\right].$$

The maximum deflection obtained with the one-parameter Ritz method is in 10% error.

In Galerkin's method, we first compute the residual using the functions in Eqs. (8.92a,b). We have ($\phi_0 = 0$):

$$\begin{aligned}\frac{dW_2}{dr} &= c_1\frac{d\phi_1}{dr} + c_2\frac{d\phi_2}{dr}\\ &= \frac{4}{a}\left(\frac{r}{a}\right)\left[1 - \left(\frac{r}{a}\right)^2\right]c_1 + \frac{5}{a}\left[-2\left(\frac{r}{a}\right) + \left(\frac{r}{a}\right)^4\right]c_2,\\ \frac{d^2W_2}{dr^2} &= \frac{4}{a^2}\left[1 - 3\left(\frac{r}{a}\right)^2\right]c_1 + \frac{10}{a^2}\left[-1 + 2\left(\frac{r}{a}\right)^3\right]c_2,\\ \nabla^2 W_2 &\equiv \frac{d^2W_2}{dr^2} + \frac{1}{r}\frac{dW_2}{dr} = \frac{8}{a^2}\left[1 - 2\left(\frac{r}{a}\right)^2\right]c_1 + \frac{5}{a^2}\left[-4 + 5\left(\frac{r}{a}\right)^3\right]c_2,\\ \nabla^4 W_2 &\equiv \frac{d^2}{dr^2}(\nabla^2 W_2) + \frac{1}{r}\frac{d}{dr}(\nabla^2 W_2) = -\frac{64}{a^4}c_1 + \frac{225}{a^4}\left(\frac{r}{a}\right)c_2.\end{aligned}$$

Also, note that

$$Q_r(r) \equiv -D\frac{d}{dr}\left[\frac{1}{r}\frac{d}{dr}\left(r\frac{dw_0}{dr}\right)\right] = D\left[-\frac{32}{a^3}\left(\frac{r}{a}\right)c_1 + \frac{75}{a^3}\left(\frac{r}{a}\right)^2 c_2\right],$$

$$\frac{d}{dr}(rQ_r) = D\frac{d}{dr}\left\{r\frac{d}{dr}\left[\frac{1}{r}\frac{d}{dr}\left(r\frac{dW_2}{dr}\right)\right]\right\} = D\left[-\frac{64}{a^3}\left(\frac{r}{a}\right)c_1 + \frac{225}{a^3}\left(\frac{r}{a}\right)^2 c_2\right].$$

Hence the coefficients a_{ij} and b_i of Eqs. (8.92a,b) have the values

$$a_{11} = -\frac{64D}{6a^2}, \qquad a_{12} = \frac{120D}{7a^2}, \qquad a_{21} = -\frac{176D}{7a^2},$$

$$a_{22} = \frac{225D}{8a^2}, \qquad b_1 = -\frac{q_0 a^2}{6}, \qquad b_2 = -\frac{11 q_0 a^2}{28},$$

and the solution of the Galerkin equations gives

$$c_1 = \frac{q_0 a^4}{64D}, \qquad c_2 = 0.$$

Therefore, the two-parameter solution is the same as the one-parameter solution, which coincides with the exact solution:

$$w_0(r) = \frac{q_0 a^4}{64D}\left[1 - \left(\frac{r}{a}\right)^2\right]^2.$$

The next example illustrates the use of Betti/Maxwell reciprocity theorem in obtaining the deflections of circular plates under asymmetric loading.

Example 8.6 Consider a circular plate of radius a with an axisymmetric boundary condition, and subjected to an asymmetric loading of the type (see Fig. 8.9):

$$q(r,\theta) = q_0 + q_1\frac{r}{a}\cos\theta, \tag{8.93}$$

where q_0 represents the uniform part of the load for which the solution can be determined for various axisymmetric boundary conditions. In particular, the deflection of a clamped circular plate under a point load at the center is given by Eq. (8.50) and that of a simply supported circular plate under a point load at the center is given by Eq. (8.59). Here we wish to use the Betti/Maxwell reciprocity theorem to determine the center deflection of a clamped plate under asymmetric distributed load.

By Maxwell's theorem, the work done by a point load ($Q_0 = 1$) at the center of the plate due to the deflection (at the center) w_c caused by the distributed load $q(r,\theta)$ is equal to the work done by the distributed load $q(r,\theta)$ in moving through the displacement $w_0(r)$ caused by the point load at the center. Hence, the center deflection of a clamped circular plate under asymmetric load (8.93) is

$$w_c = \frac{a^2}{16\pi D}\int_0^{2\pi}\int_0^a q(r,\theta)\left[1 - \frac{r^2}{a^2}\left(1 - 2\log\frac{r}{a}\right)\right] r\,dr\,d\theta = \frac{q_0 a^4}{64D}. \tag{8.94}$$

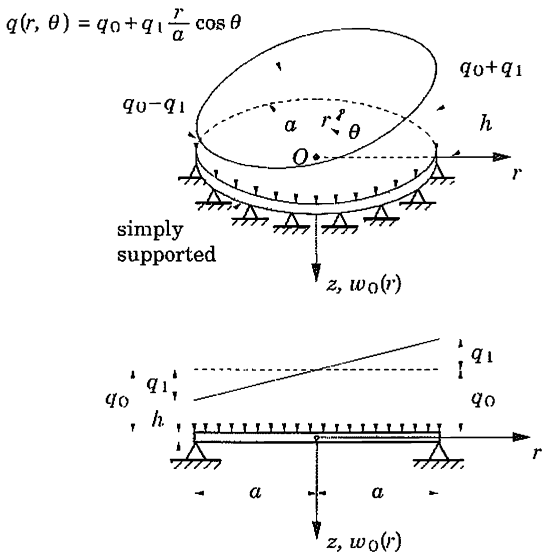


Figure 8.9 A circular plate subjected to an asymmetric loading.

Variational Solutions: Vibration Determination of the natural frequencies of circular plates with various boundary conditions by analytical means leads to complicated equations involving Bessel functions (see [8,24–28]), and the solutions are difficult to obtain, requiring the use of approximate methods of solution. Here we use the Ritz method to determine the natural frequencies.

The equation of motion of an isotropic plate is given by (see Reddy [8])

$$D\nabla^2\nabla^2 w_0 + kw_0 + I_0\frac{\partial^2 w_0}{\partial t^2} - I_2\frac{\partial^2}{\partial t^2}\left(\nabla^2 w_0\right) = 0, \tag{8.95a}$$

where $I_0 = \rho h$ and $I_2 = \rho h^3/12$ are the principal and rotatory inertias, respectively, and the Laplace operator ∇^2 is defined in the polar coordinates as

$$\nabla^2 = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}. \tag{8.95b}$$

For free harmonic motion (i.e., natural vibration), the deflection can be expressed as

$$w_0(r, \theta, t) = w(r, \theta)\cos\omega t, \tag{8.96}$$

where ω is the circular frequency of vibration (radians per unit time), and w is a function of only r and θ. Substituting Eq. (8.96) into Eq. (8.95a), we obtain

$$D\nabla^2\nabla^2 w + kw - I_0\omega^2 w + I_2\omega^2\nabla^2 w = 0. \tag{8.97}$$

The weak form of Eq. (8.97) is given by

$$0=\int_\Omega \left\{ D\frac{\partial^2 w}{\partial r^2}\frac{\partial^2 \delta w}{\partial r^2} + D\frac{\nu}{r}\left(\frac{\partial w}{\partial r}\frac{\partial^2 \delta w}{\partial r^2} + \frac{\partial \delta w}{\partial r}\frac{\partial^2 w}{\partial r^2} \right.\right.$$
$$\left. + \frac{1}{r}\frac{\partial^2 w}{\partial \theta^2}\frac{\partial^2 \delta w}{\partial r^2} + \frac{1}{r}\frac{\partial^2 \delta w}{\partial \theta^2}\frac{\partial^2 w}{\partial r^2}\right) + kw\delta w$$
$$+ D\left(\frac{1}{r}\frac{\partial w}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2}\right)\left(\frac{1}{r}\frac{\partial \delta w}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \delta w}{\partial \theta^2}\right)$$
$$+ 2(1-\nu)D\left(\frac{1}{r}\frac{\partial^2 w}{\partial r \partial \theta} - \frac{1}{r^2}\frac{\partial w}{\partial \theta}\right)\left(\frac{1}{r}\frac{\partial^2 \delta w}{\partial r \partial \theta} - \frac{1}{r^2}\frac{\partial \delta w}{\partial \theta}\right)$$
$$\left. -\omega^2\left[I_0 w\delta w + I_2\left(\frac{\partial w}{\partial r}\frac{\partial \delta w}{\partial r} + \frac{1}{r^2}\frac{\partial w}{\partial \theta}\frac{\partial \delta w}{\partial \theta}\right)\right]\right\} r\, dr d\theta. \tag{8.98}$$

Assume an N-parameter Ritz approximation of the form

$$w(r,\theta) \approx W_N(r,\theta) = \sum_{j=1}^{N} c_j \phi_j(r,\theta). \tag{8.99}$$

Substituting Eq. (8.99) into Eq. (8.98), we obtain

$$0 = \sum_{j=1}^{N}\left(a_{ij}^{(1)} + a_{ij}^{(2)} - \omega^2 m_{ij}\right) c_j, \tag{8.100}$$

where $a_{ij}^{(1)}$, $a_{ij}^{(2)}$, and m_{ij} are defined as

$$a_{ij}^{(1)} = D\int_\Omega \left[\frac{\partial^2 \phi_i}{\partial r^2}\frac{\partial^2 \phi_j}{\partial r^2} + \left(\frac{1}{r}\frac{\partial \phi_i}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \phi_i}{\partial \theta^2}\right)\left(\frac{1}{r}\frac{\partial \phi_j}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \phi_j}{\partial \theta^2}\right)\right.$$
$$+ \frac{\nu}{r}\left(\frac{\partial \phi_i}{\partial r}\frac{\partial^2 \phi_j}{\partial r^2} + \frac{\partial \phi_j}{\partial r}\frac{\partial^2 \phi_i}{\partial r^2} + \frac{1}{r}\frac{\partial^2 \phi_i}{\partial \theta^2}\frac{\partial^2 \phi_j}{\partial r^2} + \frac{1}{r}\frac{\partial^2 \phi_j}{\partial \theta^2}\frac{\partial^2 \phi_i}{\partial r^2}\right)$$
$$\left. +2(1-\nu)\left(\frac{1}{r}\frac{\partial^2 \phi_i}{\partial r \partial \theta} - \frac{1}{r^2}\frac{\partial \phi_i}{\partial \theta}\right)\left(\frac{1}{r}\frac{\partial^2 \phi_j}{\partial r \partial \theta} - \frac{1}{r^2}\frac{\partial \phi_j}{\partial \theta}\right)\right] r\, dr d\theta, \tag{8.101a}$$

$$a_{ij}^{(2)} = k\int_\Omega \phi_i \phi_j \, r\, dr d\theta, \tag{8.101b}$$

$$m_{ij} = \int_\Omega \left[I_0 \phi_i \phi_j + I_2\left(\frac{\partial \phi_i}{\partial r}\frac{\partial \phi_j}{\partial r} + \frac{1}{r^2}\frac{\partial \phi_i}{\partial \theta}\frac{\partial \phi_j}{\partial \theta}\right)\right] r\, dr d\theta. \tag{8.101c}$$

In matrix notation, Eq. (8.100) has the form of an eigenvalue problem:

$$\left(\left[A^{(1)} + A^{(2)}\right] - \omega^2[M]\right)\{c\} = \{0\}. \tag{8.102}$$

For a nontrivial solution, the coefficient matrix in Eq. (8.102) should be singular, i.e.,

$$\left|\left[A^{(1)}\right] + \left[A^{(2)}\right] - \omega^2[M]\right| = 0. \tag{8.103}$$

The above formulation is valid also for annular plates.

Example 8.7 As an example, consider free vibration of a clamped solid circular plate. For a one-parameter ($N = 1$) Ritz solution, let

$$\phi_1(r, \theta) = f_1(r)\cos n\theta \tag{8.104}$$

and compute $a_{11}^{(1)}$, $a_{11}^{(2)}$, and m_{11} as

$$a_{11}^{(1)} = D\int_0^{2\pi}\int_0^a \left\{\left[f_1''f_1'' + \left(\frac{1}{r}f_1' - \frac{n^2}{r^2}f_1\right)^2 + 2\nu\frac{1}{r}\left(f_1'f_1'' - \frac{n^2}{r}f_1f_1''\right)\right]\right.$$
$$\left. \times \cos^2 n\theta + 2(1-\nu)\left(\frac{n}{r}f_1' - \frac{n}{r^2}f_1\right)^2 \sin^2 n\theta\right\} r\,dr d\theta, \tag{8.105a}$$

$$a_{11}^{(2)} = k\int_0^{2\pi}\int_0^a f_1 f_1 \cos^2 n\theta\; r\,dr d\theta, \tag{8.105b}$$

$$m_{11} = \int_0^{2\pi}\int_0^a \left[\left(I_0 f_1 f_1 + f_1' f_1'\right)\cos^2 n\theta + I_2\frac{n^2}{r^2}f_1 f_1 \sin^2 n\theta\right] r\,dr d\theta. \tag{8.105c}$$

The conditions on the approximation functions are

$$\phi_i(r, \theta) = \frac{\partial \phi_i}{\partial r} = 0 \text{ at } r = a, \qquad \text{and} \qquad \frac{\partial \phi_i}{\partial r} = 0 \text{ at } r = 0, \tag{8.106}$$

which translate into the conditions $f_1(a) = f_1'(a) = f_1'(0) = 0$. Clearly, the choice

$$f_1(r) = 1 - 3\left(\frac{r}{a}\right)^2 + 2\left(\frac{r}{a}\right)^3 \tag{8.107}$$

satisfies the conditions. Since

$$f_1' = -\frac{6r}{a^2} + \frac{6r^2}{a^3}, \qquad f_1'' = -\frac{6}{a^2} + \frac{12r}{a^3},$$

it is clear that the integrals (for $n > 0$)

$$\int_0^a \frac{1}{r} f_1 f_1''\, dr, \qquad \int_0^a \frac{1}{r^3} f_1' f_1\, dr$$

required in $a_{11}^{(1)}$ of Eq. (8.105a) do not exist because of the logarithmic singularity. Thus $f_1(r)$ defined in Eq. (8.107) is not admissible for $n > 0$. The next function that satisfies the boundary conditions $f_1(a) = f_1'(a) = f_1'(0) = 0$ is

$$f_1(r) = \left[1 - \left(\frac{r}{a}\right)^2\right]^2 = 1 - 2\left(\frac{r}{a}\right)^2 + \left(\frac{r}{a}\right)^4, \tag{8.108}$$

which is also not admissible for $n > 0$. The next admissible function is

$$f_1(r) = \frac{r}{a}\left[1 - \left(\frac{r}{a}\right)^2\right]^2 = \frac{r}{a} - 2\left(\frac{r}{a}\right)^3 + \left(\frac{r}{a}\right)^5, \tag{8.109}$$

which will not present any problem in evaluating the integrals for $n > 0$.

Here we seek the fundamental frequency corresponding to the axisymmetric mode, $n = 0$. For this case, we can use the function in (8.107) and obtain

$$a_{11}^{(1)} = 2\pi D \int_0^a \left(f_1'' f_1'' + \frac{1}{r^2} f_1' f_1' + 2\nu \frac{1}{r} f_1' f_1''\right) r\, dr = 2\pi \frac{9D}{a^2},$$

$$a_{11}^{(2)} = 2\pi k \int_0^a f_1 f_1\, r\, dr = 2\pi \frac{3ka^2}{35},$$

$$m_{11} = 2\pi \int_0^a \left(I_0 f_1 f_1 + I_2 f_1' f_1'\right) r\, dr = 2\pi \left(\frac{3a^2}{35} I_0 + \frac{3}{5} I_2\right).$$

Hence

$$\omega^2 = \left(\frac{9D}{a^2} + \frac{3ka^2}{35}\right)\left[\frac{3a^2}{35} I_0 + \frac{3}{5} I_2\right]^{-1} = \frac{D}{a^4 I_0}\left[\frac{105 + (ka^4/D)}{1 + 7(I_2/I_0 a^2)}\right]. \tag{8.110}$$

Clearly, rotatory inertia has the effect of reducing the frequency of vibration while the elastic foundation modulus increases it. For a clamped isotropic plate without elastic foundation (i.e., $k = 0$), the frequency parameter becomes ($I_2 = I_0 h^2/12$):

$$\lambda^2 \equiv \omega a^2 \sqrt{\frac{I_0}{D}} = 10.247 \left[\frac{1}{1 + 0.583 \dfrac{h^2}{a^2}}\right]^{1/2}. \tag{8.111}$$

For very thin plates, say $h/a = 0.01$, the effect of rotatory inertia is negligible. Even for $h/a = 0.1$, the effect is less than 1%. Note that the one-parameter Ritz solution differs from the analytical solution $\omega = 10.216$ listed in Table 5.5.1 (for $m = n = 0$) of Reddy [8] by less than 0.5%!

For $n = 1$, the function in Eq. (8.109) may be used. We obtain

$$a_{11}^{(1)} = \frac{\pi D}{a^2}\left[6 + \frac{2}{3} + 2\left(\frac{1}{2} - \frac{7}{6}\right)\nu + \frac{4}{3}(1-\nu)\right],$$

$$a_{11}^{(2)} = \frac{\pi a^2}{60}k, \qquad m_{11} = \pi\left(\frac{a^2}{60}I_0 + \frac{1}{6}I_2\right).$$

The frequency parameter is given by

$$\lambda^2 = \omega a^2\sqrt{\frac{I_0}{D}} = \left[\frac{480 + \dfrac{a^4 k}{60D}}{1 + \dfrac{10I_2}{a^2 I_0}}\right]^{1/2} \tag{8.112}$$

For a clamped isotropic plate without elastic foundation, the frequency for the case $m = 0,\ n = 1$, becomes ($I_2 = I_0h^2/12$):

$$\lambda^2 = \omega a^2\sqrt{\frac{I_0}{D}} = 21.909\left[\frac{1}{1 + \dfrac{5}{6}\dfrac{h^2}{a^2}}\right]^{1/2} \tag{8.113}$$

When rotatory inertia is neglected, the frequency predicted by Eq. (8.113) differs from the analytical solution $\omega = 21.26$ listed in Table 5.5.1 (for $m = 0, n = 1$) of Reddy [8] by only 3%.

For other values of n, one must select functions that are admissible (i.e., allow evaluation of the integrals). Generally, higher values of n require higher-order functions $f_i(r)$.

Variational Solutions: Buckling Next, we consider buckling of circular plates under uniform compression $\hat{N}$ in the middle plane of the plate (see Fig. 8.7). The governing equation is

$$D\nabla^2\nabla^2 w + \hat{N}\nabla^2 w = 0. \tag{8.114}$$

The weak form of this equation is a slight modification of that given in Eq. (8.98):

$$\begin{aligned}
0 = \int_\Omega \Bigg\{ & D\frac{\partial^2 w}{\partial r^2}\frac{\partial^2 \delta w}{\partial r^2} + D\frac{\nu}{r}\left(\frac{\partial w}{\partial r}\frac{\partial^2 \delta w}{\partial r^2} + \frac{\partial \delta w}{\partial r}\frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial^2 w}{\partial \theta^2}\frac{\partial^2 \delta w}{\partial r^2}\right.\\
& \left. + \frac{1}{r}\frac{\partial^2 \delta w}{\partial \theta^2}\frac{\partial^2 w}{\partial r^2}\right) + D\left(\frac{1}{r}\frac{\partial w}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2}\right)\left(\frac{1}{r}\frac{\partial \delta w}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \delta w}{\partial \theta^2}\right)\\
& + 2(1-\nu)D\left(\frac{1}{r}\frac{\partial^2 w}{\partial r \partial \theta} - \frac{1}{r^2}\frac{\partial w}{\partial \theta}\right)\left(\frac{1}{r}\frac{\partial^2 \delta w}{\partial r \partial \theta} - \frac{1}{r^2}\frac{\partial \delta w}{\partial \theta}\right)\\
& - \hat{N}\frac{\partial \delta w}{\partial r}\frac{\partial w}{\partial r}\Bigg\} r\, dr d\theta.
\end{aligned} \tag{8.115a}$$

The N-parameter Ritz approximation (8.99) results in

$$\left([A] - \hat{N}[G]\right)\{c\} = \{0\}, \tag{8.115b}$$

where a_{ij} are the same as $a_{ij}^{(1)}$ in Eq. (8.101a) (with $k = 0$) and g_{ij} are given by

$$g_{ij} = \int_0^{2\pi} \int_0^a \frac{\partial \phi_i}{\partial r} \frac{\partial \phi_j}{\partial r}\, r\, dr d\theta. \tag{8.115c}$$

Example 8.8 As an example, consider a clamped circular plate under uniform compression. Since we are interested in the minimum buckling load, which occurs in the axisymmetric mode ($n = 0$), we can use the one-parameter approximation (see Example 8.6) $W_1(r) = c_1 f_1(r)$, where $f_1(r)$ is defined in Eq. (8.107). We obtain

$$\frac{9D}{a^2} - \hat{N}\frac{3}{5} = 0 \quad \rightarrow \quad \hat{N} = \frac{15D}{a^2},$$

which differs from the exact solution in (8.71), $14.682(D/a^2)$, by 2.18%.

8.2.3 Governing Equations in Rectangular Coordinates

Here we consider the classical theory of plates in rectangular Cartesian coordinates. We wish to develop the total Lagrangian functional and to use Hamilton's principle to derive the equations of motion and natural boundary conditions for linear bending of a plate according to the Kirchhoff hypotheses. Stretching deformation is not considered as it can be uncoupled from bending deformation for the linear problem.

Let us denote, as before, the undeformed midplane of the plate with the symbol Ω_0, and let it coincide with the xy-plane of the coordinate system with the z coordinate along the thickness of the plate. The total domain of the plate is symbolically expressed as the tensor product $\Omega = \Omega_0 \times (-h/2, h/2)$. The boundary of the total domain Ω consists of the top surface $S_t(z = h/2)$, bottom surface $S_b(z = -h/2)$, and the edge $\bar{\Gamma} \equiv \Gamma \times (-h/2, h/2)$. In general, $\bar{\Gamma}$ is a curved surface, with outward normal $\hat{\mathbf{n}} = n_x \hat{\mathbf{e}}_x + n_y \hat{\mathbf{e}}_y$, where n_x and n_y are the direction cosines of the unit normal (see Fig. 8.10).

We begin with the following displacement field [cf. Eq. (8.7)], which is a direct result of the Kirchhoff hypotheses in the pure bending case (see Fig. 8.11):

$$u(x, y, z, t) = -z\frac{\partial w_0}{\partial x}, \qquad v(x, y, z, t) = -z\frac{\partial w_0}{\partial y}, \qquad w(x, y, z, t) = w_0(x, y, t), \tag{8.116}$$

where (u, v, w) denote the displacements of a material point in the (x, y, z) coordinate directions at time t.

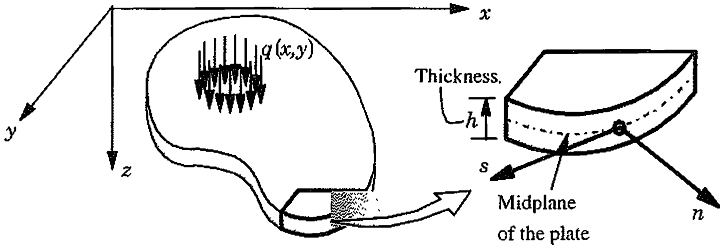


Figure 8.10 Geometry of a plate with curved boundary.

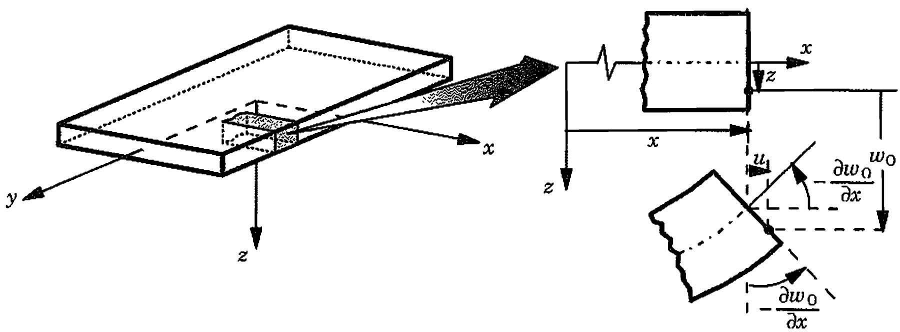


Figure 8.11 Kinematics of deformation of the classical plate theory in rectangular Cartesian coordinates.

Assuming small strains and displacements, the linear strains can be computed using Eq. (3.23). For the displacement field in Eq. (8.116), the linear strains are

$$\varepsilon_{xx} = -z\frac{\partial^2 w_0}{\partial x^2}, \qquad \varepsilon_{xy} = -z\frac{\partial^2 w_0}{\partial x \partial y}, \qquad \varepsilon_{yy} = -z\frac{\partial^2 w_0}{\partial y^2},$$
$$\varepsilon_{xz} = 0, \qquad \varepsilon_{yz} = 0, \qquad \varepsilon_{zz} = 0. \tag{8.117}$$

Note that the transverse strains $(\varepsilon_{xz}, \varepsilon_{yz}, \varepsilon_{zz})$ are identically zero in the classical plate theory.

Hamilton's principle can be used in two different forms: one way is to construct the Lagrangian functional $L = (U + V - K)$ and use Hamilton's principle (the dynamic version of the principle of "minimum" potential energy):

$$0 = \delta \int_{t_1}^{t_2} L\, dt = \delta \int_{t_1}^{t_2} (U + V - K)\, dt, \tag{8.118a}$$

where (U, V, K) are the total strain energy, potential energy due to applied loads, and kinetic energy, respectively. The other way is to construct only the virtual Lagrangian $\delta L = \delta U + \delta V - \delta K$ and use Hamilton's principle in the form (the dynamic version of the principle of virtual displacements):

$$0 = \int_{t_1}^{t_2} \delta L \, dt = \int_{t_1}^{t_2} (\delta U + \delta V - \delta K) \, dt. \tag{8.118b}$$

The first way makes use of the constitutive equations to write U and the assumption that the forces are conservative to write V, much the same way as in the principle of minimum total potential energy. The second way is general and does not require the use of constitutive relations or the assumption that the forces are derivable from a potential. Here we use the latter, but also give the expressions for (U, V, K) for completeness.

The virtual strain energy δU is given by

$$\begin{aligned}\delta U &= \int_{\Omega_0} \int_{-h/2}^{h/2} \left(\sigma_{xx}\delta\varepsilon_{xx} + \sigma_{yy}\delta\varepsilon_{yy} + 2\sigma_{xy}\delta\varepsilon_{xy}\right) dz dx dy \\ &= -\int_{\Omega_0} \left(M_{xx}\frac{\partial^2 \delta w_0}{\partial x^2} + M_{yy}\frac{\partial^2 \delta w_0}{\partial y^2} + 2M_{xy}\frac{\partial^2 \delta w_0}{\partial x \partial y}\right) dx dy,\end{aligned} \tag{8.119}$$

where (M_{xx}, M_{yy}, M_{xy}) are the *moments* per unit length (see Fig. 8.12):

$$\begin{Bmatrix} M_{xx} \\ M_{yy} \\ M_{xy} \end{Bmatrix} = \int_{-h/2}^{h/2} \begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{Bmatrix} z \, dz. \tag{8.120}$$

To write U, we assume linear elastic behavior of the plate material and write [see Eqs. (3.40) and (3.41)]

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{Bmatrix} = \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ 2\varepsilon_{xy} \end{Bmatrix} \tag{8.121}$$

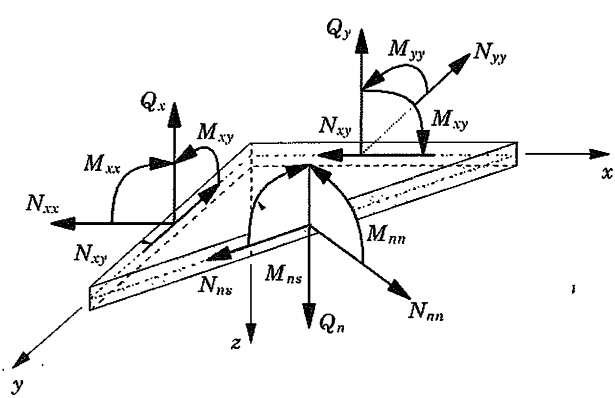


Figure 8.12 Definitions of moments and shear forces.

for an isotropic material and

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{Bmatrix} = \begin{bmatrix} Q_{11} & Q_{12} & 0 \\ Q_{12} & Q_{22} & 0 \\ 0 & 0 & Q_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ 2\varepsilon_{xy} \end{Bmatrix} \tag{8.122a}$$

for an orthotropic material, where

$$Q_{11} = \frac{E_1}{1-\nu_{12}\nu_{21}}, \qquad Q_{12} = \frac{\nu_{12}E_2}{1-\nu_{12}\nu_{21}}, \qquad Q_{22} = \frac{E_2}{1-\nu_{12}\nu_{21}},$$

$$Q_{66} = G_{12}, \qquad \nu_{21} = \nu_{12}\frac{E_2}{E_1}. \tag{8.122b}$$

Then substituting Eq. (8.117) into Eq. (8.122a), and the result into Eq. (8.120), gives

$$\begin{aligned} M_{xx} &= -\left(D_{11}\frac{\partial^2 w_0}{\partial x^2} + D_{12}\frac{\partial^2 w_0}{\partial y^2}\right), \\ M_{yy} &= -\left(D_{22}\frac{\partial^2 w_0}{\partial y^2} + D_{12}\frac{\partial^2 w_0}{\partial x^2}\right), \\ M_{xy} &= -2D_{66}\frac{\partial^2 w_0}{\partial x \partial y}, \end{aligned} \tag{8.123a}$$

where

$$D_{ij} = \frac{h^3}{12}Q_{ij}, \qquad h = \text{plate thickness}. \tag{8.123b}$$

For an isotropic plate we have ($E_1 = E_2 = E$, $\nu_{12} = \nu_{21} = \nu$, and $G_{12} = G = E/[2(1+\nu)]$):

$$D_{11} = D_{22} = D = \frac{Eh^3}{12(1-\nu^2)}, \qquad D_{12} = \nu D, \qquad 2D_{66} = (1-\nu)D. \tag{8.124}$$

The strain energy of an orthotropic plate is

$$\begin{aligned} U &= \frac{1}{2}\int_V (\sigma_{xx}\varepsilon_{xx} + \sigma_{yy}\varepsilon_{yy} + 2\sigma_{xy}\varepsilon_{xy})\, dV \\ &= \frac{1}{2}\int_{\Omega_0}\int_{-h/2}^{h/2}\left[D_{11}\varepsilon_{xx}^2 + D_{22}\varepsilon_{yy}^2 + 2D_{12}\varepsilon_{xx}\varepsilon_{yy} + 4D_{66}\varepsilon_{xy}^2\right] dz\,dx\,dy \\ &= \frac{1}{2}\int_{\Omega_0}\left[D_{11}\left(\frac{\partial^2 w_0}{\partial x^2}\right)^2 + D_{22}\left(\frac{\partial^2 w_0}{\partial y^2}\right)^2 + 2D_{12}\frac{\partial^2 w_0}{\partial x^2}\frac{\partial^2 w_0}{\partial y^2}\right. \\ &\quad \left. + 4D_{66}\left(\frac{\partial^2 w_0}{\partial x \partial y}\right)^2\right] dx\,dy \end{aligned} \tag{8.125}$$

The virtual work done by the applied distributed load $q(x, y)$ on the top surface, $z = h/2$, the applied transverse edge force $\hat{V}_n$, the applied normal edge moment $\hat{M}_{nn}$ on boundary Γ_2 (a portion of the total boundary Γ of Ω_0), and the applied in-plane compressive and shear forces $(\hat{N}_{xx}, \hat{N}_{yy}, \hat{N}_{xy})$ is

$$\begin{aligned}\delta V = -\Bigg\{ \int_{\Omega_0} \Bigg[\hat{N}_{xx} \frac{\partial w_0}{\partial x} \frac{\partial \delta w_0}{\partial x} + \hat{N}_{yy} \frac{\partial w_0}{\partial y} \frac{\partial \delta w_0}{\partial y} \\ + \hat{N}_{xy} \left(\frac{\partial w_0}{\partial x} \frac{\partial \delta w_0}{\partial y} + \frac{\partial w_0}{\partial y} \frac{\partial \delta w_0}{\partial x} \right) \Bigg] dxdy \\ + \int_{\Omega_0} q(x, y) \delta w_0 \, dxdy + \int_{\Gamma_2} \left(\hat{V}_n \delta w_0 - \hat{M}_{nn} \frac{\partial \delta w_0}{\partial n} \right) ds \Bigg\}. \end{aligned} \tag{8.126}$$

The potential energy due to applied loads is

$$\begin{aligned}V = -\Bigg\{ \frac{1}{2} \int_{\Omega_0} \left[\hat{N}_{xx} \left(\frac{\partial w_0}{\partial x} \right)^2 + \hat{N}_{yy} \left(\frac{\partial w_0}{\partial y} \right)^2 + \hat{N}_{xy} \frac{\partial w_0}{\partial x} \frac{\partial w_0}{\partial y} \right] dxdy \\ + \int_{\Omega_0} q(x, y) w_0 \, dxdy + \int_{\Gamma_2} \left(\hat{V}_n w_0 - \hat{M}_{nn} \frac{\partial w_0}{\partial n} \right) ds \Bigg\}. \end{aligned} \tag{8.127}$$

The virtual kinetic energy is given by

$$\begin{aligned}\delta K &= \int_{\Omega_0} \int_{-h/2}^{h/2} \rho \left(\dot{u} \delta \dot{u} + \dot{v} \delta \dot{v} + \dot{w} \delta \dot{w} \right) dz \, dxdy \\ &= \int_{\Omega_0} \int_{-h/2}^{h/2} \rho \Bigg[\left(-z \frac{\partial \dot{w}_0}{\partial x} \right) \left(-z \frac{\partial \delta \dot{w}_0}{\partial x} \right) + \left(-z \frac{\partial \dot{w}_0}{\partial y} \right) \left(-z \frac{\partial \delta \dot{w}_0}{\partial y} \right) \\ &\qquad + \dot{w}_0 \delta \dot{w}_0 \Bigg] dz \, dxdy \\ &= \int_{\Omega_0} \left[I_0 \dot{w}_0 \delta \dot{w}_0 + I_2 \left(\frac{\partial \dot{w}_0}{\partial x} \frac{\partial \delta \dot{w}_0}{\partial x} + \frac{\partial \dot{w}_0}{\partial y} \frac{\partial \delta \dot{w}_0}{\partial y} \right) \right] dxdy, \end{aligned} \tag{8.128}$$

where ρ is the mass density and the superposed dot on a variable indicates time derivative, $\dot{w}_0 = \partial w_0 / \partial t$, and (I_0, I_2) are the mass moments of inertia:

$$\begin{Bmatrix} I_0 \\ I_2 \end{Bmatrix} = \int_{-h/2}^{h/2} \begin{Bmatrix} 1 \\ z^2 \end{Bmatrix} \rho \, dz = \rho_0 \begin{Bmatrix} h \\ h^3/12 \end{Bmatrix}. \tag{8.129}$$

The kinetic energy is

$$K = \frac{1}{2} \int_{\Omega_0} \left\{ I_0 \left(\dot{w}_0 \right)^2 + I_2 \left[\left(\frac{\partial \dot{w}_0}{\partial x} \right)^2 + \left(\frac{\partial \dot{w}_0}{\partial y} \right)^2 \right] \right\} dxdy. \tag{8.130}$$

Now we have all the elements in place to apply Hamilton's principle (8.118b). We have

$$
\begin{aligned}
0 = \int_{t_1}^{t_2} \int_{\Omega_0} \bigg\{ & -\left(M_{xx} \frac{\partial^2 \delta w_0}{\partial x^2} + M_{yy} \frac{\partial^2 \delta w_0}{\partial y^2} + 2M_{xy} \frac{\partial^2 \delta w_0}{\partial x \partial y} \right) \\
& - \left[I_0 \dot{w}_0 \delta \dot{w}_0 + I_2 \left(\frac{\partial \dot{w}_0}{\partial x} \frac{\partial \delta \dot{w}_0}{\partial x} + \frac{\partial \dot{w}_0}{\partial y} \frac{\partial \delta \dot{w}_0}{\partial y} \right) \right] \\
& - \left[\hat{N}_{xx} \frac{\partial w_0}{\partial x} \frac{\partial \delta w_0}{\partial x} + \hat{N}_{yy} \frac{\partial w_0}{\partial y} \frac{\partial \delta w_0}{\partial y} \right. \\
& \left. + \hat{N}_{xy} \left(\frac{\partial w_0}{\partial x} \frac{\partial \delta w_0}{\partial y} + \frac{\partial w_0}{\partial y} \frac{\partial \delta w_0}{\partial x} \right) \right] \bigg\} \, dxdydt \\
- \int_{t_1}^{t_2} & \left[\int_{\Omega_0} q \delta w_0 \, dxdy - \int_{\Gamma_2} \left(\hat{V}_n \delta w_0 - \hat{M}_{nn} \frac{\partial \delta w_0}{\partial n} \right) ds \right] dt, \qquad (8.131a)
\end{aligned}
$$

Integrating terms by parts to relieve δw_0 of any differentiation, we obtain

$$
\begin{aligned}
0 = \int_{t_1}^{t_2} \bigg\{ & \int_{\Omega_0} \left[-\left(\frac{\partial^2 M_{xx}}{\partial x^2} + \frac{\partial^2 M_{yy}}{\partial y^2} + 2 \frac{\partial^2 M_{xy}}{\partial x \partial y} \right) + I_0 \ddot{w}_0 - I_2 \left(\frac{\partial^2 \ddot{w}_0}{\partial x^2} + \frac{\partial^2 \ddot{w}_0}{\partial y^2} \right) \right. \\
& \left. + \frac{\partial}{\partial x} \left(\hat{N}_{xx} \frac{\partial w_0}{\partial x} + \hat{N}_{xy} \frac{\partial w_0}{\partial y} \right) + \frac{\partial}{\partial y} \left(\hat{N}_{xy} \frac{\partial w_0}{\partial x} + \hat{N}_{yy} \frac{\partial w_0}{\partial y} \right) - q \right] \delta w_0 \, dxdy \\
& - \oint_{\Gamma} \left[M_{xx} n_x \frac{\partial \delta w_0}{\partial x} + M_{yy} n_y \frac{\partial \delta w_0}{\partial y} + M_{xy} n_x \frac{\partial \delta w_0}{\partial y} + M_{xy} n_y \frac{\partial \delta w_0}{\partial x} \right. \\
& - \left(\frac{\partial M_{xx}}{\partial x} n_x + \frac{\partial M_{yy}}{\partial y} n_y + \frac{\partial M_{xy}}{\partial x} n_y + \frac{\partial M_{xy}}{\partial y} n_x \right) \delta w_0 \\
& + I_2 \left(\frac{\partial \dot{w}_0}{\partial x} n_x + \frac{\partial \dot{w}_0}{\partial y} n_y \right) + \left(\hat{N}_{xx} \frac{\partial w_0}{\partial x} + \hat{N}_{xy} \frac{\partial w_0}{\partial y} \right) n_x \\
& \left. + \left(\hat{N}_{xy} \frac{\partial w_0}{\partial x} + \hat{N}_{yy} \frac{\partial w_0}{\partial y} \right) n_y \right] \delta w_0 \, ds - \int_{\Gamma_2} \left(\hat{V}_n \delta w_0 - \hat{M}_{nn} \frac{\partial \delta w_0}{\partial n} \right) ds \bigg\} dt,
\end{aligned}
$$

(8.131b)

where all terms evaluated at $t = t_1$ and $t = t_2$ are zero (by assumption) and not included in the above statement. The Euler–Lagrange equation is clearly

$$
\begin{aligned}
& -\left(\frac{\partial^2 M_{xx}}{\partial x^2} + 2 \frac{\partial^2 M_{xy}}{\partial y \partial x} + \frac{\partial^2 M_{yy}}{\partial y^2} \right) - q \\
& + \frac{\partial}{\partial x} \left(\hat{N}_{xx} \frac{\partial w_0}{\partial x} + \hat{N}_{xy} \frac{\partial w_0}{\partial y} \right) + \frac{\partial}{\partial y} \left(\hat{N}_{xy} \frac{\partial w_0}{\partial x} + \hat{N}_{yy} \frac{\partial w_0}{\partial y} \right) \\
& + I_0 \ddot{w}_0 - I_2 \left(\frac{\partial^2 \ddot{w}_0}{\partial x^2} + \frac{\partial^2 \ddot{w}_0}{\partial y^2} \right) = 0. \qquad (8.132)
\end{aligned}
$$

Inspection of the boundary terms in (8.131b) indicates that w_0, $\partial w_0/\partial x$, and $\partial w_0/\partial y$ are the primary variables, and

$$\left(\bar{Q}_x + I_2\frac{\partial \ddot{w}_0}{\partial x}\right)n_x + \left(\bar{Q}_y + I_2\frac{\partial \ddot{w}_0}{\partial y}\right)n_y, \qquad \text{and}$$

$$M_{xx}n_x + M_{xy}n_y, \qquad M_{xy}n_x + M_{yy}n_y, \tag{8.133a}$$

are the secondary variables of the theory. Here $\bar{Q}_x$ and $\bar{Q}_y$ are defined as

$$\begin{aligned}
\bar{Q}_x &= Q_x - \left(\hat{N}_{xx}\frac{\partial w_0}{\partial x} + \hat{N}_{xy}\frac{\partial w_0}{\partial y}\right),\\
\bar{Q}_y &= Q_y - \left(\hat{N}_{xy}\frac{\partial w_0}{\partial x} + \hat{N}_{yy}\frac{\partial w_0}{\partial y}\right),\\
Q_x &\equiv \frac{\partial M_{xx}}{\partial x} + \frac{\partial M_{xy}}{\partial y},\\
Q_y &\equiv \frac{\partial M_{xy}}{\partial x} + \frac{\partial M_{yy}}{\partial y}.
\end{aligned} \tag{8.133b}$$

In order to cast the boundary conditions on an edge arbitrarily oriented in the xy-plane, we convert the derivatives with respect to x and y to those in terms of the normal and tangential coordinates (n, s). If the unit outward normal vector $\hat{\mathbf{n}}$ is oriented at an angle θ counterclockwise from the positive x-axis, then its direction cosines are $n_x = \cos\theta$ and $n_y = \sin\theta$. Hence the transformation between the coordinate system (n, s) and (x, y) is given by

$$\begin{aligned}
\hat{\mathbf{e}}_x &= \cos\theta\ \hat{\mathbf{e}}_n - \sin\theta\ \hat{\mathbf{e}}_s = n_x\ \hat{\mathbf{e}}_n - n_y\ \hat{\mathbf{e}}_s,\\
\hat{\mathbf{e}}_y &= \sin\theta\ \hat{\mathbf{e}}_n + \cos\theta\ \hat{\mathbf{e}}_s = n_y\ \hat{\mathbf{e}}_n + n_x\ \hat{\mathbf{e}}_s.
\end{aligned} \tag{8.134}$$

Hence the normal and tangential derivatives $(\partial w_0/\partial n, \partial w_0/\partial s)$ are related to the derivatives $(\partial w_0/\partial x, \partial w_0/\partial y)$ by

$$\frac{\partial w_0}{\partial x} = n_x\frac{\partial w_0}{\partial n} - n_y\frac{\partial w_0}{\partial s}, \qquad \frac{\partial w_0}{\partial y} = n_y\frac{\partial w_0}{\partial n} + n_x\frac{\partial w_0}{\partial s}. \tag{8.135}$$

Using Eq. (8.135), we can rewrite the boundary expressions in (8.131b) as

$$\begin{aligned}
0 = \oint_\Gamma \bigg\{&(V_x n_x + V_y n_y)\delta w_0 - \left(M_{xx}n_x^2 + 2M_{xy}n_x n_y + M_{yy}n_y^2\right)\frac{\partial \delta w_0}{\partial n}\\
&- \left[\left(M_{yy} - M_{xx}\right)n_x n_y + M_{xy}\left(n_x^2 - n_y^2\right)\right]\frac{\partial \delta w_0}{\partial s}\bigg\}\, ds\\
- \int_{\Gamma_2}&\left(\hat{V}_n \delta w_0 - \hat{M}_{nn}\frac{\partial \delta w_0}{\partial n}\right) ds
\end{aligned}$$

$$= \oint_{\Gamma_2} \left[\left(V_x n_x + V_y n_y + \frac{\partial M_{ns}}{\partial s} - \hat{V}_n \right) \delta w_0 + \left(\hat{M}_{nn} - M_{nn} \right) \frac{\partial \delta w_0}{\partial n} \right] ds. \tag{8.136}$$

Here, $\delta w_0 = 0$ and $(\partial \delta w_0/\partial n) = 0$ on Γ_1 where w_0 and $(\partial w_0/\partial n)$ are specified, and

$$V_x = Q_x - \left(\hat{N}_{xx} \frac{\partial w_0}{\partial x} + \hat{N}_{xy} \frac{\partial w_0}{\partial y} \right) + I_2 \frac{\partial \dot{w}_0}{\partial x}, \tag{8.137a}$$

$$V_y = Q_y - \left(\hat{N}_{xy} \frac{\partial w_0}{\partial x} + \hat{N}_{yy} \frac{\partial w_0}{\partial y} \right) + I_2 \frac{\partial \dot{w}_0}{\partial x}, \tag{8.137b}$$

$$M_{nn} = M_{xx} n_x^2 + 2 M_{xy} n_x n_y + M_{yy} n_y^2, \tag{8.138a}$$

$$M_{ns} = \left(M_{yy} - M_{xx} \right) n_x n_y + M_{xy} \left(n_x^2 - n_y^2 \right). \tag{8.138b}$$

Note that the expression involving $\partial \delta w_0/\partial s$ was integrated by parts to give

$$-\oint_{\Gamma} M_{ns} \frac{\partial \delta w_0}{\partial s} \, ds = \oint_{\Gamma} \frac{\partial M_{ns}}{\partial s} \delta w_0 \, ds - [M_{ns} \delta w_0]_{\Gamma}, \tag{8.139}$$

and the term $[M_{ns} \delta w_0]_{\Gamma}$ is set to zero since the end points of a closed smooth curve coincide. For plates with corners (i.e., for polygonal plates), concentrated forces of magnitude

$$F_c = -2 M_{ns} \tag{8.140}$$

will be produced at the corners. The factor of 2 appears because M_{ns} from two sides of the corner are added there. Finally, the natural boundary conditions are

$$V_x n_x + V_y n_y + \frac{\partial M_{ns}}{\partial s} - \hat{V}_n = 0, \qquad M_{nn} - \hat{M}_{nn} = 0 \quad \text{on } \Gamma_2. \tag{8.141}$$

The first of the two boundary conditions is known as the *Kirchhoff free-edge condition*. Thus, the effective shear force is

$$V_n \equiv V_x n_x + V_y n_y + \frac{\partial M_{ns}}{\partial s}. \tag{8.142}$$

Equation (8.132) can be expressed in terms of the transverse deflection with the help of moment-deflection relations (8.123a). We have

$$\begin{aligned} &\left[D_{11} \frac{\partial^4 w_0}{\partial x^4} + 2(D_{66} + 2D_{12}) \frac{\partial^4 w_0}{\partial x^2 \partial y^2} + D_{22} \frac{\partial^4 w_0}{\partial y^4} \right] \\ &\quad + \frac{\partial}{\partial x} \left(\hat{N}_{xx} \frac{\partial w_0}{\partial x} + \hat{N}_{xy} \frac{\partial w_0}{\partial y} \right) + \frac{\partial}{\partial y} \left(\hat{N}_{xy} \frac{\partial w_0}{\partial x} + \hat{N}_{yy} \frac{\partial w_0}{\partial y} \right) \\ &\quad = q - I_0 \ddot{w}_0 + I_2 \left(\frac{\partial^2 \ddot{w}_0}{\partial x^2} + \frac{\partial^2 \ddot{w}_0}{\partial y^2} \right), \end{aligned} \tag{8.143a}$$

for orthotropic plates, and

$$D\left(\frac{\partial^4 w_0}{\partial x^4}+2\frac{\partial^4 w_0}{\partial x^2\partial y^2}+\frac{\partial^4 w_0}{\partial y^4}\right)+\frac{\partial}{\partial x}\left(\hat{N}_{xx}\frac{\partial w}{\partial x}+\hat{N}_{xy}\frac{\partial w}{\partial y}\right)$$
$$+\frac{\partial}{\partial y}\left(\hat{N}_{xy}\frac{\partial w_0}{\partial x}+\hat{N}_{yy}\frac{\partial w_0}{\partial y}\right)=q-I_0\ddot{w}_0+I_2\left(\frac{\partial^2\ddot{w}_0}{\partial x^2}+\frac{\partial^2\ddot{w}_0}{\partial y^2}\right), \quad (8.143b)$$

for isotropic plates, where $D = Eh^3/[12(1-\nu^2)]$.

8.2.4 Navier Solutions of Rectangular Plates

In this section we briefly discuss Navier solutions of isotropic rectangular plates for bending, natural vibration, buckling, and transient response. Navier solutions can be developed for simply supported rectangular plates in the following cases:

(1) bending solutions under arbitrary transverse load $q(x, y)$;
(2) natural frequencies;
(3) buckling loads under in-plane biaxial compressive loads $\hat{N}_{yy} = \gamma\hat{N}_{xx} = \gamma N_0$; and
(4) spatial part of the solution for the transient case.

Keeping the scope of this book in mind, only a brief discussion of analytical solutions is presented. For additional information, the reader may consult the books on plates [8,22,24–29], especially those by Reddy [8] and Timoshenko and Woinowsky-Krieger [24].

The simply supported boundary conditions for a rectangular plate, for the choice of coordinate system shown in Fig. 8.13, are

$$w_0(0, y, t) = 0, \qquad w_0(a, y, t) = 0, \qquad w_0(x, 0, t) = 0, \qquad w_0(x, b, t) = 0, \quad (8.144a)$$

$$M_{xx}(0, y, t) = 0, \qquad M_{xx}(a, y, t) = 0, \qquad M_{yy}(x, 0, t) = 0, \qquad M_{yy}(x, b, t) = 0. \quad (8.144b)$$

In Navier's method the displacement w_0 is expanded in double sine series with unknown coefficients:

$$w_0(x, y, t) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} W_{mn}(t)\sin\alpha_m x\sin\beta_n y, \quad (8.145a)$$

where

$$\alpha_m = \frac{m\pi}{a}, \qquad \beta_n = \frac{n\pi}{b}. \quad (8.145b)$$

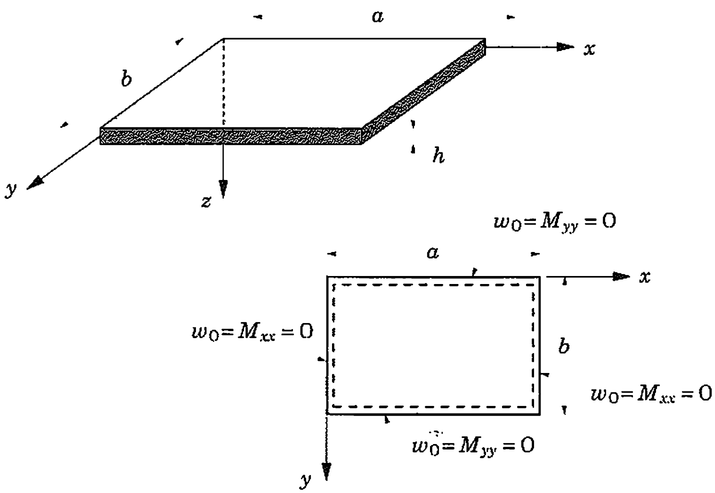


Figure 8.13 Geometry and coordinate system for a rectangular plate.

Here W_{mn} denote the coefficients (that depend on time t) to be determined such that Eq. (8.143b) is satisfied everywhere in the domain of the plate for all $t > 0$. The choice of expansion (8.145a) is dictated by the fact that the double sine series satisfies the simply supported boundary conditions in (8.144a,b). Substitution of the expansion (8.145a) into the governing equation (8.143a) or (8.143b) shows that the mechanical load q may also be expanded in double sine series:

$$q(x, y, t) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} q_{mn}(t) \sin \alpha_m x \sin \beta_n y, \tag{8.146a}$$

$$q_{mn}(t) = \frac{4}{ab} \int_0^b \int_0^a q(x, y, t) \sin \alpha_m x \sin \beta_n y \, dx dy. \tag{8.146b}$$

The coefficients q_{mn} for various types of loads can be calculated using Eq. (8.146b) (see Table 8.3 for typical loads; see Reddy [7,8]).

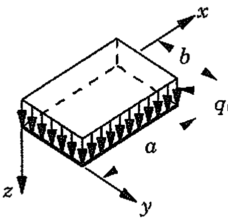
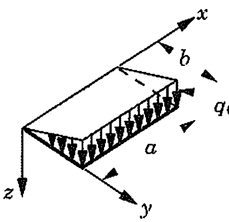
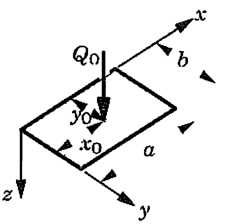
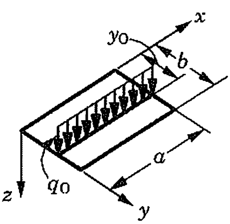
Substituting Eqs. (8.145a) and (8.146a) into Eq. (8.143b), we obtain

$$0 = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \Big\{ D\big(\alpha_m^4 + 2\alpha_m^2 \beta_n^2 + \beta_n^4\big) W_{mn} - \big(\alpha_m^2 + \gamma \beta_n^2\big) N_0 - q_{mn} + \Big[I_0 + I_2\big(\alpha_m^2 + \beta_n^2\big)\Big] \ddot{W}_{mn} \Big\} \sin \alpha_m x \sin \beta_n y \tag{8.147}$$

for any (x, y) and $t > 0$. Hence, it follows that

$$\Big[I_0 + I_2\big(\alpha_m^2 + \beta_n^2\big)\Big] \frac{d^2 W_{mn}}{dt^2} + \Big[D\big(\alpha_m^2 + \beta_n^2\big)^2 - \big(\alpha_m^2 + \gamma \beta_n^2\big) N_0\Big] W_{mn} = q_{mn}, \tag{8.148}$$

Table 8.3 Coefficients in the double sine series expansion of loads in Navier's method

<table>
<tr><th>Loading</th><th>Coefficients, q_{mn}</th></tr>
<tr><td></td><td>Uniform load
$q(x, y) = q_0$
$q_{mn} = \frac{16q_0}{\pi^2 mn}$
$(m, n = 1, 3, 5, \ldots)$</td></tr>
<tr><td></td><td>Hydrostatic load
$q(x, y) = q_0 \frac{y}{b}$
$q_{mn} = \frac{8q_0}{\pi^2 mn}(-1)^{n+1}$
$(m = 1, 3, 5, \ldots)$
$(n = 1, 2, 3, \ldots)$</td></tr>
<tr><td></td><td>Point load
[i.e., Q_0 at (x_0, y_0)]
$q_{mn} = \frac{4Q_0}{ab} \sin \frac{m\pi x_0}{a} \sin \frac{n\pi y_0}{b}$
$(m, n = 1, 2, 3, \ldots)$</td></tr>
<tr><td></td><td>Line load
$q(x, y) = q_0\, \delta(y - y_0)$
$q_{mn} = \frac{8q_0}{\pi bm} \sin \frac{n\pi y_0}{b}$
$(m = 1, 3, 5, \ldots)$
$(n = 1, 2, 3, \ldots)$</td></tr>
</table>

for every pair of m and n and $t > 0$. Equation (8.148) can be specialized to static bending, natural vibration, or buckling of simply supported rectangular plates. For the transient case, the ordinary differential equation in time, Eq. (8.148), must be solved subject to initial conditions on the deflection w_0 and velocity $\dot{w}_0$. We discuss these cases next for isotropic plates.

Bending For the static bending response under applied transverse load $q(x, y)$ and in-plane biaxial compressive forces $(\hat{N}_0, \gamma \hat{N}_0)$, the coefficients W_{mn} (which are constants) in the displacement expansion (8.145a) can be obtained from Eq. (8.148) by setting time derivative terms to zero:

$$W_{mn} = \frac{q_{mn}}{\left[D\left(\alpha_m^2 + \beta_n^2\right)^2 - \left(\alpha_m^2 + \gamma\beta_n^2\right)N_0\right]} \tag{8.149a}$$

and the static solution becomes

$$w_0(x, y) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} W_{mn} \sin\alpha_m x \sin\beta_n y, \tag{8.149b}$$

Note that the compressive force N_0 has the effect of increasing the deflection (or reduce the deflection when the in-plane forces are tensile).

When the in-plane compressive loads are zero, we have

$$W_{mn} = \frac{q_{mn}}{D\left(\alpha_m^2 + \beta_n^2\right)^2} = \frac{b^4}{D\pi^4}\frac{q_{mn}}{(m^2s^2 + n^2)^2}, \tag{8.150a}$$

where s is the plate aspect ratio, $s = b/a$, and the solution is

$$w_0(x, y) = \frac{b^4}{D\pi^4}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{q_{mn}}{(m^2s^2 + n^2)^2}\sin\alpha_m x \sin\beta_n y. \tag{8.150b}$$

The bending moments can be calculated from ($s = b/a$):

$$\begin{aligned}
M_{xx} &= D\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{\pi^2}{b^2}\left(m^2s^2 + \nu n^2\right)W_{mn}\sin\alpha_m x \sin\beta_n y,\\
M_{yy} &= D\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{\pi^2}{b^2}\left(\nu m^2s^2 + n^2\right)W_{mn}\sin\alpha_m x \sin\beta_n y,\\
M_{xy} &= -(1-\nu)\frac{s\pi^2}{b^2}D\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} mn W_{mn}\cos\alpha_m x \cos\beta_n y.
\end{aligned} \tag{8.151}$$

The shear forces Q_x and Q_y can be computed using

$$Q_x = \frac{\partial M_{xx}}{\partial x} + \frac{\partial M_{xy}}{\partial y} = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} S_{xx} W_{mn}\cos\alpha_m x \sin\beta_n y, \tag{8.152a}$$

$$Q_y = \frac{\partial M_{xy}}{\partial x} + \frac{\partial M_{yy}}{\partial y} = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} S_{yy} W_{mn}\sin\alpha_m x \cos\beta_n y, \tag{8.152b}$$

where

$$S_{xx} = D\left(\alpha_m^3 + \nu\alpha_m\beta_n^2\right), \qquad S_{yy} = D\left(\nu\alpha_m^3 + \alpha_m\beta_n^2\right). \tag{8.152c}$$

The effective shear forces (i.e., reaction forces) V_x and V_y along the simply supported edges $x = a$ and $y = b$, respectively, can be calculated using

$$V_x(a, y) = Q_x + \frac{\partial M_{xy}}{\partial y} = \frac{\partial M_{xx}}{\partial x} + 2\frac{\partial M_{xy}}{\partial y}$$

$$= \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} (-1)^m \hat{S}_{xx} W_{mn} \sin \beta_n y, \tag{8.153a}$$

$$V_y(x, b) = Q_y + \frac{\partial M_{xy}}{\partial x} = 2\frac{\partial M_{xy}}{\partial x} + \frac{\partial M_{yy}}{\partial y}$$

$$= \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} (-1)^n \hat{S}_{yy} W_{mn} \sin \alpha_m x, \tag{8.153b}$$

where

$$\hat{S}_{xx} = D\left[\alpha_m^3 + (2-\nu)\alpha_m \beta_n^2\right], \qquad \hat{S}_{yy} = D\left[\beta_n^3 + (2-\nu)\alpha_m^2 \beta_n\right]. \tag{8.153c}$$

Thus, the distribution of the reaction forces along the edges follows a sinusoidal form. Similar expressions hold for $V_x(0, y)$ and $V_y(x, 0)$.

In addition to the reactions in Eqs. (8.153a,b) along the edges, the plate experiences concentrated forces at the corners of a rectangular plate due to the twisting moment M_{xy} per unit length (which has the dimensions of a force). The concentrated force at the corner $x = a$ and $y = b$ is given by

$$F_c = -2M_{xy} = 2(1-\nu)D\frac{\partial^2 w_0}{\partial x \partial y}$$

$$= 2(1-\nu)D\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\left(\frac{m\pi}{a}\right)\left(\frac{n\pi}{b}\right)(-1)^{m+n} W_{mn}. \tag{8.154}$$

For the pure bending case considered here, the stresses in a simply supported rectangular plate are given by

$$\begin{Bmatrix}\sigma_{xx}\\ \sigma_{yy}\\ \sigma_{xy}\end{Bmatrix} = -\frac{Ez}{(1-\nu^2)}\begin{bmatrix}1 & \nu & 0\\ \nu & 1 & 0\\ 0 & 0 & \frac{1-\nu}{2}\end{bmatrix}\begin{Bmatrix}\frac{\partial^2 w_0}{\partial x^2}\\ \frac{\partial^2 w_0}{\partial y^2}\\ 2\frac{\partial^2 w_0}{\partial x \partial y}\end{Bmatrix},$$

$$= z\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} W_{mn}\begin{Bmatrix}R_{xx}\sin\alpha_m x\,\sin\beta_n y\\ R_{yy}\sin\alpha_m x\,\sin\beta_n y\\ -R_{xy}\cos\alpha_m x\,\cos\beta_n y\end{Bmatrix}, \tag{8.155a}$$

where

$$R_{xx} = \frac{\pi^2 E}{b^2(1-\nu^2)}\left(m^2 s^2 + \nu n^2\right),$$
$$R_{yy} = \frac{\pi^2 E}{b^2(1-\nu^2)}\left(\nu m^2 s^2 + n^2\right), \tag{8.155b}$$
$$R_{xy} = mns\,\frac{\pi^2 E}{b^2(1+\nu)}.$$

The maximum stresses occur at $(x, y, z) = (a/2, b/2, \pm h/2)$.

In the classical plate theory, the transverse stresses $(\sigma_{xz}, \sigma_{yz}, \sigma_{zz})$ are identically zero when computed from the constitutive equations because the transverse shear strains are zero. However, they can be computed using the 3D stress equilibrium equations for any $-h/2 \leq z \leq h/2$:

$$\sigma_{xz} = -\int_{-h/2}^{z}\left(\frac{\partial\sigma_{xx}}{\partial x} + \frac{\partial\sigma_{xy}}{\partial y}\right)dz + C_1(x, y),$$
$$\sigma_{yz} = -\int_{-h/2}^{z}\left(\frac{\partial\sigma_{xy}}{\partial x} + \frac{\partial\sigma_{yy}}{\partial y}\right)dz + C_2(x, y), \tag{8.156}$$
$$\sigma_{zz} = -\int_{-h/2}^{z}\left(\frac{\partial\sigma_{xz}}{\partial x} + \frac{\partial\sigma_{yz}}{\partial y}\right)dz + C_3(x, y),$$

where the stresses σ_{xx}, σ_{xy}, and σ_{yy} are known from Eq. (8.155a) and C_i are functions to be determined using the conditions $\sigma_{xz}(x, y, -h/2) = \sigma_{yz}(x, y, -h/2) = \sigma_{zz}(x, y, -h/2) = 0$. We obtain $C_i = 0$ and

$$\sigma_{xz} = \frac{h^2}{8}\left[1 - \left(\frac{2z}{h}\right)^2\right]\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} S_{mn}^{xz}\cos\frac{m\pi x}{a}\sin\frac{n\pi y}{b}, \tag{8.157a}$$

$$\sigma_{yz} = \frac{h^2}{8}\left[1 - \left(\frac{2z}{h}\right)^2\right]\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} S_{mn}^{yz}\sin\frac{m\pi x}{a}\cos\frac{n\pi y}{b}, \tag{8.157b}$$

$$\sigma_{zz} = -\frac{h^3}{48}\left\{\left[1 + \left(\frac{2z}{h}\right)^3\right] - 3\left[1 + \left(\frac{2z}{h}\right)\right]\right\} \times \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} S_{mn}^{zz}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}, \tag{8.157c}$$

where

$$S_{mn}^{xz} = S_{13}W_{mn}, \qquad S_{mn}^{yz} = S_{23}W_{mn}, \qquad S_{mn}^{zz} = S_{33}W_{mn}, \tag{8.158a}$$

with S_{ij} defined by

$$S_{13} = \frac{E}{(1-\nu^2)}\left(\alpha_m^3 + \alpha_m \beta_n^2\right),$$
$$S_{23} = \frac{E}{(1-\nu^2)}\left(\beta_n^3 + \alpha_m^2 \beta_n\right), \tag{8.158b}$$
$$S_{33} = \frac{E}{(1-\nu^2)}\left(\alpha_m^2 + \beta_n^2\right)^2.$$

Note that σ_{xz} and σ_{yz} are zero and $\sigma_{zz} = q$ at the top surface of the plate ($z = h/2$). The transverse shear stress σ_{xz} is the maximum at $(x, y, z) = (0, b/2, 0)$, σ_{yz} is the maximum at $(x, y, z) = (a/2, 0, 0)$, and the transverse normal stress σ_{zz} is the maximum at $(x, y, z) = (a/2, b/2, h/2)$.

Example 8.9 For an isotropic rectangular plate subjected to a uniformly distributed load, the deflection is given by

$$w_0(x,y) = \frac{16q_0b^4}{D\pi^6}\sum_{n=1,3,\ldots}^{\infty}\sum_{m=1,3,\ldots}^{\infty}\frac{1}{mn(m^2s^2+n^2)^2}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}. \tag{8.159}$$

The maximum of w_0 and M_{xx} are given by ($s = b/a$)

$$w_{max} = w_0\left(\frac{a}{2},\frac{b}{2}\right) = \frac{16q_0b^4}{D\pi^6}\sum_{n=1,3,\ldots}^{\infty}\sum_{m=1,3,\ldots}^{\infty}\frac{(-1)^{((m+n)/2)-1}}{mn(m^2s^2+n^2)^2}, \tag{8.160a}$$

$$(M_{xx})_{max} = M_{xx}\left(\frac{a}{2},\frac{b}{2}\right) = \frac{16q_0b^2}{D\pi^4}\sum_{n=1,3,\ldots}^{\infty}\sum_{m=1,3,\ldots}^{\infty}(-1)^{((m+n)/2)-1} \times \frac{(m^2s^2+\nu n^2)^2}{mn(m^2s^2+n^2)^2}. \tag{8.160b}$$

For a square plate ($b = a$) we have

$$w_{max} = \frac{16q_0a^4}{D\pi^6}\sum_{n=1,3,\ldots}^{\infty}\sum_{m=1,3,\ldots}^{\infty}\frac{(-1)^{((m+n)/2)-1}}{mn(m^2+n^2)^2}, \tag{8.161a}$$

$$(M_{xx})_{max} = \frac{16q_0b^2}{D\pi^4}\sum_{n=1,3,\ldots}^{\infty}\sum_{m=1,3,\ldots}^{\infty}(-1)^{((m+n)/2)-1}\frac{(m^2+\nu n^2)^2}{mn(m^2+n^2)^2}. \tag{8.161b}$$

A one-term solution is given by ($\nu = 0.3$)

$$w_{max} = \frac{4q_0a^4}{D\pi^6} = 0.00416\frac{q_0a^4}{D},$$

$$(M_{xx})_{max} = 0.05338q_0a^2, \qquad (\sigma_{xx})_{max} = 0.3203\frac{q_0a^2}{h^2}.$$

The deflection is about 2.4% in error compared to the solution obtained with $m, n = 1, 3, \ldots, 9$. Thus, the series in Eq. (8.161a) converges rapidly. The expressions for bending moments do not converge as rapidly. For $m, n = 1, 3, \ldots, 29$, we obtain

$$w_{max} = 0.004062\frac{q_0 a^4}{D}, \tag{8.162a}$$

$$(M_{xx})_{max} = 0.04789 q_0 a^2, \qquad (\sigma_{xx})_{max} = 0.2816\frac{q_0 a^2}{h^2}. \tag{8.162b}$$

For an isotropic rectangular plate under a point load Q_0 at (x_0, y_0), the deflection is given by

$$w_0(x, y) = \frac{4Q_0 b^2 s}{D\pi^4} \sum_{n=1,3,\ldots}^{\infty} \sum_{m=1,3,\ldots}^{\infty} \frac{\sin\dfrac{m\pi x_0}{a}\sin\dfrac{n\pi y_0}{b}}{(m^2 s^2 + n^2)^2} \sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}. \tag{8.163}$$

The center deflection when the load is applied at the center is given by

$$w_0\left(\frac{a}{2}, \frac{b}{2}\right) = \frac{4Q_0 b^2 s}{D\pi^4} \sum_{n=1,3,\ldots}^{\infty} \sum_{m=1,3,\ldots}^{\infty} \frac{1}{(m^2 s^2 + n^2)^2}. \tag{8.164}$$

In the case of a square plate, the center deflection becomes

$$w_{max} = \frac{4Q_0 a^2}{D\pi^4} \sum_{n=1,3,\ldots}^{\infty} \sum_{m=1,3,\ldots}^{\infty} \frac{1}{(m^2 + n^2)^2}. \tag{8.165}$$

The first term of the series yields ($\nu = 0.3$)

$$w_{max} = 0.01027\frac{Q_0 a^2}{D} = 0.1121\frac{Q_0 a^2}{Eh^3},$$

$$(M_{xx})_{max} = 0.1317 Q_0 a, \qquad (\sigma_{xx})_{max} = 0.7903\frac{Q_0 a}{h^2}.$$

Taking the first four terms (i.e., $m, n = 1, 3$) of the series, we obtain

$$w_{max} = 0.01121\frac{Q_0 a^2}{D} = 0.1225\frac{Q_0 a^2}{Eh^3},$$

$$(M_{xx})_{max} = 0.199 Q_0 a, \qquad (\sigma_{xx})_{max} = 1.194\frac{Q_0 a}{h^2}.$$

The deflection is about 3.4% in error compared to the solution 0.0116 $(Q_0 a^2/D)$ obtained using $m, n = 1, 3, \ldots, 19$.

Table 8.4 Transverse deflections and stresses in isotropic ($\nu = 0.3$) square and rectangular plates subjected to various types of loads

Load	$\bar{w}$	$\bar{\sigma}_{xx}$	$\bar{\sigma}_{yy}$	$\bar{\sigma}_{xy}$	$\bar{\sigma}_{xz}$	$\bar{\sigma}_{yz}$
			Square Plates ($s = b/a = 1$)			
SL	0.0280	0.1976	0.1976	0.1064	0.2387	0.2387
UL (19)[a]	0.0444	0.2873	0.2873	0.1946	0.4909	0.4909
HL (19)	0.0222	0.1436	0.1436	0.0775	0.2455	0.1353
PL (29)	0.1266	2.4350	2.4350	0.3658	1.0010	1.0010
			Rectangular Plates ($s = b/a = 3$)			
SL	0.0908	0.5088	0.2024	0.1149	0.4297	0.1432
UL (19)	0.1336	0.7130	0.2433	0.2830	0.7221	0.5110
HL (19)	0.0668	0.3565	0.1217	0.0579	0.3610	0.0636
PL (29)	0.1845	2.3523	1.8828	0.0566	0.9032	0.2257

[a]The number in parentheses denotes the maximum values of m and n used to evaluate the series.

Table 8.4 contains the nondimensionalized maximum transverse deflections and stresses of square and rectangular plates under various types of loads. The transverse deflection and stresses are nondimensionalized as follows:

$$\bar{w} = w_0(0,0)\left(\frac{Eh^3}{a^4 q_0}\right); \qquad \bar{\sigma}_{xx} = \sigma_{xx}(a/2, b/2, h/2)\left(\frac{h^2}{a^2 q_0}\right),$$

$$\bar{\sigma}_{yy} = \sigma_{yy}(a/2, b/2, h/2)\left(\frac{h^2}{a^2 q_0}\right); \qquad \bar{\sigma}_{xy} = \sigma_{xy}(a, b, -h/2)\left(\frac{h^2}{a^2 q_0}\right),$$

$$\bar{\sigma}_{xz} = \sigma_{xz}(0, b/2, 0)\left(\frac{h}{a q_0}\right); \qquad \bar{\sigma}_{yz} = \sigma_{yz}(a/2, 0, 0)\left(\frac{h}{a q_0}\right). \tag{8.166}$$

The loads considered are sinusoidal (SL), uniform (UL), hydrostatic (HL), or central point load (PL). Since convergence for derivatives of a function is slower than the function itself, the convergence is slower for stresses, which are calculated using the derivatives of the deflection. In the case of the point load, convergence for stresses will not be reached due to the stress singularity at the center of the plate.

Natural Vibration For natural vibration, the deflection is assumed to be periodic in time:

$$w_0(x, y, t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} W_{mn}^0 \sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}\, e^{i\omega_{mn}t} \tag{8.167a}$$

or

$$W_{mn}(t) = W_{mn}^0 e^{i\omega_{mn}t}, \tag{8.167b}$$

where $i = \sqrt{-1}$ and ω_{mn} is the frequency of natural vibration associated with the (m, n)th mode shape (W^0_{mn} is the amplitude of vibration):

$$\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}. \tag{8.168}$$

Substituting (8.167b) into Eq. (8.148), we obtain, for nonzero W^0_{mn}, the result

$$D\left(\alpha_m^2+\beta_n^2\right)^2-\left(\alpha_m^2+\gamma\beta_n^2\right)N_0-\omega_{mn}^2\left[I_0+\left(\alpha_m^2+\beta_n^2\right)I_2\right]=0, \tag{8.169}$$

where $\alpha_m = m\pi/a$ and $\beta_n = n\pi/b$. Solving Eq. (8.169) for the natural frequency, we obtain

$$\omega_{mn}^2=\frac{\pi^2}{\tilde{I}_0 b^2}\left[\frac{\pi^2 D}{b^2}\left(m^2s^2+n^2\right)^2-N_0\left(m^2s^2+\gamma n^2\right)\right],\qquad s=\frac{b}{a}, \tag{8.170}$$

where

$$\tilde{I}_0=I_0+I_2\left[\left(\frac{m\pi}{a}\right)^2+\left(\frac{n\pi}{b}\right)^2\right]. \tag{8.171}$$

For different values of m and n, there corresponds a unique frequency ω_{mn} and mode shape given by Eq. (8.168). The in-plane compressive force as well as the rotatory (or rotary) inertia I_2 have the effect of reducing the magnitude of the frequency of vibration and their relative effect depends on m and n, and the plate thickness-to-side ratio h/b. For most plates with $h/b < 0.1$, the rotary inertia may be neglected.

The smallest value of ω_{mn} is called the *fundamental frequency*. When the rotatory inertia I_2 is neglected, the frequency of a rectangular isotropic plate without in-plane forces N_0 reduces to ($I_0 = \rho h$)

$$\omega_{mn}=\frac{\pi^2}{b^2}\sqrt{\frac{D}{\rho h}}\left(m^2s^2+n^2\right),\qquad s=\frac{b}{a}. \tag{8.172}$$

For a square isotropic plate without rotary inertia, the frequency is

$$\omega_{mn}=\frac{\pi^2}{a^2}\sqrt{\frac{D}{\rho h}}\left(m^2+n^2\right). \tag{8.173}$$

The fundamental frequency of a square isotropic plate is

$$\omega_{11}=\frac{2\pi^2}{a^2}\sqrt{\frac{D}{\rho h}}. \tag{8.174}$$

Table 8.5 contains nondimensionalized frequencies $\bar{\omega}_{mn}$ of isotropic ($\nu = 0.25$) plates for $(m, n) = (1, 1)$, $(1, 2)$, and $(2, 1)$. The effect of rotary inertia is negligible for plates with $h/b < 0.1$.

Table 8.5 Nondimensionalized natural frequencies $\bar{\omega}_{mn}$ of simply supported isotropic plates[a] [$\bar{\omega}_{mn} = \omega_{mn}(b^2/\pi^2)\sqrt{\rho h/D}$]

	$\bar{\omega}_{11}$			$\bar{\omega}_{12}$			$\bar{\omega}_{21}$		
b/a	w/o	0.01	0.1	w/o	0.01	0.1	w/o	0.01	0.1
0.5	1.250	1.250	1.249	4.250	4.249	4.243	2.000	2.000	1.998
1.0	2.000	2.000	1.998	5.000	4.999	4.990	5.000	4.999	4.990
1.5	3.250	3.250	3.246	6.250	6.248	6.234	10.000	9.996	9.959
2.0	5.000	4.999	4.990	8.000	7.997	7.974	17.000	16.988	16.882
2.5	7.250	7.248	7.228	10.250	10.246	10.207	26.000	25.972	25.726
3.0	10.000	9.996	9.959	13.000	12.993	12.931	37.000	36.944	36.450

[a] w/o = without rotary inertia; the second and third columns contain frequencies when the rotary inertia is included for $h/b = 0.01$ and 0.1, respectively.

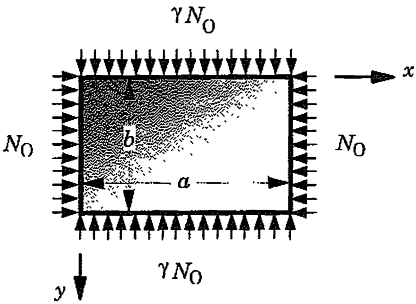


Figure 8.14 Biaxial compression of a rectangular plate ($\hat{N}_{xx} = N_0$ and $N_{yy}^0 = \gamma N_0$).

Buckling Analysis For buckling of rectangular plates under in-plane biaxial compressive loads (see Fig. 8.14), Eq. (8.148) reduces, after omitting the time derivative and load terms, to

$$\left[D\left(\alpha_m^2 + \beta_n^2\right)^2 - \left(\alpha_m^2 + \gamma\beta_n^2\right)N_0\right] W_{mn} = 0,$$

which gives, for nonzero W_{mn}, the result

$$N_0(m,n) = \frac{\pi^2 D}{b^2}\frac{(s^2m^2 + n^2)^2}{s^2m^2 + \gamma n^2}, \tag{8.175}$$

where s is the plate aspect ratio $s = (b/a)$. For each choice of m and n there corresponds a unique value of N_0. The *critical buckling load* is the smallest of $N_0(m,n)$. For a given plate this value is dictated by a particular combination of the values of m and n, the value of γ, plate geometry, and material properties. Next we present critical buckling loads for various cases.

Biaxial Compression of a Plate For a rectangular isotropic plate under biaxial compression ($\gamma = 1$), the buckling load is

$$N_0(m, n) = \frac{\pi^2 D}{b^2}\left(m^2 s^2 + n^2\right), \tag{8.176}$$

where s is the plate aspect ratio $s = b/a$. Clearly, the critical buckling load occurs at $m = n = 1$ and it is equal to

$$N_{cr} = (1 + s^2)\frac{\pi^2 D}{b^2}, \qquad s = \frac{b}{a}, \tag{8.177a}$$

and for a square plate it reduces to

$$N_{cr} = \frac{2\pi^2 D}{b^2}. \tag{8.177b}$$

Biaxial Loading of a Plate When the edges $x = 0, a$ of a square plate are subjected to compressive load $\hat{N}_{xx} = N_0$ and the edges $y = 0, b$ are subjected to tensile load $\hat{N}_{yy} = -\gamma N_0$, Eq. (8.175) becomes

$$N_0(m, n) = \frac{\pi^2 D}{a^2}\left[\frac{(m^2 + n^2)^2}{m^2 - \gamma n^2}\right], \tag{8.178}$$

when $\gamma n^2 < m^2$. The minimum buckling load occurs for $n = 1$:

$$N_0(m, 1) = \left(\frac{\pi^2 D}{b^2}\right)\frac{(m^2 s^2 + 1)^2}{m^2 s^2 - \gamma}. \tag{8.179a}$$

In theory, the minimum of $N_0(m, 1)$ occurs when $m^2 s^2 = 1 + 2\gamma$. For a square plate with $\gamma = 0.5$, we find

$$N_0(1, 1) = \frac{8\pi^2 D}{a^2}, \qquad N_0(2, 1) = 7.1429\frac{\pi^2 D}{a^2} = N_{cr}. \tag{8.179b}$$

Uniaxial Compression of a Rectangular Plate When a rectangular plate is subjected to uniform compressive load N_0 on edges $x = 0$ and $x = a$, i.e., when $\gamma = 0$, the buckling load is given by

$$N_0(m, n) = \frac{\pi^2 a^2 D}{m^2}\left(\frac{m^2}{a^2} + \frac{n^2}{b^2}\right)^2, \tag{8.180}$$

$$N_0(m, 1) = \frac{\pi^2 D}{a^2}\left(m + \frac{1}{m}\frac{a^2}{b^2}\right)^2. \tag{8.181}$$

Table 8.6 Effect of plate aspect ratio on the nondimensionalized buckling loads $\bar{N}$ of simply supported (SSSS) rectangular plates under uniform axial compression ($\gamma = 0$) and biaxial compression ($\gamma = 1$)

$\gamma = 0$				$\gamma = 1$			
a/b	$\bar{N}$	a/b	$\bar{N}$	a/b	$\bar{N}$	a/b	$\bar{N}$
0.5	6.250	2.0	4.000$^{(2,1)}$	0.5	5.000	2.0	1.250
1.0	4.000	2.5	4.134$^{(3,1)}$	1.0	2.000	2.5	1.160
1.5	4.340$^{(2,1)}$[a]	3.0	4.000$^{(3,1)}$	1.5	1.444	3.0	1.111

[a] Denotes mode numbers (m, n) at which the critical buckling load occurred; $(m, n) = (1, 1)$ for all other cases.

For a given aspect ratio, two different modes, m_1 and m_2, will have the same buckling load when $\sqrt{m_1 m_2} = a/b$. In particular, the point of intersection of curves m and $m + 1$ occurs for aspect ratios

$$\frac{a}{b} = \sqrt{2}, \sqrt{6}, \sqrt{12}, \sqrt{20}, \ldots, \sqrt{m^2 + m}.$$

Thus, there is a mode change at these aspect ratios from m half-waves to $m + 1$ half-waves. Putting $m = 1$ in Eq. (8.181), we find

$$N_{cr} = \frac{\pi^2 D}{b^2} \left(\frac{a}{b} + \frac{b}{a} \right)^2 . \tag{8.182}$$

For a square plate we obtain

$$N_{cr} = \frac{4\pi^2 D}{b^2} . \tag{8.183}$$

Table 8.6 shows the effect of plate aspect ratio and modulus ratio (orthotropy) on the critical buckling loads $\bar{N} = N_{cr} b^2/(\pi^2 D)$ of rectangular isotropic ($\nu = 0.3$) plates under uniform axial compression ($\gamma = 0$) and biaxial compression ($\gamma = 1$). In all cases, the critical buckling mode is $(m, n) = (1, 1)$, except as indicated.

Transient Analysis The determination of the solution $w_0(x, y, t)$ of Eq. (8.143a) or (8.143b) for all times $t > 0$ under an applied load $q(x, y, t)$ and known initial conditions

$$w_0(x, y, 0) = d_0(x, y), \qquad \frac{\partial w_0}{\partial t}(x, y, 0) = v_0(x, y) \quad \text{for all } x \text{ and } y, \tag{8.184}$$

where d_0 and v_0 are the initial displacement and velocity, respectively, is termed the *transient response*. The transient response of simply supported rectangular plates can be determined by assuming a solution of the form in Eq. (8.145a) and determining the solution $W_{mn}(t)$ of the ordinary differential equation in (8.148). It is necessary to expand the nonzero initial displacement and velocity field in double sine series:

$$d_0(x, y) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} D_{mn} \sin \alpha_m x \sin \beta_n y, \tag{8.185a}$$

$$v_0(x, y) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} V_{mn} \sin \alpha_m x \sin \beta_n y, \tag{8.185b}$$

where $\alpha_m = m\pi/a$, $\beta_n = n\pi/b$, and D_{mn} and V_{mn} are given by

$$D_{mn} = \frac{4}{ab}\int_0^b \int_0^a d(x, y) \sin \alpha_m x \sin \beta_n y \, dxdy, \tag{8.186a}$$

$$V_{mn} = \frac{4}{ab}\int_0^b \int_0^a v(x, y) \sin \alpha_m x \sin \beta_n y \, dxdy. \tag{8.186b}$$

Equation (8.148) has the general form

$$K_{mn} W_{mn}(t) + M_{mn}\frac{d^2 W_{mn}}{dt^2} = q_{mn}(t), \tag{8.187a}$$

where

$$K_{mn} = D\left(\alpha_m^2 + \beta_n^2\right)^2 - N_0\left(\alpha_m^2 + \gamma\beta_n^2\right), \qquad M_{mn} = I_0 + I_2\left(\alpha_m^2 + \beta_n^2\right). \tag{8.187b}$$

The second-order differential equation (8.187a) can be solved either exactly or numerically. The numerical solutions are often based on the finite difference methods.

To solve Eq. (8.187a) exactly, we first write it in the form

$$\frac{d^2 W_{mn}}{dt^2} + \left(\frac{K_{mn}}{M_{mn}}\right) W_{mn} = \frac{1}{M_{mn}} q_{mn}(t) \equiv \hat{q}_{mn}(t). \tag{8.188}$$

The solution of Eq. (8.188) is given by

$$W_{mn}(t) = C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t} + W_{mn}^p(t), \tag{8.189}$$

where C_1 and C_2 are constants to be determined using the initial conditions, $W_{mn}^p(t)$ is the particular solution

$$W_{mn}^p(t) = \int^t \frac{r_1(\tau) r_2(t) - r_1(t) r_2(\tau)}{r_1(\tau)\dot{r}_2(\tau) - \dot{r}_1(\tau) r_2(\tau)} \hat{q}_{mn}(\tau)\, d\tau, \tag{8.190a}$$

with $r_1(t) = e^{\lambda_1 t}$ and $r_2(t) = e^{\lambda_2 t}$, and λ_1 and λ_2 are the roots of the equation

$$\lambda^2 + \frac{K_{mn}}{M_{mn}} = 0; \qquad \lambda_1 = -i\mu,\ \lambda_2 = i\mu,\ i = \sqrt{-1},\ \mu = \sqrt{\frac{K_{mn}}{M_{mn}}}. \tag{8.190b}$$

The solution becomes

$$W_{mn}(t) = A\cos\mu t + B\sin\mu t + W_{mn}^p(t), \tag{8.191a}$$

$$W_{mn}^p(t) = \frac{1}{2i\mu}\left(e^{i\mu t}\int^t e^{-i\mu\tau}\hat{q}_{mn}(\tau)\,d\tau - e^{-i\mu t}\int^t e^{i\mu\tau}\hat{q}_{mn}(\tau)\,d\tau\right). \tag{8.191b}$$

Once the load distribution is known, the solution can be determined from Eq. (8.189).

For a step loading, $q_{mn}(t) = q_{mn}^0 H(t)$, where $H(t)$ denotes the Heaviside step function, Eq. (8.191a) takes the form

$$W_{mn}(t) = A_{mn}\cos\mu t + B_{mn}\sin\mu t + \frac{1}{K_{mn}}q_{mn}^0. \tag{8.192a}$$

Using the initial conditions (8.184), we obtain

$$A_{mn} = D_{mn} - \frac{1}{K_{mn}}q_{mn}^0, \qquad B_{mn} = \frac{V_{mn}}{\mu}. \tag{8.192b}$$

Thus the final solution is given by

$$w_0(x, y, t) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\left[D_{mn}\cos\mu t + \frac{V_{mn}}{\mu}\sin\mu t + \frac{q_{mn}^0}{K_{mn}}(1-\cos\mu t)\right]$$
$$\times \sin\alpha_m x\sin\beta_n y. \tag{8.193}$$

The coefficients q_{mn}^0 were given in Table 8.3 for various types of load distributions. The same holds for D_{mn} and V_{mn}.

The exact solution of the differential equation (8.188) can also be obtained using the Laplace transform method.

8.2.5 Lévy Solutions of Rectangular Plates

The deflection of a rectangular plate with two opposite edges ($x = 0, a$) simply supported and the other two edges ($-b/2 \le y \le b/2$) having arbitrary boundary conditions (see Fig. 8.15) can be represented in the form (see the semidiscretization method of Examples 7.17–7.19):

$$w_0(x, y) = \sum_{m=1}^{\infty} W_m(y)\sin\alpha_m x, \tag{8.194}$$

which satisfies the simply supported boundary conditions on edges $x = 0$ and $x = a$. Substituting the expansion (8.194) into the equilibrium equation of the plate,

$$D\left(\frac{\partial^4 w_0}{\partial x^4} + 2\frac{\partial^4 w_0}{\partial x^2\partial y^2} + \frac{\partial^4 w_0}{\partial y^4}\right) = q, \tag{8.195}$$

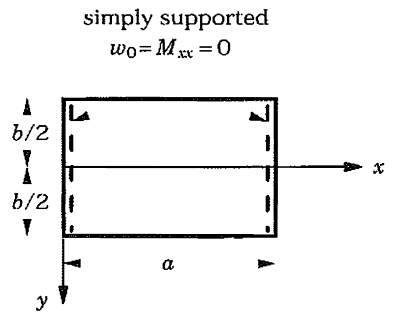


Figure 8.15 Geometry and coordinate system for a rectangular plate with sides $x = 0$, a simply supported.

we obtain an ordinary differential equation for $W_m(y)$:

$$D\left(\alpha_m^4 W_m - 2\alpha_m^2 \frac{d^2 W_m}{dy^2} + \frac{d^4 W_m}{dy^4}\right) = q_m, \tag{8.196}$$

where q_m is defined by

$$q_m(y) = \frac{2}{a}\int_0^a q(x, y) \sin \alpha_m x \, dx, \qquad \alpha_m = \frac{m\pi}{a}. \tag{8.197}$$

Table 8.7 contains the coefficients q_m for various types of loads. The solution of Eq. (8.196) may be determined exactly or numerically. Here we discuss its exact solution. One may also solve it using the Ritz method (see Reddy [8]).

The solution W_m consists of the homogeneous solution W_m^h and particular solution W_m^p. The homogeneous solution of Eq. (8.196) is

$$W_m^h(y) = (A_m + B_m y) \cosh \alpha_m y + (C_m + D_m y) \sinh \alpha_m y. \tag{8.198}$$

The particular solution W_m^p is given by the solution of

$$\alpha_m^4 W_m^p - 2\alpha_m^2 \frac{d^2 W_m^p}{dy^2} + \frac{d^4 W_m^p}{dy^4} = \frac{q_m}{D}. \tag{8.199}$$

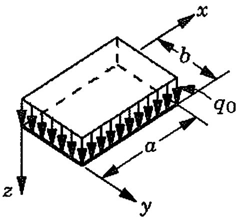
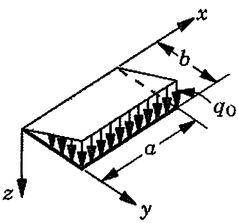
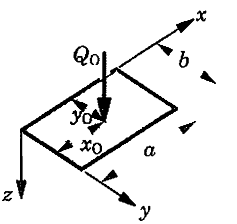
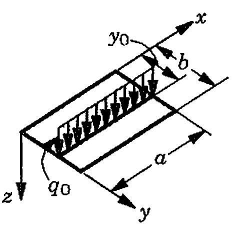
Assuming solution of Eq. (8.199) in the form

$$W_m^p(y) = \sum_{n=1}^{\infty} W_{mn} \sin \beta_n y, \qquad \beta_n = \frac{n\pi}{b}, \tag{8.200}$$

expanding the load q_m also in the same form:

$$q_m(y) = \sum_{n=1}^{\infty} q_{mn} \sin \beta_n y, \tag{8.201}$$

Table 8.7 Coefficients in the single trigonometric series expansion of loads in the Lévy method

Loading	Coefficients, $q_n(x)$
	**Uniform load** $q = q_0$ $q_n = \dfrac{4q_0}{n\pi}$ $(n = 1, 3, 5, \ldots)$
	**Hydrostatic load** $q(x, y) = q_0 \dfrac{y}{b}$ $q_n = \dfrac{2q_0}{n\pi}(-1)^{n+1}$ $(n = 1, 2, 3, \ldots)$
	**Point load** Q_0 at (x_0, y_0) $q_n = \dfrac{2Q_0}{b}\delta(x - x_0)\sin\dfrac{n\pi y_0}{b}$ $(n = 1, 2, 3, \ldots)$
	**Line load** $q(x, y) = q_0\delta(y - y_0)$ $q_n = \dfrac{2q_0}{b}\sin\dfrac{n\pi y_0}{b}$ $(n = 1, 2, 3, \ldots)$

and substituting both into Eq. (8.199), we obtain

$$W_{mn} = \frac{q_{mn}}{d_{mn}}, \qquad d_{mn} = D\left(\alpha_m^2 + \beta_n^2\right)^2. \tag{8.202}$$

The particular solution becomes

$$W_m^p(y) = \sum_{n=1}^{\infty} \frac{q_{mn}}{d_{mn}} \sin\frac{n\pi y}{b}, \tag{8.203}$$

and the complete solution becomes

$$w_0(x, y) = \sum_{m=1}^{\infty} \left[(A_m + B_m y) \cosh \alpha_m y + (C_m + D_m y) \sinh \alpha_m y + W_m^p \right] \sin \alpha_m x, \tag{8.204}$$

where W_m^p is given by Eq. (8.203). The constants A_m, B_m, C_m, and D_m must be determined for the particular set of boundary conditions on edges $y = \pm b/2$.

Example 8.10 Consider a rectangular plate simply supported on all four edges and subjected to distributed bending moments along edges $y = \pm b/2$ (see Fig. 8.16):

$$f(x) = \sum_{m=1}^{\infty} M_m \sin \alpha_m x, \qquad \alpha_m = \frac{m\pi}{a}, \tag{8.205}$$

with the coefficients M_m known from

$$M_m = \frac{2}{a} \int_0^a f(x) \sin \alpha_m x \, dx. \tag{8.206}$$

We wish to determine the deflections and bending moments throughout the plate using the Lévy solution procedure. The general solution (8.204) for rectangular plates with simply supported edges $x = 0, a$ is valid here except that the distributed load is zero: $q = 0$ (hence, the particular solution is zero).

The boundary conditions on edges $y = \pm b/2$ are

$$w_0 = 0, \qquad M_{yy} = f(x). \tag{8.207a}$$

Since $w_0 = 0$ on $y = \pm b/2$, the condition $M_{yy} = f$ on $y = \pm b/2$ reduces to

$$-D \frac{\partial^2 w_0}{\partial y^2} = f(x). \tag{8.207b}$$

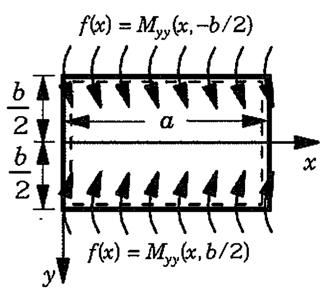


Figure 8.16 A simply supported rectangular plate with distributed moments along edges $y = \pm b/2$.

Due to the symmetry of the problem about the x axis, $w_0(x, y)$ must be an even function of y. This implies that $B_m = C_m = 0$, and Eq. (8.204) becomes

$$w_0(x, y) = \sum_{m=1}^{\infty} (A_m \cosh \alpha_m y + D_m y \sinh \alpha_m y) \sin \alpha_m x. \tag{8.208}$$

Using the boundary conditions (8.207a), we obtain

$$A_m = -\frac{b}{2} D_m \tanh \hat{\alpha}_m, \qquad D_m = -\frac{a M_m}{2m\pi D \cosh \hat{\alpha}_m}, \qquad \hat{\alpha}_m = \alpha_m \frac{b}{2}. \tag{8.209}$$

The deflection becomes

$$w_0(x, y) = \frac{a}{2\pi D} \sum_{m=1}^{\infty} M_m \frac{\sin \alpha_m x}{m \cosh \hat{\alpha}_m} \left(\frac{b}{2} \tanh \hat{\alpha}_m \cosh \alpha_m y - y \sinh \alpha_m y \right). \tag{8.210}$$

The distribution of bending moments for the problem can be computed using the definitions in Eq. (8.123a).

Nondimensional deflections and bending moments of simply supported rectangular plates with uniformly distributed moments of intensity $f = M_0$ (or $M_m = 4M_0/m\pi$, $m = 1, 3, 5, \ldots$) are presented in Table 8.8 for various aspect ratios. The results were computed using $m = 1, 3, \ldots, 9$. The nondimensionalizations used are as follows:

$$\begin{aligned} &\text{For } \frac{b}{a} < 1: \quad \bar{w} = w_0(a/2, 0)\frac{D}{M_0 b^2}, \qquad \bar{M} = M(a/2, 0)\frac{1}{M_0}, \\ &\text{For } \frac{b}{a} \geq 1: \quad \bar{w} = w_0(a/2, 0)\frac{D}{M_0 a^2}, \qquad \bar{M} = M(a/2, 0)\frac{1}{M_0}. \end{aligned} \tag{8.211}$$

Example 8.11 Consider a rectangular plate with edges $x = 0, a$ simply supported and edges $y = \pm b/2$ clamped, and subjected to distributed transverse load (see Fig. 8.17). The clamped boundary conditions on edges $y = \pm b/2$ are

$$w_0 = 0, \qquad \frac{\partial w_0}{\partial y} = 0. \tag{8.212}$$

Table 8.8 Nondimensionalized center deflections and bending moments of simply supported plates with applied edge moment along $y = \pm b/2$

Variable	$b/a \rightarrow$	0.5	0.75	1.0	1.5	2.0	3.0
$\bar{w}$		0.0965	0.0620	0.0368	0.0280	0.0174	0.0055
$\bar{M}_{xx}$		0.3873	0.4240	0.3938	0.2635	0.1530	0.0446
$\bar{M}_{yy}$		0.7701	0.4764	0.2562	0.0465	−0.0103	−0.0148

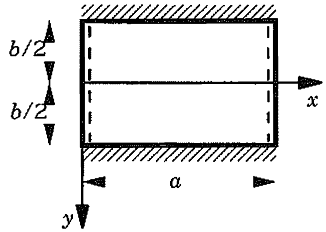


Figure 8.17 Geometry and coordinate system for a rectangular plate with sides $x = 0, a$ simply supported and $y = \pm b/2$ clamped.

The general solution is given by

$$w_0(x, y) = \sum_{m=1}^{\infty} \left(A_m \cosh \alpha_m y + D_m y \sinh \alpha_m y + \sum_{n=1}^{\infty} W_{mn} \sin \beta_n y \right) \sin \alpha_m x, \tag{8.213}$$

and its derivative is

$$\frac{\partial w_0}{\partial y} = \sum_{m=1}^{\infty} \left(\alpha_m A_m \sinh \alpha_m y + D_m \sinh \alpha_m y + D_m \alpha_m y \cosh \alpha_m y \right.$$
$$\left. + \sum_{n=1}^{\infty} \beta_n W_{mn} \cos \beta_n y \right) \sin \alpha_m x. \tag{8.214}$$

Boundary conditions in Eq. (8.212) give

$$A_m = -\frac{b}{2} D_m \tanh \hat{\alpha}_m - Q_m, \qquad D_m = -\frac{a M_m}{2m\pi D \cosh \hat{\alpha}_m},$$
$$Q_m = \frac{\hat{q}_m}{\cosh \hat{\alpha}_m}, \qquad \hat{q}_m = \sum_{n=1}^{\infty} (-1)^n W_{mn},$$
$$M_m = \hat{\alpha}_m + (1 - \hat{\alpha}_m \tanh \hat{\alpha}_m) \tanh \hat{\alpha}_m,$$
$$\alpha_m = \frac{m\pi}{a}, \qquad \beta_n = \frac{n\pi}{b}, \qquad \hat{\alpha}_m = \frac{b}{2}\alpha_m, \tag{8.215}$$

where W_{mn} is defined in Eq. (8.202):

$$W_{mn} = \frac{q_{mn}}{D\left(\alpha_m^2 + \beta_n^2\right)^2},$$

and q_{mn} are defined in Table 8.3 for various types of loads.

Table 8.9 contains nondimensionalized deflections and bending moments,

$$\bar{w} = w_0 \left(\frac{D}{q_0 a^4} \right) \times 10^2, \qquad \bar{M} = M \left(\frac{1}{q_0 a^2} \right) \times 10, \tag{8.216}$$

Table 8.9 Maximum nondimensional deflections and bending moments in isotropic ($\nu = 0.3$) rectangular plates with edges $x = 0, a$ simply supported and edges $y = \pm b/2$ clamped, and subjected to distributed loads

Variable	$\frac{b}{a}=3$	$\frac{b}{a}=2$	$\frac{b}{a}=1$	$\frac{a}{b}=\frac{3}{2}$	$\frac{a}{b}=2$	$\frac{a}{b}=3$
			Uniform load, $q = q_0$			
$\bar{w}(a/2,0)$	1.1681	0.8445	0.1917	0.2476	0.2612	0.2619
$\bar{M}_{xx}(a/2,0)$	1.1442	0.8693	0.2445	0.1794	0.1441	0.1302
$\bar{M}_{yy}(a/2,0)$	0.4214	0.4738	0.3326	0.4067	0.4214	0.4201
$-\bar{M}_{xx}(a/2,b/2)$	0.3740	0.3574	0.2097	0.2470	0.2535	0.2525
$-\bar{M}_{yy}(a/2,b/2)$	1.2467	1.1915	0.6990	0.8233	0.8451	0.8417
			Hydrostatic load, $q = q_0(x/a)$			
$\bar{w}(a/2,0)$	0.5841	0.4222	0.0959	0.1238	0.1306	0.1309
$\bar{M}_{xx}(a/2,0)$	0.5721	0.4346	0.1222	0.0897	0.0721	0.0651
$\bar{M}_{yy}(a/2,0)$	0.2107	0.2369	0.1663	0.2033	0.2107	0.2100
$-\bar{M}_{xx}(a/2,b/2)$	0.1870	0.1787	0.1048	0.1235	0.1267	0.1263
$-\bar{M}_{yy}(a/2,b/2)$	0.6233	0.5957	0.3495	0.4117	0.4225	0.4209
$\bar{w}(3a/4,0)$	0.4368	0.3218	0.0841	0.1313	0.1614	0.1884
$\bar{M}_{xx}(3a/4,b/2)$	0.5085	0.4097	0.1640	0.1665	0.1509	0.1153
$\bar{M}_{yy}(3a/4,b/2)$	0.1803	0.1997	0.1562	0.2276	0.2705	0.3055
$-\bar{M}_{xx}(a/2,3b/4)$	0.1635	0.1576	0.1042	0.1443	0.1670	0.1842
$-\bar{M}_{yy}(a/2,3b/4)$	0.5449	0.5254	0.3475	0.4810	0.5566	0.6140

$$\bar{w} = w_0\left(\frac{D}{q_0 b^4}\right) \times 10^2, \qquad \bar{M} = M\left(\frac{1}{q_0 b^2}\right) \times 10, \tag{8.217}$$

for various aspect ratios. The first set of nondimensionalizations is used for $b \geq a$ (the first three columns of values) and the second set is used for $a > b$ (the last three columns of values). The values are obtained using the first five terms of the series ($m = 1, 3, \ldots, 9$).

8.2.6 Variational Solutions: Bending

Here we discuss application of the Ritz method to the bending of plates with various boundary conditions. The virtual work statement for an orthotropic rectangular plate is

$$\begin{aligned} 0 = \int_\Omega \Bigg[& D_{11}\frac{\partial^2 w_0}{\partial x^2}\frac{\partial^2 \delta w_0}{\partial x^2} + D_{12}\left(\frac{\partial^2 w_0}{\partial y^2}\frac{\partial^2 \delta w_0}{\partial x^2} + \frac{\partial^2 w_0}{\partial x^2}\frac{\partial^2 \delta w_0}{\partial y^2}\right) \\ & + 4D_{66}\frac{\partial^2 w_0}{\partial x \partial y}\frac{\partial^2 \delta w_0}{\partial x \partial y} + D_{22}\frac{\partial^2 w_0}{\partial y^2}\frac{\partial^2 \delta w_0}{\partial y^2} - q\delta w_0\Bigg] dxdy \\ & - \oint_\Gamma \left(-\hat{M}_{nn}\frac{\partial \delta w_0}{\partial n} + \hat{V}_n \delta w_0\right) ds, \end{aligned} \tag{8.218}$$

where $\hat{M}_{nn}$ and $\hat{V}_n$ are the applied edge moment and effective shear force, respectively, on the boundary Γ of the domain Ω (the midplane of the plate).

We seek an N-parameter Ritz solution in the form

$$w_0(x, y) \approx \sum_{j=1}^{N} c_j \phi_j(x, y), \tag{8.219}$$

and substitute it into Eq. (8.218) to obtain

$$[R]\{c\} = \{F\}, \tag{8.220}$$

where

$$R_{ij} = \int_\Omega \left[D_{11} \frac{\partial^2 \phi_i}{\partial x^2} \frac{\partial^2 \phi_j}{\partial x^2} + D_{12} \left(\frac{\partial^2 \phi_i}{\partial y^2} \frac{\partial^2 \phi_j}{\partial x^2} + \frac{\partial^2 \phi_i}{\partial x^2} \frac{\partial^2 \phi_j}{\partial y^2} \right) \right.$$
$$\left. + 4D_{66} \frac{\partial^2 \phi_i}{\partial x \partial y} \frac{\partial^2 \phi_j}{\partial x \partial y} + D_{22} \frac{\partial^2 \phi_i}{\partial y^2} \frac{\partial^2 \phi_j}{\partial y^2} \right] dxdy, \tag{8.221a}$$

$$F_i = \int_\Omega q \phi_i \, dxdy + \oint_\Gamma \left(\hat{V}_n \phi_i - \hat{M}_{nn} \frac{\partial \phi_i}{\partial n} \right) ds. \tag{8.221b}$$

For rectangular plates, it is convenient to express the Ritz approximation in the form

$$w_0(x, y) \approx W_N(x, y) = \sum_{i=1}^{N} \sum_{j=1}^{N} c_{ij} \, \phi_{ij}(x, y) = \sum_{i=1}^{N} \sum_{j=1}^{N} c_{ij} X_i(x) Y_j(y), \tag{8.222}$$

where $\phi_{ij}(x, y)$ is expressed as a tensor product of the one-dimensional functions X_i and Y_j of x and y, respectively. Substituting Eq. (8.222) into Eq. (8.218) ($\delta w_{0,n} = \delta w_0 = 0$ or $M_{nn} = 0$, and $\hat{V}_n = 0$ on Γ), we obtain

$$0 = \sum_{i=1}^{N} \sum_{j=1}^{N} \left\{ \int_0^b \int_0^a \left[D_{11} \frac{d^2 X_i}{dx^2} Y_j \frac{d^2 X_p}{dx^2} Y_q + 4D_{66} \frac{dX_i}{dx} \frac{dY_j}{dy} \frac{dX_p}{dx} \frac{dY_q}{dy} \right. \right.$$
$$+ D_{12} \left(X_i \frac{d^2 Y_j}{dy^2} \frac{d^2 X_p}{dx^2} Y_q + \frac{d^2 X_i}{dx^2} Y_j X_p \frac{d^2 Y_q}{dy^2} \right)$$
$$\left. \left. + D_{22} X_i \frac{d^2 Y_j}{dy^2} X_p \frac{d^2 Y_q}{dy^2} \right] dxdy \right\} c_{ij} - \int_0^b \int_0^a q X_p Y_q \, dxdy$$
$$\equiv R_{(pq)(ij)} c_{ij} - F_{pq}, \tag{8.223}$$

where

$$R_{(pq)(ij)} = \int_0^b \int_0^a \left[D_{11} \frac{d^2 X_i}{dx^2} Y_j \frac{d^2 X_p}{dx^2} Y_q + 4D_{66} \frac{dX_i}{dx}\frac{dY_j}{dy}\frac{dX_p}{dx}\frac{dY_q}{dy} + D_{12}\left(X_i \frac{d^2 Y_j}{dy^2}\frac{d^2 X_p}{dx^2} Y_q + \frac{d^2 X_i}{dx^2} Y_j X_p \frac{d^2 Y_q}{dy^2}\right) + D_{22} X_i \frac{d^2 Y_j}{dy^2} X_p \frac{d^2 Y_q}{dy^2} \right] dxdy, \quad (8.224a)$$

$$F_{pq} = \int_0^b \int_0^a q X_p Y_q \, dxdy. \quad (8.224b)$$

One choice for X_i and Y_j is to use algebraic polynomials. A second choice is provided by the characteristic equations of beams. Both sets of functions are given here for typical boundary conditions (see Fig. 8.18). The notation CFSF, for example, means that edge $x = 0$ is clamped (C), edge $x = a$ is free (F), edge $y = 0$ is simply supported (S), and edge $y = b$ is free. In the case of characteristic polynomials, the roots λ_i are to be determined by solving a transcendental equation. The first few values of $\lambda_i a$ are given in Table 8.10, which is the same as Table 4.5.1 of Reddy [8].

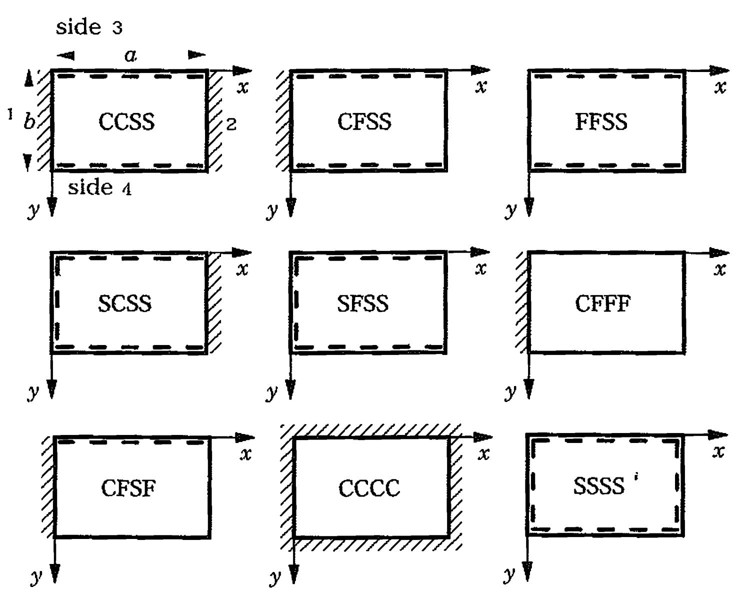


Figure 8.18 Rectangular plates with various boundary conditions.

Table 8.10 Values of the constants and eigenvalues for natural vibration of plate strips with various boundary conditions ($\lambda_n^4 \equiv \omega_n^2 I_0/D = (e_n/a)^4$). The classical plate theory without rotary inertia is used

End Conditions at $x = 0$ and $x = a$	Characteristic Equation and Values of $e_n \equiv \lambda_n a$
• Hinged–Hinged (S–S)	$\sin e_n = 0,\ e_n = n\pi$
• Fixed–Fixed (C–C)	$\cos e_n \cosh e_n - 1 = 0$ $e_n = 4.730, 7.853, \ldots$
• Fixed–Free (C–F)	$\cos e_n \cosh e_n + 1 = 0$ $e_n = 1.875, 4.694, \ldots$
• Free–Free (F–F)	$\cos e_n \cosh e_n - 1 = 0$ $e_n = 4.730, 7.853, \ldots$
• Hinged–Fixed (S–C)	$\tan e_n = \tanh e_n$ $e_n = 3.927, 7.069, \ldots$
• Hinged–Free (S–F)	$\tan e_n = \tanh e_n$ $e_n = 3.927, 7.069, \ldots$

CCSS Plates

Algebraic Polynomials

$$X_i(x) = \left(\frac{x}{a}\right)^{i+1} - 2\left(\frac{x}{a}\right)^{i+2} + \left(\frac{x}{a}\right)^{i+3},$$
$$Y_j(y) = \left(\frac{y}{b}\right)^{j} - \left(\frac{y}{b}\right)^{j+1}. \tag{8.225}$$

Characteristic Polynomials

$$X_i(x) = \sin\lambda_i x - \sinh\lambda_i x + \alpha_i\left(\cosh\lambda_i x - \cos\lambda_i x\right),$$
$$Y_j(y) = \sin\frac{n\pi y}{b}, \tag{8.226a}$$

where λ_i are the roots of the characteristic equation

$$\cos\lambda_i a \cosh\lambda_i a - 1 = 0, \tag{8.226b}$$

and α_i are defined by

$$\alpha_i = \frac{\sinh\lambda_i a - \sin\lambda_i a}{\cosh\lambda_i a - \cos\lambda_i a}. \tag{8.226c}$$

CFSS Plates

Algebraic Polynomials

$$X_i(x) = \left(\frac{x}{a}\right)^{i+1}, \qquad Y_j(y) = \left(\frac{y}{b}\right)^{j} - \left(\frac{y}{b}\right)^{j+1}. \tag{8.227}$$

Characteristic Polynomials

$$X_i(x) = \sin\lambda_i x - \sinh\lambda_i x + \alpha_i\left(\cosh\lambda_i x - \cos\lambda_i x\right),$$
$$Y_j(y) = \sin\frac{n\pi y}{b}, \tag{8.228a}$$

$$\cos\lambda_i a \cosh\lambda_i a + 1 = 0, \qquad \alpha_i = \frac{\sinh\lambda_i a + \sin\lambda_i a}{\cosh\lambda_i a + \cos\lambda_i a}. \tag{8.228b}$$

FFSS Plates

Algebraic Polynomials ($N > 1$)

$$X_i(x) = \left(\frac{x}{a}\right)^{i-1}, \qquad Y_j(y) = \left(\frac{y}{b}\right)^j - \left(\frac{y}{b}\right)^{j+1}. \tag{8.229}$$

Characteristic Polynomials

$$X_i(x) = \sin\lambda_i x + \sinh\lambda_i x - \alpha_i\left(\cosh\lambda_i x + \cos\lambda_i x\right),$$
$$Y_j(y) = \sin\frac{n\pi y}{b}, \tag{8.230a}$$

$$\cos\lambda_i a \cosh\lambda_i a - 1 = 0, \qquad \alpha_i = \frac{\sinh\lambda_i a - \sin\lambda_i a}{\cosh\lambda_i a - \cos\lambda_i a}. \tag{8.230b}$$

SCSS Plates

Algebraic Polynomials

$$X_i(x) = \left(\frac{x}{a}\right)\left[1 - \left(\frac{x}{a}\right)\right]^{i+1}, \qquad Y_j(y) = \left(\frac{y}{b}\right)^j - \left(\frac{y}{b}\right)^{j+1}. \tag{8.231}$$

Characteristic Polynomials

$$X_i(x) = \sinh\lambda_i a \sin\lambda_i x + \sin\lambda_i a \sinh\lambda_i x,$$
$$Y_j(y) = \sin\frac{n\pi y}{b}, \tag{8.232a}$$

$$\tan\lambda_i a - \tanh\lambda_i a = 0. \tag{8.232b}$$

SFSS Plates

Algebraic Polynomials

$$X_i(x) = \left(\frac{x}{a}\right)^i, \qquad Y_j(y) = \left(\frac{y}{b}\right)^j - \left(\frac{y}{b}\right)^{j+1}. \tag{8.233}$$

Characteristic Polynomials

$$X_i(x) = \sinh \lambda_i a \sin \lambda_i x - \sin \lambda_i a \sinh \lambda_i x,$$
$$Y_j(y) = \sin \frac{n\pi y}{b}, \tag{8.234a}$$
$$\tan \lambda_i a - \tanh \lambda_i a = 0. \tag{8.234b}$$

CFFF Plates (N > 1)

Algebraic Polynomials

$$X_i(x) = \left(\frac{x}{a}\right)^{i+1}, \qquad Y_j(y) = \left(\frac{y}{b}\right)^{j-1}. \tag{8.235}$$

Characteristic Polynomials

$$X_i(x) = \sin \lambda_i x - \sinh \lambda_i x + \alpha_i (\cosh \lambda_i x - \cos \lambda_i x),$$
$$Y_j(y) = \sin \mu_j y + \sinh \mu_j y - \beta_i (\cosh \mu_j y + \cos \mu_j y), \tag{8.236a}$$
$$\cos \lambda_i a \cosh \lambda_i a + 1 = 0, \qquad \cos \mu_j b \cosh \mu_j b - 1 = 0, \tag{8.236b}$$
$$\alpha_i = \frac{\sinh \lambda_i a + \sin \lambda_i a}{\cosh \lambda_i a + \cos \lambda_i a}, \qquad \beta_j = \frac{\sinh \mu_j b - \sin \mu_i b}{\cosh \mu_j b - \cos \mu_j b}. \tag{8.236c}$$

CFSF Plates

Algebraic Polynomials

$$X_i(x) = \left(\frac{x}{a}\right)^{i+1}, \qquad Y_j(y) = \left(\frac{y}{b}\right)^{j}. \tag{8.237}$$

Characteristic Polynomials

$$X_i(x) = \sin \lambda_i x - \sinh \lambda_i x + \alpha_i \left(\cosh \lambda_i x - \cos \lambda_i x\right),$$
$$Y_j(y) = \sinh \mu_j b \sin \mu_j y - \sin \mu_j b \sinh \mu_j y, \tag{8.238a}$$
$$\cos \lambda_i a \cosh \lambda_i a + 1 = 0, \qquad \tan \mu_j b - \tanh \mu_j b = 0, \tag{8.238b}$$
$$\alpha_i = \frac{\sinh \lambda_i a + \sin \lambda_i a}{\cosh \lambda_i a + \cos \lambda_i a}. \tag{8.238c}$$

CCCC Plates

Algebraic Polynomials

$$X_i(x) = \left(\frac{x}{a}\right)^{i+1} - 2\left(\frac{x}{a}\right)^{i+2} + \left(\frac{x}{a}\right)^{i+3}, \tag{8.239a}$$
$$Y_j(y) = \left(\frac{y}{b}\right)^{j+1} - 2\left(\frac{y}{b}\right)^{j+2} + \left(\frac{y}{b}\right)^{j+3}. \tag{8.239b}$$

Characteristic Polynomials

$$X_i(x) = \sin\lambda_i x - \sinh\lambda_i x + \alpha_i(\cosh\lambda_i x - \cos\lambda_i x),$$
$$Y_j(y) = \sin\lambda_j y - \sinh\lambda_j y + \alpha_j(\cosh\lambda_j y - \cos\lambda_j y), \tag{8.240a}$$

$$\cos\lambda_i a\cosh\lambda_i a - 1 = 0, \tag{8.240b}$$

$$\alpha_i = \frac{\sinh\lambda_i a - \sin\lambda_i a}{\cosh\lambda_i a - \cos\lambda_i a} = \frac{\cosh\lambda_i a - \cos\lambda_i a}{\sinh\lambda_i a + \sin\lambda_i a}. \tag{8.240c}$$

The plate types presented above indicate how one can construct the approximation functions for any combination of fixed, hinged, and free boundary conditions on the four edges of a rectangular plate. The difficult task is to evaluate the integrals of these functions as required in Eq. (8.221a,b). One may use a symbolic manipulator, such as *Mathematica* or *Maple*, to evaluate the integrals. Of course, one may also use trigonometric functions, which have the orthogonality property, for certain boundary conditions. In general, the Ritz method for general rectangular plates with arbitrary boundary conditions is algebraically more complicated than a numerical method, such as the finite element method.

Example 8.12 As an example of the application of the Ritz method to rectangular plates with arbitrary boundary conditions, we consider a CCCC plate subjected to distributed transverse load $q(x, y)$. The approximation functions of Eqs. (8.239a,b) or (8.240a–c) may be used in Eq. (8.222) (see Fig. 8.18 for the coordinate system).

First we consider the algebraic functions in Eqs. (8.239a,b) with $N = 1$ and $q = q_0$. Equation (8.223) for this case takes the form

$$0 = \left[\left(\frac{4}{5a^3}\right)\left(\frac{b}{630}\right)D_{11} + 4D_{66}\left(\frac{2}{105a}\right)\left(\frac{2}{105b}\right)\right.$$
$$\left. + 2D_{12}\left(-\frac{2}{105a}\right)\left(-\frac{2}{105b}\right) + \left(\frac{a}{630}\right)\left(\frac{4}{5b^3}\right)D_{22}\right]c_{11} - \left(\frac{ab}{900}\right)q_0$$

or

$$\left[\frac{7}{a^4}D_{11} + \frac{4}{a^2b^2}(D_{12} + 2D_{66}) + \frac{7}{b^4}D_{22}\right]c_{11} = \frac{49}{8}q_0.$$

The one-parameter Ritz solution becomes

$$W_{11}(x, y) = \left(\frac{49}{8}\right)\frac{q_0a^4\left[\frac{x}{a} - \left(\frac{x}{a}\right)^2\right]^2\left[\frac{y}{b} - \left(\frac{y}{b}\right)^2\right]^2}{7D_{11} + 4(D_{12} + 2D_{66})s^2 + 7D_{22}s^4},$$

where $s = a/b$ denotes the plate aspect ratio. The maximum deflection occurs at $x = a/2$ and $y = b/2$:

$$W_{11}\left(\frac{a}{2}, \frac{b}{2}\right) = \frac{0.00342q_0a^4}{D_{11} + 0.5714(D_{12} + 2D_{66})s^2 + D_{22}s^4}. \tag{a}$$

Next we use the characteristic functions in Eqs. (8.240a) for $N = 1$ ($\lambda_1 = 4.73/a$ and $\alpha_1 = 1.0178$):

$$X_1(x) = \sin\frac{4.73x}{a} - \sinh\frac{4.73x}{a} + 1.0178\left(\cosh\frac{4.73x}{a} - \cos\frac{4.73x}{a}\right),$$

$$Y_1(y) = \sin\frac{4.73y}{b} - \sinh\frac{4.73y}{b} + 1.0178\left(\cosh\frac{4.73y}{b} - \cos\frac{4.73y}{b}\right).$$

Evaluating the integrals, we obtain

$$\left[\frac{537.181b}{a^3}D_{11} + \frac{324.829}{ab}(D_{12} + 2D_{66}) + \frac{537.181a}{b^3}D_{22}\right]c_{11} = 0.715q_0ab.$$

The maximum deflection is given by ($X_1(a/2) = Y_1(b/2) = 1.6164$):

$$W_{11}\left(\frac{a}{2}, \frac{b}{2}\right) = \frac{0.00348q_0a^4}{D_{11} + 0.6047(D_{12} + 2D_{66})s^2 + D_{22}s^4}. \tag{b}$$

For an isotropic square plate, the maximum deflection in Eq. (a) becomes $W_{11}(a/2, b/2) = 0.00134(q_0a^4/D)$, whereas Eq. (b) gives $W_{11}(a/2, b/2) = 0.00133(q_0a^4/D)$. The "exact" solution (see Timoshenko and Woinowski-Krieger [24], p. 202) is $0.00126(q_0a^4/D)$.

Example 8.13 Consider a uniformly loaded, isotropic, rectangular plate ($a \times b$) that has as its edges $x = 0$ and $x = a$ simply supported, the edge $y = 0$ clamped, and the edge $y = b$ free (i.e., a SSCF plate; see Fig. 8.19). The geometric boundary conditions of the problem are

$$w_0(0, y) = w_0(a, y) = w_0(x, 0) = \left.\frac{\partial w_0}{\partial y}\right|_{(x,0)} = 0. \tag{a}$$

We wish to determine a one-parameter Ritz solution to the problem.

We choose an approximation of the form

$$w_0(x, y) \approx W_1(x, y) = c_1\phi_1(x, y), \qquad \phi_1 = \left(\frac{y}{b}\right)^2 \sin\frac{\pi x}{a}, \tag{b}$$

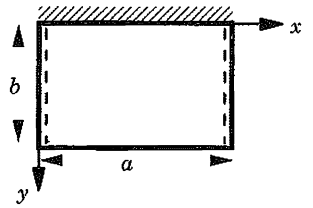


Figure 8.19 A rectangular plate with sides $x = 0$, a simply supported, side $y = 0$ clamped, and side $y = b$ free.

which satisfies the geometric boundary conditions of the problem. Substituting Eq. (b) into the total potential energy expression [see Eqs. (8.125) and (8.127)],

$$\Pi(w_0) = \frac{D}{2}\int_0^b \int_0^a \left[\left(\frac{\partial^2 w_0}{\partial x^2}\right)^2 + \left(\frac{\partial^2 w_0}{\partial y^2}\right)^2 + 2\nu\frac{\partial^2 w_0}{\partial x^2}\frac{\partial^2 w_0}{\partial y^2} + 2(1-\nu)\left(\frac{\partial^2 w_0}{\partial x \partial y}\right)^2\right] dxdy - \int_0^b \int_0^a q_0 w_0 \, dxdy,$$

we obtain

$$\begin{aligned}\Pi(c_1) =& \frac{Dc_1^2}{2}\int_0^a \int_0^b \left\{\left[-\left(\frac{\pi}{a}\right)^2\left(\frac{y}{b}\right)^2 \sin\frac{\pi x}{a} + \frac{2}{b^2}\sin\frac{\pi x}{a}\right]^2 \right.\\ &\left. + 2(1-\nu)\left[\frac{2}{b^2}\left(\frac{\pi}{a}\right)^2\left(\frac{y}{b}\right)^2 \sin^2\frac{\pi x}{a} + \left(\frac{\pi}{a}\frac{2y}{b^2}\cos\frac{\pi x}{a}\right)^2\right]\right\} dxdy \\ & - c_1 \int_0^a \int_0^b q_0 \left(\frac{y}{b}\right)^2 \sin\frac{\pi x}{a}\, dxdy \\ =& \frac{D}{2}c_1^2\left\{\int_0^a \sin^2\frac{\pi x}{a}\,dx \int_0^b \left[\left(\frac{2}{b^2} - \frac{\pi^2}{a^2 b^2}y^2\right)^2 + \frac{4(1-\nu)\pi^2}{a^2 b^4}y^2\right] dy \right.\\ &\left. + \frac{8(1-\nu)\pi^2}{a^2 b^4}\int_0^a \cos^2\frac{\pi x}{a}\,dx \int_0^b y^2\,dy\right\} - c_1 q_0 \int_0^a \sin\frac{\pi x}{a}\,dx \int_0^b \frac{y^2}{b^2}\,dy \\ =& \frac{Dc_1^2 a}{4}\left[\frac{4}{b^3} + \frac{\pi^4 b}{5a^4} - \frac{4\pi^2}{3a^2 b} + \frac{4(1-\nu)\pi^2}{a^2 b}\right] - \frac{2abc_1 q_0}{3\pi}. \end{aligned} \tag{c}$$

Using the principle of minimum total potential energy, $\delta\Pi = (d\Pi/dc_1)\delta c_1 = 0$, we obtain

$$c_1 = \frac{20 q_0 b^4}{D\pi\left[60 + 3\pi^4 s^4 + 20\pi^2 s^2(2-3\nu)\right]}, \qquad s = b/a,$$

and the solution is given by

$$W_1(x,y) = \frac{20 q_0 b^4}{D\pi\left[60 + 3\pi^4 s^4 + 20\pi^2 s^2(2-3\nu)\right]}\left(\frac{y}{b}\right)^2 \sin\frac{\pi x}{a}. \tag{d}$$

The maximum deflection is

$$W_{max} = W_1\left(\frac{a}{2}, b\right) = c_1, \tag{e}$$

which, for a square plate with $\nu = 0.3$, is equal to $0.01118(q_0 a^4/D)$.

Example 8.14 Here we solve the clamped plate problem using the semidiscrete approximation method of Chapter 7 (see Example 7.19). Consider an isotropic, clamped rectangular plate of dimensions $2a \times 2b$, and subjected to uniformly distributed load q_0. We seek a one-parameter solution of the form

$$W_1(x, y) = c_1(x)\phi_1(y) = c_1(x)(y^2 - b^2)^2. \tag{a}$$

For convenience, the origin of the coordinate system is taken at the center of the plate. The approximation $W_1(x, y)$ must satisfy the boundary conditions (see Fig. 8.20)

$$W_1 = 0, \qquad \partial W_1/\partial x = 0 \quad \text{at } x = \pm a, \tag{b}$$

$$W_1 = 0, \qquad \partial W_1/\partial y = 0 \quad \text{at } y = \pm b. \tag{c}$$

The boundary conditions in (c) are satisfied (because of the choice of ϕ_1) for any x. To satisfy the boundary conditions in (b) for any y, the function $c_1(x)$ is required to satisfy the conditions

$$c_1 = 0, \qquad \frac{dc_1}{dx} = 0 \quad \text{at } x = \pm a. \tag{d}$$

The Galerkin integral of Eq. (8.195) over the domain $(-b, b)$ is

$$0 = \int_{-b}^{b} \left[D\left(\frac{\partial^4 W_1}{\partial x^4} + 2\frac{\partial^4 W_1}{\partial x^2 \partial y^2} + \frac{\partial^4 W_1}{\partial y^4} \right) - q_0 \right] \phi_1(y)\, dy,$$

$$0 = D\int_{-b}^{b} \left[(y^2 - b^2)^2 \frac{d^4 c_1}{dx^4} + 2(12y^2 - 4b^2)\frac{d^2 c_1}{dx^2} + 24c_1 - \frac{q_0}{D} \right] (y^2 - b^2)^2\, dy.$$

After performing the integration, we obtain

$$b^5 \left(\frac{256b^4}{315}\frac{d^4 c_1}{dx^4} - \frac{512b^2}{105}\frac{d^2 c_1}{dx^2} + \frac{384}{15}c_1 - \frac{16}{15}\frac{q_0}{D} \right) = 0$$

or

$$\frac{d^4 c_1}{d\xi^4} - 6\frac{d^2 c_1}{d\xi^2} + \frac{63}{2}c_1 = \frac{21}{16}\frac{q_0}{D}, \tag{e}$$

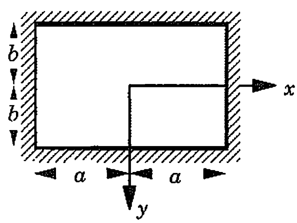


Figure 8.20 A rectangular plate with sides $x = \pm a$ and $y = \pm b$ clamped.

where $\xi = x/b$. The homogeneous solution of Eq. (e) can be calculated using the roots of the characteristic equation associated with Eq. (e):

$$d^4 - 6d^2 + \frac{63}{2} = 0 \rightarrow (d^2)_{1,2} = 3 \pm \sqrt{9 - (63/2)} = 3 \pm i4.7434,$$

where $i = \sqrt{-1}$. To determine the roots d_1 through d_4, we use the polar form

$$(d^2)_{1,2} = x \pm iy = r(\cos\theta \pm i\sin\theta) = re^{\pm i\theta},$$

where $x = 3$, $y = 4.7434$, $r = \sqrt{x^2 + y^2} = 5.6125$, and $\theta = \tan^{-1}(y/x) = 57.69°$. Therefore, we have

$$\begin{aligned}(d)_{1-4} &= \pm\sqrt{r}e^{\pm i\theta/2} = \pm\sqrt{r}\left(\cos\frac{\theta}{2} \pm i\sin\frac{\theta}{2}\right)\\ &= \pm(2.075 \pm i1.143),\\ d_1 &= 2.075 + i1.143, \qquad d_2 = 2.075 - i1.143,\\ d_3 &= -(2.075 + i1.143), \qquad d_4 = -(2.075 - i1.143).\end{aligned}$$

Therefore, the homogeneous solution is given by

$$\begin{aligned}c_{1h} &= \bar{A}_1 e^{(\alpha+i\beta)\xi} + \bar{A}_2 e^{(\alpha-i\beta)\xi} + \bar{A}_3 e^{-(\alpha+i\beta)\xi} + \bar{A}_4 e^{-(\alpha-i\beta)\xi}\\ &= A_1\cosh\alpha\xi\cos\beta\xi + A_2\cosh\alpha\xi\sin\beta\xi + A_3\sinh\alpha\xi\sin\beta\xi\\ &\quad + A_4\sinh\alpha\xi\cos\beta\xi,\end{aligned} \tag{f}$$

where $\alpha = 2.075$, $\beta = 1.143$, and $\xi = x/b$. The particular solution is given by

$$c_{1p} = \frac{2}{63}\left(\frac{21}{16}\frac{q_0}{D}\right) = \frac{q_0}{24D}.$$

The general solution of Eq. (e) becomes

$$c_1(\xi) = c_{1h} + c_{1p}. \tag{g}$$

Because of the symmetry of the expected solution about the y-axis, it follows that $A_2 = A_4 = 0$. The remaining constants A_1 and A_3 can be determined using boundary conditions in (d) on c_1 at $x = \pm a$. We obtain

$$A_1\cosh k_1\cos k_2 + A_3\sinh k_1\sin k_2 = -\frac{q_0}{24D},$$

$$\begin{aligned}A_1&\left(\frac{\alpha}{b}\sinh k_1\cos k_2 - \frac{\beta}{b}\cosh k_1\sin k_2\right)\\ &+ A_3\left(\frac{\alpha}{b}\cosh k_1\sin k_2 + \frac{\beta}{b}\sinh k_1\cos k_2\right) = 0,\end{aligned}$$

where $k_1 = \alpha a/b$ and $k_2 = \beta a/b$. Solving these equations, we obtain

$$A_1 = \frac{\mu_1}{\mu_0}\left(\frac{q_0}{24D}\right), \qquad A_3 = \frac{\mu_2}{\mu_0}\left(\frac{q_0}{24D}\right),$$

where $\alpha = 2.075$, $\beta = 1.143$, and

$$\begin{aligned} \mu_0 &= \beta \sinh k_1 \cosh k_1 + \alpha \sin k_2 \cos k_2, \\ \mu_1 &= -(\alpha \cosh k_1 \sin k_2 + \beta \sinh k_1 \cos k_2), \\ \mu_2 &= \alpha \sinh k_1 \cos k_2 - \beta \cosh k_1 \sin k_2. \end{aligned} \tag{h}$$

Thus, the one-parameter solution becomes

$$W_1(x, y) = \frac{q_0}{24D}\left[\left(\frac{\mu_1}{\mu_0}\cosh\frac{\alpha x}{b}\cos\frac{\beta x}{b} + \frac{\mu_2}{\mu_0}\sinh\frac{\alpha x}{b}\sin\frac{\beta x}{b}\right) + 1\right](y^2 - b^2)^2. \tag{i}$$

The maximum deflection occurs at the center of the plate; for a square plate of dimensions $2a \times 2a$, the maximum deflection is given by ($k_1 \to \alpha = 2.075$, $k_2 \to \beta = 1.143$):

$$W_{max} = W_1(0, 0) = \frac{q_0 a^4}{24D}\left(1 + \frac{\mu_1}{\mu_0}\right) = 0.4977\frac{q_0 a^4}{24D} = 0.0207\frac{q_0 a^4}{D}.$$

For a square plate of dimension $a \times a$, this reduces to (divide the above result with 16):

$$W_{max} = 0.00129\frac{q_0 a^4}{D},$$

which is closer to the "exact" solution $0.00126(q_0 a^4/D)$ than that obtained in Example 8.12.

Example 8.15 Much of the discussion presented in Example 8.14 is also valid for a rectangular plate with edges $y = \pm b$ clamped and $x = \pm a$ simply supported (see Fig. 8.21) and subjected to a uniformly distributed load. The general solution in Eq. (f) is valid with the following boundary conditions on c_1:

$$c_1(\pm a) = \left(\frac{d^2 c_1}{dx^2}\right)_{x=\pm a} = 0. \tag{a}$$

These conditions give ($\alpha = 2.075$ and $\beta = 1.143$):

$$A_1 \cosh k_1 \cos k_2 + A_3 \sinh k_1 \sin k_2 = -\frac{q_0}{24D},$$

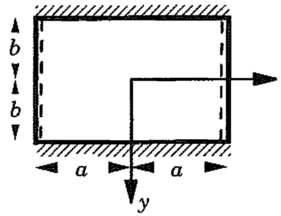


Figure 8.21 A rectangular plate with sides $x = \pm a$ simply supported and sides $y = \pm b$ clamped.

$$A_1 \sinh k_1 \sin k_2 - A_3 \cosh k_1 \cos k_2 = \frac{q_0}{24D}\left(\frac{\beta^2 - \alpha^2}{2\alpha\beta}\right).$$

The solution of these equations is given by

$$\begin{aligned} A_1 &= -\frac{q_0}{24D}\frac{\cosh k_1 \cos k_2 + a_0 \sinh k_1 \sin k_2}{\cosh^2 k_1 \cos^2 k_2 + \sinh^2 k_1 \sin^2 k_2}, \\ A_3 &= \frac{q_0}{24D}\frac{a_0 \cosh k_1 \cos k_2 + \sinh k_1 \sin k_2}{\cosh^2 k_1 \cos^2 k_2 + \sinh^2 k_1 \sin^2 k_2}, \end{aligned} \tag{b}$$

where $a_0 = \left(\beta^2 - \alpha^2\right)/2\alpha\beta$. For a square plate, the maximum deflection is given by

$$W_{max} = W_1(0, 0) = 0.0248\frac{q_0 a^4}{D}.$$

Example 8.16 The Galerkin solution of a simply supported rectangular plate using trigonometric functions coincides with the Navier solution. To show this, we begin with the approximation [see Eq. (8.145a)]

$$w_0(x, y, t) \approx W_N(x, y, t) = \sum_{i=1}^{N}\sum_{j=1}^{N} c_{ij}(t) \sin\frac{i\pi x}{a}\sin\frac{j\pi y}{b},$$

which satisfies all of the boundary conditions in Eq. (8.144a,b). Substituting W_N into the (semidiscrete) Galerkin integral:

$$0 = \int_0^b \int_0^a \left[D\left(\frac{\partial^4 W_N}{\partial x^4} + 2\frac{\partial^4 W_N}{\partial x^2 \partial y^2} + \frac{\partial^4 W_N}{\partial y^4}\right)\right.$$
$$\left. - q + I_0 \ddot{W}_N - I_2\left(\frac{\partial^2 \ddot{W}_N}{\partial x^2} + \frac{\partial^2 \ddot{W}_N}{\partial y^2}\right)\right] \sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}\, dx dy$$

for $m, n = 1, 2, \ldots, N$, we obtain

$$0 = \int_0^b \int_0^a \left\{ \sum_{i=1}^{N} \sum_{j=1}^{N} \left[D\left(\alpha_i^2 + \beta_j^2\right)^2 c_{ij} + \left[I_0 + I_2\left(\alpha_i^2 + \beta_j^2\right)\right] \frac{d^2 c_{ij}}{dt^2} \right] \times \sin\frac{i\pi x}{a} \sin\frac{j\pi y}{b} - q \right\} \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b}\, dxdy.$$

Since we can write

$$q(x, y, t) = \sum_{i=1}^{N} \sum_{j=1}^{N} q_{ij}(t) \sin\frac{i\pi x}{a} \sin\frac{j\pi y}{b},$$

and due to the orthogonality of the trigonometric functions

$$\int_0^a \sin\frac{i\pi x}{a} \sin\frac{m\pi x}{a}\, dx = \begin{cases} \frac{a}{2} & \text{for } i = m, \\ 0 & \text{for } i \neq m, \end{cases}$$

we have

$$0 = \frac{ab}{4} \left\{ D\left(\alpha_m^2 + \beta_n^2\right)^2 c_{mn} + \left[I_0 + I_2\left(\alpha_m^2 + \beta_n^2\right)\right] \frac{d^2 c_{mn}}{dt^2} - q_{mn} \right\}.$$

The expression in the curl brackets is identical to Eq. (8.148) with $c_{mn} = W_{mn}$ and $N_0 = 0$. Hence, for $N \to \infty$ the Galerkin solution is the same as the Navier solution. The use of the same approximation functions in the Ritz method will also produce the same result.

Example 8.17 Consider an isotropic equilateral triangular plate (see Fig. 8.22) subjected to uniform transverse load of intensity q_0. We wish to determine the deflection using the Ritz method. Since the essential boundary conditions require $w_0 = 0$ on

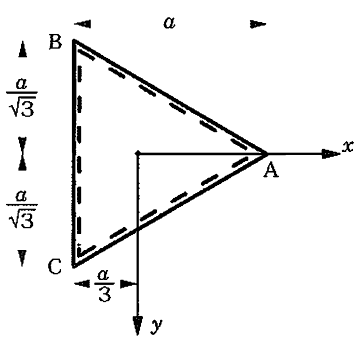


Figure 8.22 An equilateral triangular plate with simply supported edges.

the edges, the approximation functions should vanish on the edges. A natural choice for the first approximation function is the product of the equations of the edges themselves.

The equations of the edges, with respect to the coordinate system shown, are

$$\begin{aligned} \frac{x}{a}+\frac{1}{3}&=0 &\quad &\text{on BC,}\\ \frac{1}{\sqrt{3}}\frac{x}{a}+\frac{y}{a}-\frac{2}{3\sqrt{3}}&=0 &\quad &\text{on AC,}\\ \frac{1}{\sqrt{3}}\frac{x}{a}-\frac{y}{a}-\frac{2}{3\sqrt{3}}&=0 &\quad &\text{on AB.}\end{aligned} \tag{a}$$

Hence a one-parameter Ritz approximation is of the form

$$\begin{aligned} W_1(x,y)&=c_1\left(\frac{x}{a}+\frac{1}{3}\right)\left(\frac{1}{\sqrt{3}}\frac{x}{a}+\frac{y}{a}-\frac{2}{3\sqrt{3}}\right)\left(\frac{1}{\sqrt{3}}\frac{x}{a}-\frac{y}{a}-\frac{2}{3\sqrt{3}}\right)\\ &=\frac{1}{3}c_1\left[\frac{4}{27}-\left(\frac{x}{a}\right)^2-\left(\frac{y}{a}\right)^2-3\left(\frac{x}{a}\right)\left(\frac{y}{a}\right)^2+\left(\frac{x}{a}\right)^3\right]\\ &\equiv \bar{c}_1\phi_1(x,y), \qquad \bar{c}_1=\tfrac{1}{3}c_1. \end{aligned} \tag{b}$$

First note that

$$\frac{\partial\phi_1}{\partial x}=-\frac{2x}{a^2}-\frac{3y^2}{a^3}+\frac{3x^2}{a^3}, \qquad \frac{\partial\phi_1}{\partial y}=-\frac{2y}{a^2}-\frac{6xy}{a^3},$$

$$\frac{\partial^2\phi_1}{\partial x^2}=-\frac{2}{a^2}+\frac{6x}{a^3}, \qquad \frac{\partial^2\phi_1}{\partial y^2}=-\frac{2}{a^2}-\frac{6x}{a^3}, \qquad \frac{\partial^2\phi_1}{\partial x\partial y}=-\frac{6y}{a^3}.$$

Substituting the above expressions into Eq. (8.221a,b), we obtain [$D_{11}=D_{22}=D$, $D_{12}=\nu D$, and $2D_{66}=(1-\nu)D$]:

$$\begin{aligned} R_{11}&=D\int_T\left[\frac{\partial^2\phi_1}{\partial x^2}\frac{\partial^2\phi_1}{\partial x^2}+\nu\left(\frac{\partial^2\phi_1}{\partial y^2}\frac{\partial^2\phi_1}{\partial x^2}+\frac{\partial^2\phi_1}{\partial x^2}\frac{\partial^2\phi_1}{\partial y^2}\right)\right.\\ &\qquad\left.+2(1-\nu)\frac{\partial^2\phi_1}{\partial x\partial y}\frac{\partial^2\phi_1}{\partial x\partial y}+\frac{\partial^2\phi_1}{\partial y^2}\frac{\partial^2\phi_1}{\partial y^2}\right]dxdy\\ &=D\left(\frac{4}{a^4}I_{00}+\frac{36}{a^6}I_{20}-\frac{24}{a^5}I_{10}\right)+2\nu D\left(\frac{2}{a^4}I_{00}+\frac{6}{a^5}I_{10}-\frac{36}{a^6}I_{20}\right)\\ &\quad+2(1-\nu)D\frac{36}{a^6}I_{02}+D\left(\frac{1}{a^4}I_{00}+\frac{36}{a^6}I_{20}+\frac{12}{a^5}I_{10}\right), \end{aligned}$$

$$F_1 = \int_T q_0 \left[\frac{4}{27} - \left(\frac{x}{a}\right)^2 - \left(\frac{y}{a}\right)^2 - 3\left(\frac{x}{a}\right)\left(\frac{y}{a}\right)^2 + \left(\frac{x}{a}\right)^3\right] dxdy$$
$$= q_0 \left(\frac{4A_0}{27} - \frac{I_{20}}{a^2} - \frac{I_{02}}{a^2} - \frac{3I_{12}}{a^3} + \frac{I_{30}}{a^3}\right),$$

where

$$I_{mn} \equiv \int_T x^m y^n \, dxdy, \qquad I_{00} = A_0, \qquad I_{10} = A_0\bar{x}, \qquad I_{01} = A_0\bar{y},$$
$$\bar{x} = \frac{1}{3}\sum_{i=1}^{3} x_i, \qquad \bar{y} = \frac{1}{3}\sum_{i=1}^{3} y_i, \qquad I_{11} = \frac{A_0}{12}\left(\sum_{i=1}^{3} x_i y_i + 9\bar{x}\bar{y}\right), \tag{c}$$
$$I_{20} = \frac{A_0}{12}\left(\sum_{i=1}^{3} x_i^2 + 9\bar{x}^2\right), \qquad I_{02} = \frac{A_0}{12}\left(\sum_{i=1}^{3} y_i^2 + 9\bar{y}^2\right).$$

(x_i, y_i) are the coordinates of the three vertices of a triangle, and A_0 is the area of a triangle ($A_0 = a^2/\sqrt{3}$). For the triangle in Fig. 8.22, we have

$$x_1 = \frac{2a}{3}, \qquad x_2 = x_3 = -\frac{a}{3}, \qquad y_1 = 0, \qquad y_2 = -y_3 = \frac{a}{\sqrt{3}},$$
$$\bar{x} = \bar{y} = 0, \qquad I_{10} = I_{01} = I_{11} = 0, \qquad I_{20} = I_{02} = \frac{A_0 a^2}{18},$$
$$I_{30} = \frac{A_0 a^3}{135}, \qquad I_{12} = -\frac{A_0 a^3}{135}.$$

Substituting these values into the expressions for R_{11} and F_1, we obtain

$$R_{11} = \frac{16DA_0}{a^4}, \qquad F_1 = \frac{q_0 A_0}{15}; \quad \rightarrow \quad \bar{c}_1 = \frac{q_0 a^4}{240D}.$$

Thus the one-parameter Ritz solution becomes

$$W_1(x, y) = \frac{q_0 a^4}{240D}\left[\frac{4}{27} - \left(\frac{x}{a}\right)^2 - \left(\frac{y}{a}\right)^2 - 3\left(\frac{x}{a}\right)\left(\frac{y}{a}\right)^2 + \left(\frac{x}{a}\right)^3\right]. \tag{d}$$

For an N-parameter Ritz approximation, one may use

$$\phi_i(x, y) = p_{(i-1)}(x, y)\phi_{(i-1)}(x, y), \qquad i = 2, 3, \ldots, N,$$

where $p_i(x, y)$ is a polynomial of degree i:

$$p_1(x, y) = \frac{x}{a} + \frac{y}{a}, \qquad p_2(x, y) = \frac{x}{a}\frac{y}{a} + \frac{x^2}{a^2} + \frac{y^2}{a^2}, \ldots .$$

The exact deflection is given by [which can be verified by substitution into the governing equation (8.195)]

$$w_0(x, y) = \frac{q_0 a^4}{64D}\left[\frac{4}{27} - \left(\frac{x}{a}\right)^2 - \left(\frac{y}{a}\right)^2 - 3\left(\frac{x}{a}\right)\left(\frac{y}{a}\right)^2 + \left(\frac{x}{a}\right)^3\right] \times \left[\frac{4}{9} - \left(\frac{x}{a}\right)^2 - \left(\frac{y}{a}\right)^2\right]. \tag{e}$$

Example 8.18 Consider a clamped, isotropic elliptic plate with major and minor axes $2a$ and $2b$, respectively (see Fig. 8.23). We wish to obtain a one-parameter Galerkin solution for the case in which the plate is subjected to uniformly distributed load q_0. The boundary conditions are

$$w_0 = 0, \qquad \frac{\partial w_0}{\partial n} = 0 \qquad \left(\text{or } \frac{\partial w_0}{\partial x} = 0, \quad \frac{\partial w_0}{\partial y} = 0\right). \tag{8.241}$$

As before, we use the equation for the boundary of the ellipse but square it to satisfy the deflection as well as slope boundary conditions:

$$W_1(x, y) = c_1 \phi_1(x, y) = c_1\left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}\right)^2. \tag{8.242}$$

We have

$$\frac{\partial \phi_1}{\partial x} = -\frac{4x}{a^2}\left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}\right), \qquad \frac{\partial \phi_1}{\partial y} = -\frac{4y}{b^2}\left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}\right),$$

$$\frac{\partial^4 \phi_1}{\partial x^4} = \frac{24}{a^4}, \qquad \frac{\partial^4 \phi_1}{\partial x^2 \partial y^2} = \frac{8}{a^2 b^2}, \qquad \frac{\partial^4 \phi_1}{\partial y^4} = \frac{24}{b^4}.$$

Clearly, the boundary conditions in Eq. (8.241) are satisfied by the choice. Substituting into the expression

$$\mathcal{R} = D\left(\frac{\partial^4 W_1}{\partial x^4} + 2\frac{\partial^4 W_1}{\partial x^2 \partial y^2} + \frac{\partial^4 W_1}{\partial y^4}\right) - q_0,$$

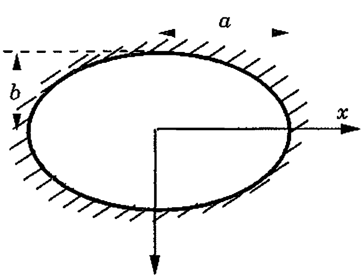


Figure 8.23 A clamped elliptic plate.

we obtain

$$\mathcal{R} = c_1 D\left(\frac{24}{a^4} + 2\frac{8}{a^2b^2} + \frac{24}{b^4}\right) - q_0, \tag{8.243}$$

which is independent of x and y. In this case, there is no need to evaluate the weighted integral of the residual

$$0 = \int_{-b}^{b}\int_{-a}^{a} \mathcal{R}\phi_1(x, y)\, dxdy,$$

as it yields only a constant multiple of the expression in Eq. (8.243). Thus we have

$$c_1 = \frac{q_0}{8D\left[(3/a^4) + (2/a^2b^2) + (3/b^4)\right]},$$

and the Galerkin solution, which coincides with the exact solution, is given by

$$w_0(x, y) = \frac{q_0 a^4}{8D(3s^4 + 2s^2 + 3)}\left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}\right)^2, \qquad s = \frac{a}{b}. \tag{8.244}$$

When $a = b$, the solution reduces to that of a circular plate ($x^2 + y^2 = a^2$).

8.2.7 Variational Solutions: Vibration

This section is dedicated to the study of natural vibrations of rectangular plates by the Ritz method. Equation (8.143), without the applied loads q and $(\hat{N}_{xx}, \hat{N}_{yy}, \hat{N}_{xy})$, reduces to

$$D\left(\frac{\partial^4 w_0}{\partial x^4} + \frac{\partial^4 w_0}{\partial x^2 \partial y^2} + \frac{\partial^4 w_0}{\partial y^4}\right)$$
$$+ I_0\frac{\partial^2 w_0}{\partial t^2} - I_2\left(\frac{\partial^4 w_0}{\partial t^2 \partial x^2} + \frac{\partial^4 w_0}{\partial t^2 \partial y^2}\right) = 0, \tag{8.245a}$$

where

$$I_0 = \rho_0 h, \qquad I_2 = \frac{\rho_0 h^3}{12}. \tag{8.245b}$$

For natural vibration, the solution is assumed to be periodic:

$$w_0(x, y, t) = w(x, y)e^{i\omega t},$$

where $i = \sqrt{-1}$ and ω is the frequency of natural vibration associated with mode shape w, and Eq. (8.245a) takes the form

$$D\left(\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4}\right) - \omega^2\left[I_0 w - I_2\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right)\right] = 0. \tag{8.246}$$

We wish to find values of ω such that Eq. (8.246) has a nontrivial solution w (mode shape).

The weak form of Eq. (8.246) is

$$0 = \int_0^b \int_0^a \left\{D\left[\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 \delta w}{\partial x^2} + \nu\left(\frac{\partial^2 w}{\partial y^2}\frac{\partial^2 \delta w}{\partial x^2} + \frac{\partial^2 w}{\partial x^2}\frac{\partial^2 \delta w}{\partial y^2}\right) + 2(1-\nu)\frac{\partial^2 w}{\partial x \partial y}\frac{\partial^2 \delta w}{\partial x \partial y} + \frac{\partial^2 w}{\partial y^2}\frac{\partial^2 \delta w}{\partial y^2}\right] - \omega^2\left[I_0 w \delta w + I_2\left(\frac{\partial w}{\partial x}\frac{\partial \delta w}{\partial x} + \frac{\partial w}{\partial y}\frac{\partial \delta w}{\partial y}\right)\right]\right\} dxdy. \tag{8.247}$$

Note that the weak form is based on the principle of minimum total potential energy [and not Eq. (8.246) directly].

Using an N-parameter Ritz solution of the form

$$w(x, y) \approx \sum_{j=1}^{N} c_j \phi_j(x, y) \tag{8.248}$$

in Eq. (8.247), we obtain the eigenvalue problem

$$\left([R] - \omega^2[B]\right)\{c\} = \{0\}, \tag{8.249a}$$

where

$$R_{ij} = \int_\Omega D\left[\frac{\partial^2 \phi_i}{\partial x^2}\frac{\partial^2 \phi_j}{\partial x^2} + \nu\left(\frac{\partial^2 \phi_i}{\partial y^2}\frac{\partial^2 \phi_j}{\partial x^2} + \frac{\partial^2 \phi_i}{\partial x^2}\frac{\partial^2 \phi_j}{\partial y^2}\right) + 2(1-\nu)\frac{\partial^2 \phi_i}{\partial x \partial y}\frac{\partial^2 \phi_j}{\partial x \partial y} + \frac{\partial^2 \phi_i}{\partial y^2}\frac{\partial^2 \phi_j}{\partial y^2}\right] dxdy, \tag{8.249b}$$

$$B_{ij} = \int_\Omega \left[I_0 \phi_i \phi_j + I_2\left(\frac{\partial \phi_i}{\partial x}\frac{\partial \phi_j}{\partial x} + \frac{\partial \phi_i}{\partial y}\frac{\partial \phi_j}{\partial y}\right)\right] dxdy. \tag{8.249c}$$

Equation (8.249a) represents a standard eigenvalue problem, which must be solved numerically.

As discussed earlier [see Eq. (8.222)], it is convenient to express the Ritz approximation of rectangular plates in the form

$$w(x, y) \approx W_{mn}(x, y) = \sum_{i=1}^{N}\sum_{j=1}^{N} c_{ij}\, X_i(x)Y_j(y), \tag{8.250}$$

where the functions X_i and Y_j for various boundary conditions were given in Eqs. (8.225)–(8.240). These functions satisfy only the geometric boundary conditions of the problem. Substituting Eq. (8.250) into Eq. (8.247), we obtain Eq. (8.249a) in which the coefficients $R_{(ij)(kl)}$ of $[R]$ and coefficients $B_{(ij)(kl)}$ of $[B]$ are defined by

$$\begin{aligned} R_{pq,ij} = D\int_0^b\int_0^a \Bigg[& \frac{d^2X_i}{dx^2}Y_j\frac{d^2X_p}{dx^2}Y_q + 2(1-\nu)\frac{dX_i}{dx}\frac{dY_j}{dy}\frac{dX_p}{dx}\frac{dY_q}{dy} \\ & + \nu\left(X_i\frac{d^2Y_j}{dy^2}\frac{d^2X_p}{dx^2}Y_q + \frac{d^2X_i}{dx^2}Y_jX_p\frac{d^2Y_q}{dy^2}\right) \\ & + X_i\frac{d^2Y_j}{dy^2}X_p\frac{d^2Y_q}{dy^2}\Bigg]dxdy, \end{aligned} \tag{8.251a}$$

$$B_{ij,kl} = \int_0^b\int_0^a \left[I_0X_iX_kY_jY_l + I_2\left(\frac{dX_i}{dx}\frac{dX_k}{dx}Y_jY_l + X_iX_k\frac{dY_j}{dy}\frac{dY_l}{dy}\right)\right]dxdy. \tag{8.251b}$$

The size of the matrix $[R]$ is $N^2 \times N^2$, and Eq. (8.249a) can be used to calculate the first N^2 frequencies of the infinite set.

Example 8.19 For an isotropic ($\nu = 0.25$) plate simply supported on all sides, we may select the following approximation functions:

$$\phi_{ij} = X_i(x)Y_j(y), \tag{a}$$

$$X_i(x) = \left(\frac{x}{a}\right)^i - \left(\frac{x}{a}\right)^{i+1}, \qquad Y_j(x) = \left(\frac{y}{b}\right)^j - \left(\frac{y}{b}\right)^{j+1}. \tag{b}$$

For square plates with $N = 2$, the matrices $[R]$ and $[B]$ are given by

$$[R] = \begin{bmatrix} 0.4346 & 0.2173 & 0.2173 & 0.1086 \\ 0.2173 & 0.2314 & 0.1086 & 0.1157 \\ 0.2173 & 0.1086 & 0.2314 & 0.1157 \\ 0.1086 & 0.1157 & 0.1157 & 0.0993 \end{bmatrix} \times 10^{-1},$$
$$[B] = \begin{bmatrix} 1.1111 & 0.5556 & 0.5556 & 0.2778 \\ 0.5556 & 0.3175 & 0.2778 & 0.1587 \\ 0.5556 & 0.2778 & 0.3175 & 0.1587 \\ 0.2778 & 0.1587 & 0.1587 & 0.0907 \end{bmatrix} \times 10^{-5}, \tag{c}$$

where the following notation is used to store the coefficients:

$$\begin{aligned}
&R_{11} = R_{11,11}, && R_{12} = R_{11,12}, && R_{13} = R_{11,21}, && R_{14} = R_{11,22},\\
&R_{21} = R_{12,11}, && R_{22} = R_{12,12}, && R_{23} = R_{12,21}, && R_{24} = R_{12,22},\\
&R_{31} = R_{21,11}, && R_{32} = R_{21,12}, && R_{33} = R_{21,21}, && R_{34} = R_{21,22},\\
&R_{41} = R_{22,11}, && R_{42} = R_{22,12}, && R_{43} = R_{22,21}, && R_{44} = R_{22,22}.
\end{aligned}$$

The first four frequencies ($\bar{\omega}_{mn} = \omega_{mn} a^2 \sqrt{\rho h/D}$) are

$$\bar{\omega}_{11} = 20.973, \quad \bar{\omega}_{12} = 58.992, \quad \bar{\omega}_{21} = 59.007, \quad \bar{\omega}_{22} = 92.529. \tag{d}$$

The exact frequencies from Eq. (8.173) are

$$\bar{\omega}_{11} = 19.739, \quad \bar{\omega}_{12} = \bar{\omega}_{21} = 49.348, \quad \bar{\omega}_{22} = 88.826. \tag{e}$$

For larger values of N and M, the Ritz method will yield increasingly accurate and more frequencies (see Liew et al. [29]).

Example 8.20 For a rectangular plate clamped on all sides, we may select the following approximation functions:

$$\phi_{ij} = X_i(x) Y_j(y), \tag{a}$$

$$X_i(x) = \left(\frac{x}{a}\right)^{i+1} - 2\left(\frac{x}{a}\right)^{i+2} + \left(\frac{x}{a}\right)^{i+3}, \tag{b}$$

$$Y_j(y) = \left(\frac{y}{b}\right)^{j+1} - 2\left(\frac{y}{b}\right)^{j+2} + \left(\frac{y}{b}\right)^{j+3}. \tag{c}$$

For isotropic ($\nu = 0.25$) square plates with $N = 2$, the matrices $[R]$ and $[B]$ are given by

$$\begin{aligned}
[R] &= \begin{bmatrix} 0.2902 & 0.1451 & 0.1451 & 0.0725 \\ 0.1451 & 0.1007 & 0.0725 & 0.0503 \\ 0.1451 & 0.0725 & 0.1007 & 0.0503 \\ 0.0725 & 0.0503 & 0.0503 & 0.0335 \end{bmatrix} \times 10^{-3}, \\
[B] &= \begin{bmatrix} 0.2520 & 0.1260 & 0.1260 & 0.0630 \\ 0.1260 & 0.0687 & 0.0630 & 0.0344 \\ 0.1260 & 0.0630 & 0.0687 & 0.0344 \\ 0.0630 & 0.0344 & 0.0344 & 0.0187 \end{bmatrix} \times 10^{-7}.
\end{aligned} \tag{d}$$

The first four nondimensionalized frequencies ($\bar{\omega}_{mn} = \omega_{mn} a^2 \sqrt{\rho h/D}$) are

$$\bar{\omega}_{11} = 36.000, \quad \bar{\omega}_{12} = 74.296, \quad \bar{\omega}_{21} = 74.297, \quad \bar{\omega}_{22} = 108.592,$$

and the approximate frequencies (see Leissa [28]) are

$$\bar{\omega}_{11} = 35.999, \quad \bar{\omega}_{12} = 73.405, \quad \bar{\omega}_{21} = 73.405, \quad \bar{\omega}_{22} = 108.237.$$

Table 8.11 Effect of the plate aspect ratio on the nondimensionalized frequencies of clamped isotropic ($\nu = 0.25$) rectangular plates

		$\omega_{mn} a^2 \sqrt{\rho h / D}$ for values of a/b					
m	n	0.25	0.5	0.667	1.0	1.5	2.0
1	1	22.890	24.647	27.047	36.000	60.856	98.590
	2	24.196	31.867	41.899	74.296	94.273	127.466
2	1	63.466	65.234	69.297	74.297	151.419	260.938
	2	64.943	71.941	80.394	108.592	180.887	287.766

Table 8.11 contains the first four natural frequencies of clamped rectangular plates. The results are obtained with $N = 2$ [cf. Tables 4.28 and 4.29 on pages 63 and 64, respectively, of Leissa [28]; no mention is made of the Poisson's ratio used]. The mode shape is of the form

$$w(x, y) = X_1 (c_1 Y_1 + c_2 Y_2) + X_2 (c_3 Y_1 + c_4 Y_2) . \tag{c}$$

The vectors $\{c\}^i$ for the four modes in the case of $a/b = 0.25$, for example, are given by

$$\{c\}^1 = \begin{Bmatrix} 1.0 \\ 0.0 \\ 0.0 \\ 0.0 \end{Bmatrix}, \quad \{c\}^2 = \begin{Bmatrix} -0.5 \\ 1.0 \\ 0.0 \\ 0.0 \end{Bmatrix}, \quad \{c\}^3 = \begin{Bmatrix} -0.50 \\ 0.08 \\ 1.00 \\ 0.16 \end{Bmatrix}, \quad \{c\}^4 = \begin{Bmatrix} 0.25 \\ -0.50 \\ -0.50 \\ 1.00 \end{Bmatrix}.$$

The vectors $\{c\}^i$ for the four modes in the case of $a/b = 1$ are given by

$$\{c\}^1 = \begin{Bmatrix} 1.0 \\ 0.0 \\ 0.0 \\ 0.0 \end{Bmatrix}, \quad \{c\}^2 = \begin{Bmatrix} -0.5 \\ 1.0 \\ 0.0 \\ 0.0 \end{Bmatrix}, \quad \{c\}^3 = \begin{Bmatrix} -0.5 \\ 0.0 \\ 1.0 \\ 0.0 \end{Bmatrix}, \quad \{c\}^4 = \begin{Bmatrix} 0.25 \\ -0.50 \\ -0.50 \\ 1.00 \end{Bmatrix}.$$

Example 8.21 Here we consider natural vibrations of rectangular plates with sides $x = 0, a$ and $y = 0$ clamped and side $y = b$ simply supported (CCCS). For this case, the approximation functions are given by

$$X_i = \left(\frac{x}{a}\right)^{i+1} - 2\left(\frac{x}{a}\right)^{i+2} + \left(\frac{x}{a}\right)^{i+3}, \qquad Y_j = \left(\frac{y}{b}\right)^{j+1} - \left(\frac{y}{b}\right)^{j+2}. \tag{a}$$

The first four natural frequencies of rectangular plates are presented in Table 8.12 for various aspect ratios.

Additional examples of the natural vibration of rectangular plates with other boundary conditions can be found in Leissa [28], Liew et al. [29], and Reddy [8].

Table 8.12 Effect of the plate aspect ratio on the nondimensionalized frequencies of isotropic ($\nu = 0.25$) rectangular plates (CCCS)

Mode	$\omega_{mn}a^2\sqrt{\rho h/D}$ for values of a/b					
(m, n)	0.25	0.5	0.667	1.0	1.5	2.0
(1, 1)	22.840	24.215	25.913	31.849	48.217	73.573
(1, 2)	24.654	34.421	46.951	72.025	86.045	108.452
(2, 1)	63.411	64.957	66.663	86.497	179.342	310.663
(2, 2)	65.444	74.254	84.914	120.087	208.448	337.474

8.2.8 Variational Solutions: Buckling

Rectangular Plates Simply Supported Along Two Opposite Sides and Compressed in the Direction Perpendicular to Those Sides In Section 8.2.6 we discussed bending solutions of rectangular plates simply supported along two sides and having arbitrary boundary conditions on the other two sides. We used the Lévy method of solution. Here we consider the buckling of rectangular plates simply supported along edges $x = 0, a$ and subjected to uniform compression along the same edges (see Fig. 8.24). From Eq. (8.143b), by setting the time derivative and load terms to zero, and taking $\hat{N}_{xx} = N_0$, $\hat{N}_{yy} = 0$, and $\hat{N}_{xy} = 0$, we obtain

$$D\left(\frac{\partial^4 w_0}{\partial x^4} + 2\frac{\partial^4 w_0}{\partial x^2 \partial y^2} + \frac{\partial^4 w_0}{\partial y^4}\right) + \hat{N}_{xx}\frac{\partial^2 w_0}{\partial x^2} = 0. \tag{8.252}$$

In the Lévy method, we assume a solution of the form

$$w_0(x, y) = \sum_{m=1}^{\infty} W_m(y) \sin \alpha_m x, \qquad \alpha_m = \frac{m\pi}{a}. \tag{8.253}$$

That is, the plate buckles into m sinusoidal half waves. The assumed solution satisfies the simply supported conditions

$$w_0 = 0, \qquad M_{xx} \equiv -D\left(\frac{\partial^2 w_0}{\partial x^2} + \nu\frac{\partial^2 w_0}{\partial y^2}\right) = 0 \quad \text{at } x = 0, a.$$

Substitution of (8.253) in (8.252) yields the equation

$$\frac{d^4 W_m}{dy^4} - 2\alpha_m^2 \frac{d^2 W_m}{dy^2} + \left(\alpha_m^4 - \frac{\alpha_m^2}{D} N_0\right) W_m = 0, \tag{8.254}$$

which must be solved using the boundary conditions on edges $y = 0, b$. Here we wish to solve Eq. (8.254) using the Ritz method.

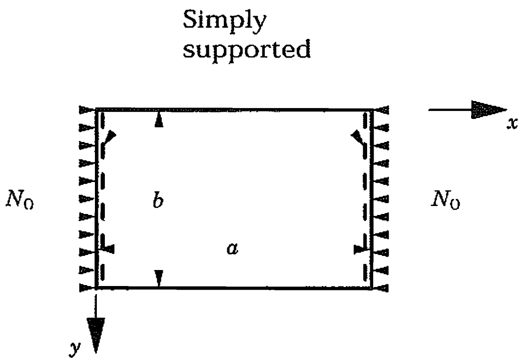


Figure 8.24 Buckling of rectangular plates simply supported along two opposite edges ($x = 0$, a) and subjected to uniform compressive forces on the same edges; other edges may have any combination of boundary conditions.

The weak form of Eq. (8.254) is

$$\int_0^b \left[\frac{d^2 W_m}{dy^2}\frac{d^2 \delta W_m}{dy^2} + 2\alpha_m^2 \frac{dW_m}{dy}\frac{d\delta W_m}{dy} + \left(\alpha_m^4 - \bar{N}_0\alpha_m^2\right) W_m \delta W_m\right] dy$$
$$+ \left[\left(\frac{d^3 W_m}{dy^3} - 2\alpha_m^2 \frac{dW_m}{dy}\right)\delta W_m - \frac{d^2 W_m}{dy^2}\frac{d\delta W_m}{dy}\right]_0^b = 0, \qquad (8.255)$$

where $\bar{N}_0 = (N_0/D)$. For a simply supported edge [$W = 0$ and $(d^2W/dy^2) = 0$] or a clamped edge [$W = 0$ and $(dW/dy) = 0$], the boundary terms in Eq. (8.255) vanish identically. However, for a free edge the vanishing of bending moment and effective shear force requires

$$\frac{d^2 W_m}{dy^2} - \nu\alpha_m^2 W_m = 0, \qquad \frac{d^3 W_m}{dy^3} - (2-\nu)\alpha_m^2 \frac{dW_m}{dy} = 0.$$

Hence, on a free edge, the boundary expression of Eq. (8.255) can be simplified to

$$\left[\left(\frac{d^3 W_m}{dy^3} - 2\alpha_m^2 \frac{dW_m}{dy}\right)\delta W_m - \frac{d^2 W_m}{dy^2}\frac{d\delta W_m}{dy}\right]_0^b$$
$$= -\nu\alpha_m^2 \left[W_m \frac{d\delta W_m}{dy} + \frac{dW_m}{dy}\delta W_m\right]_0^b,$$

so that the weak form for SSSS, SSCC, and SSCS plates becomes

$$0 = \int_0^b \left[\frac{d^2 W_m}{dy^2}\frac{d^2 \delta W_m}{dy^2} + 2\alpha_m^2 \frac{dW}{dy}\frac{d\delta W_m}{dy} + \left(\alpha_m^4 - \bar{N}_0\alpha_m^2\right) W_m \delta W_m\right] dy, \qquad (8.256a)$$

and for SSSF and SSCF plates it is given by

$$0 = \int_0^b \left[\frac{d^2 W_m}{dy^2} \frac{d^2 \delta W_m}{dy^2} + 2\alpha_m^2 \frac{dW_m}{dy} \frac{d\delta W_m}{dy} + \left(\alpha_m^4 - \bar{N}_0 \alpha_m^2\right) W_m \delta W_m \right] dy$$
$$- \nu\alpha_m^2 \left[W_m \frac{d\delta W_m}{dy} + \frac{dW_m}{dy} \delta W_m \right]_0^b . \tag{8.256b}$$

Next we assume an N-parameter Ritz approximation of the form

$$W_m(y) \approx \sum_{j=1}^{N} c_j^{(m)} \varphi_j(y), \tag{8.257}$$

where φ_j are the approximation functions [see Eqs. (8.225)–(8.240); replace x with y and a with b]. Substituting the approximation (8.257) into the weak form (8.256b), we obtain

$$([R] - N_0[B])\{c\} = \{0\}, \tag{8.258a}$$

where

$$R_{ij} = D \int_0^b \left(\frac{d^2\varphi_i}{dy^2} \frac{d^2\varphi_j}{dy^2} + 2\alpha_m^2 \frac{d\varphi_i}{dy} \frac{d\varphi_j}{dy} + \alpha_m^4 \varphi_i \varphi_j \right) dy$$
$$- \alpha_m^2 D\nu \left[\varphi_j \frac{d\varphi_i}{dy} + \frac{d\varphi_j}{dy} \varphi_i \right]_0^b , \tag{8.258b}$$

$$B_{ij} = \alpha_m^2 \int_0^b \varphi_i \varphi_j \, dy, \qquad \alpha_m = \frac{m\pi}{a}. \tag{8.258c}$$

The boundary term in R_{ij} is nonzero only for SSSF and SSCF plates. Equation (8.258a) has a nontrivial solution, $c_i \neq 0$, only if the determinant of the coefficient matrix is zero:

$$|[R] - N_0[B]| = 0. \tag{8.259}$$

Equation (8.259) yields an Nth-order polynomial in N_0 that depends on m. Hence, for any given aspect ratio a/b, the critical buckling load is the smallest value of N_0 for all m.

Table 8.13 contains numerical values of nondimensional critical buckling loads $\bar{N}_{cr} = N_{cr} b^2/(\pi^2 D)$ obtained using the algebraic approximation functions given in Eqs. (8.225)–(8.240) for various boundary conditions. The accuracy of the buckling loads predicted by the three-parameter Ritz approximation is very good and in agreement with the analytical solutions. Note that for certain aspect ratios there is a change in the buckling mode.

Table 8.13 Nondimensionalized buckling loads $\bar{N}$ of rectangular isotropic ($\nu = 0.25$) plates under uniform compression $\hat{N}_{xx} = N_0$ (the Ritz solutions)

a/b	N	SSSS	SSCC	SSCS	SSCF	SSSF	SSFF
0.5	1	6.334	7.725	7.915	4.896	4.456	4.000
	3	6.250	7.693	6.860	4.526	4.436	3.958
	E	6.250	7.691	6.853	4.518	4.404	—
1.0	1	4.258	8.606	8.149	2.050	1.456	1.000
	3	4.001	8.605	5.741	1.699	1.445	0.972
	E	4.000	8.604	5.740	1.698	1.434	—
1.5	1	4.497[a]	7.120[a]	7.040[a]	1.751	0.900	0.988
	3	4.341	7.116	5.432	1.339	0.894	0.426
	E	4.340	7.116	5.431	1.339	0.888	—
2.0	1	4.258	8.606	8.149	1.915	0.706	0.250
	3	4.001	8.605	5.741	1.386	0.702	0.238
	E	4.000	8.604	5.740	1.386	0.698	—

[a]Denotes change to the next higher mode; E denotes the "exact" solution obtained in Section 8.2.

Formulation for Rectangular Plates with Arbitrary Boundary Conditions

In this section we consider buckling of orthotropic rectangular plates using the Ritz method. Buckling problems of plates with combined bending and compression or under pure in-plane shear stress do not permit analytical solutions; therefore, it is useful to consider the Ritz method to solve such problems. The statement of the principle of virtual displacements or the minimum total potential energy for isotropic plates subjected to in-plane edge forces $\hat{N}_{xx}$, $\hat{N}_{yy}$, and $\hat{N}_{xy}$ can be expressed as

$$\begin{aligned}0 = \int_0^b \int_0^a \Bigg\{ D\Bigg[&\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 \delta w}{\partial x^2} + \nu\left(\frac{\partial^2 w}{\partial y^2}\frac{\partial^2 \delta w}{\partial x^2} + \frac{\partial^2 w}{\partial x^2}\frac{\partial^2 \delta w}{\partial y^2}\right)\\ &+ 2(1-\nu)\frac{\partial^2 w}{\partial x \partial y}\frac{\partial^2 \delta w}{\partial x \partial y} + \frac{\partial^2 w}{\partial y^2}\frac{\partial^2 \delta w}{\partial y^2}\Bigg]\\ &+ \hat{N}_{xx}\frac{\partial w}{\partial x}\frac{\partial \delta w}{\partial x} + \hat{N}_{yy}\frac{\partial w}{\partial y}\frac{\partial \delta w}{\partial y}\\ &+ \hat{N}_{xy}\left(\frac{\partial w}{\partial y}\frac{\partial \delta w}{\partial x} + \frac{\partial w}{\partial x}\frac{\partial \delta w}{\partial y}\right)\Bigg\} dxdy. \end{aligned} \tag{8.260}$$

In general, the applied edge forces are functions of position and they are independent of each other. Let

$$\hat{N}_{xx} = N_0, \qquad \hat{N}_{yy} = \gamma_1 N_0, \qquad \hat{N}_{xy} = \gamma_2 N_0, \tag{8.261}$$

where N_0 is a constant and γ_1 and γ_2 are possibly functions of position.

Using an N-parameter Ritz solution of the form

$$w(x, y) \approx \sum_{j=1}^{N} c_j \varphi_j(x, y)$$

in Eq. (8.260), we obtain

$$([R] - N_0[B])\{c\} = \{0\}, \tag{8.262a}$$

where

$$R_{ij} = D \int_0^b \int_0^a \left[\frac{\partial^2 \varphi_i}{\partial x^2} \frac{\partial^2 \varphi_j}{\partial x^2} + \nu \left(\frac{\partial^2 \varphi_i}{\partial y^2} \frac{\partial^2 \varphi_j}{\partial x^2} + \frac{\partial^2 \varphi_i}{\partial x^2} \frac{\partial^2 \varphi_j}{\partial y^2} \right) + 2(1 - \nu) \frac{\partial^2 \varphi_i}{\partial x \partial y} \frac{\partial^2 \varphi_j}{\partial x \partial y} + \frac{\partial^2 \varphi_i}{\partial y^2} \frac{\partial^2 \varphi_j}{\partial y^2} \right] dxdy, \tag{8.262b}$$

$$B_{ij} = \int_0^b \int_0^a \left[\frac{\partial \varphi_i}{\partial x} \frac{\partial \varphi_j}{\partial x} + \gamma_1 \frac{\partial \varphi_i}{\partial y} \frac{\partial \varphi_j}{\partial y} + \gamma_2 \left(\frac{\partial \varphi_i}{\partial x} \frac{\partial \varphi_j}{\partial y} + \frac{\partial \varphi_i}{\partial y} \frac{\partial \varphi_j}{\partial x} \right) \right] dxdy. \tag{8.262c}$$

As discussed earlier, it is convenient to express the Ritz approximation of rectangular plates in the form

$$w(x, y) \approx W_{mn}(x, y) = \sum_{i=1}^{M} \sum_{j=1}^{N} c_{ij} \, X_i(x) Y_j(y). \tag{8.263}$$

The functions X_i and Y_j for various boundary conditions are given in Eqs. (8.225)–(8.240). Substituting Eq. (8.263) into Eq. (8.260), we obtain Eq. (8.262a) with the following definitions of the coefficients:

$$R_{ij,kl} = D \int_0^b \int_0^a \left[\frac{d^2 X_i}{dx^2} \frac{d^2 X_k}{dx^2} Y_j Y_l + X_i X_k \frac{d^2 Y_j}{dy^2} \frac{d^2 Y_l}{dy^2} + \nu \left(X_i \frac{d^2 X_k}{dx^2} \frac{d^2 Y_j}{dy^2} Y_l + \frac{d^2 X_i}{dx^2} X_k Y_j \frac{d^2 Y_l}{dy^2} \right) + 2(1 - \nu) \frac{dX_i}{dx} \frac{dX_k}{dx} \frac{dY_j}{dy} \frac{dY_l}{dy} \right] dxdy, \tag{8.264a}$$

$$B_{ij,kl} = \int_0^b \int_0^a \left[\frac{dX_i}{dx} \frac{dX_k}{dx} Y_j Y_l + \gamma_1 X_i X_k \frac{dY_j}{dy} \frac{dY_l}{dy} + \gamma_2 \left(\frac{dX_i}{dx} X_k Y_j \frac{dY_l}{dy} + X_i \frac{dX_k}{dx} \frac{dY_j}{dy} Y_l \right) \right] dxdy. \tag{8.264b}$$

Example 8.22 Consider a simply supported plate. For the choice of algebraic functions

$$X_i(x) = \left(\frac{x}{a}\right)^i - \left(\frac{x}{a}\right)^{i+1}, \qquad Y_j(y) = \left(\frac{y}{b}\right)^i - \left(\frac{y}{b}\right)^{j+1}, \tag{a}$$

and for $M = N = 1$, we obtain

$$\begin{aligned} R_{11,11} &= D\left(\frac{2b}{15a^3} + \frac{2}{9ab} + \frac{2a}{15b^3}\right), \\ B_{11,11} &= \frac{1}{90} + \frac{\gamma_1}{90} + \gamma_2 \times 0. \end{aligned} \tag{b}$$

Note that the buckling load under in-plane shear cannot be determined with a one-parameter approximation. The buckling load under uniaxial compression along the x-axis is given by setting $\gamma_1 = 0$:

$$N_{cr}^u = D\left(\frac{12b}{a^3} + \frac{20}{ab} + \frac{12a}{b^3}\right), \tag{c}$$

and the critical buckling load under biaxial compression is given by ($\gamma_1 = 1$):

$$N_{cr}^b = D\left(\frac{6b}{a^3} + \frac{10}{ab} + \frac{6a}{b^3}\right). \tag{d}$$

For square isotropic plates, the above expressions become

$$N_{cr}^u = 44\frac{D}{a^2} = 4.458\frac{D\pi^2}{a^2}, \qquad N_{cr}^b = 22\frac{D}{a^2} = 2.229\frac{D\pi^2}{a^2}.$$

The exact values for the two cases are $4D\pi^2/a^2$ and $2D\pi^2/a^2$, respectively. The results are in about -11.5% error, and the values do not change for $N = M = 2$.

For a square isotropic plate clamped on all sides and subjected to uniaxial compression (N_{cr}^u) or biaxial compression (N_{cr}^b), the buckling loads obtained using the Ritz method with algebraic polynomials given in Eqs. (8.239a,b) are (for $N = M =$ 1 or 2):

$$N_{cr}^u = 10.943\frac{D\pi^2}{a^2}, \qquad N_{cr}^b = 5.471\frac{D\pi^2}{a^2}.$$

Example 8.23 Let us consider a simply supported rectangular plate (see Fig. 8.25) with distributed in-plane forces applied in the middle plane of the plate on sides $x = 0, a$. The distribution of the applied forces is assumed to be

$$\hat{N}_{xx} = -N_0\gamma_1 = -N_0\left(1 - c_0\frac{y}{b}\right), \tag{a}$$

where N_0 is the magnitude of the compressive force at $y = 0$ and c_0 is a parameter that defines the relative bending and compression. For example, $c_0 = 0$ corresponds

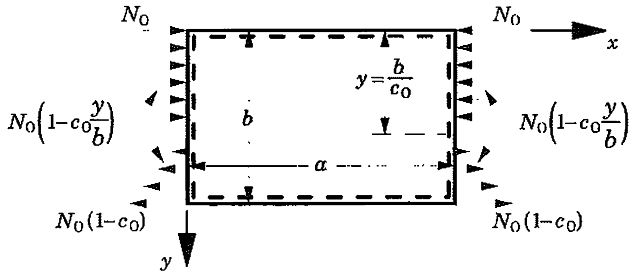


Figure 8.25 Buckling of simply supported plates under combined bending and compression.

to the case of uniformly distributed compressive force, as discussed in Section 8.2, and for $c_0 = 2$ we obtain the case of pure bending. All other values give a combination of bending and compression or tension.

We seek the deflection of the buckled plate, which is simply supported on all sides, in the form of a double sine series

$$w(x, y) \approx W_{MN} = \sum_{n=1}^{N}\sum_{m=1}^{M} c_{mn} \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b} \tag{b}$$

that satisfies the geometric as well as the force boundary conditions of the problem.

Substituting Eq. (b) for w and

$$\delta w = \sin\frac{p\pi x}{a} \sin\frac{q\pi y}{b}$$

into Eq. (8.260) with $\hat{N}_{yy} = \hat{N}_{xy} = 0$, we obtain

$$\begin{aligned} 0 = D\sum_{n=1}^{N}\sum_{m=1}^{M} c_{mn} &\left\{\left(\frac{m\pi}{a}\right)^2\left(\frac{p\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2\left(\frac{q\pi}{b}\right)^2\right.\\ &\left.+\nu\left[\left(\frac{n\pi}{b}\right)^2\left(\frac{p\pi}{a}\right)^2 + \left(\frac{m\pi}{a}\right)^2\left(\frac{q\pi}{b}\right)^2\right]\right\} I_1 \\ &+ D\sum_{n=1}^{N}\sum_{m=1}^{M} c_{mn} 2(1-\nu)\left(\frac{m\pi}{a}\right)\left(\frac{n\pi}{b}\right)\left(\frac{p\pi}{a}\right)\left(\frac{q\pi}{b}\right) I_2 \\ &-\sum_{n=1}^{N}\sum_{m=1}^{M}\sum_{q=1}^{M} c_{mq} N_0 \left(\frac{m\pi}{a}\right)\left(\frac{p\pi}{a}\right) I_{nq}, \end{aligned} \tag{c}$$

where I_1 and I_2 are nonzero only when $p = m$ and $q = n$:

$$I_1 = \int_0^b\int_0^a \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b} \sin\frac{p\pi x}{a} \sin\frac{q\pi y}{b}\, dx dy = \frac{ab}{4},$$

$$I_2 = \int_0^b\int_0^a \cos\frac{m\pi x}{a} \cos\frac{n\pi y}{b} \cos\frac{p\pi x}{a} \cos\frac{q\pi y}{b}\, dx dy = \frac{ab}{4},$$

and I_{nq} is defined as

$$I_{nq} = \int_0^b \int_0^a \left(1 - c_0\frac{y}{b}\right) \cos\frac{m\pi x}{a} \cos\frac{p\pi x}{a} \sin\frac{n\pi y}{b} \sin\frac{q\pi y}{b}\, dxdy,$$

which can be computed with the help of the following identities:

$$\begin{aligned} I_0 &\equiv \int_0^b y \sin\frac{n\pi y}{b} \sin\frac{q\pi y}{b}\, dy \\ &= \frac{b^2}{4} \quad \text{when } n = q \\ &= 0 \quad \text{when } n \neq q \text{ and } n \pm q \text{ is an even number} \\ &= -\frac{b^2}{\pi^2}\frac{2nq}{(n^2 - q^2)^2} \quad \text{when } n \neq q \text{ and } n \pm q \text{ is an odd number.} \end{aligned}$$

Thus $I_{nq} = 0$ if $p \neq m$ and

$$I_{nq} = \frac{a}{2}\left(\frac{b}{2} - \frac{c_0}{b} I_0\right),$$

when $p = m$. For any m and n, Eq. (c) becomes

$$\begin{aligned} 0 = &D\left[\left(\frac{m\pi}{a}\right)^4 + \left(\frac{n\pi}{b}\right)^4 + 2\left(\frac{n\pi}{b}\right)^2\left(\frac{m\pi}{a}\right)^2\right] c_{mn} \\ &- N_0\left(\frac{m\pi}{a}\right)^2 \left[c_{mn} - \frac{c_0}{2}\left(c_{mn} - \frac{8}{\pi^2}\sum_{q=1}^{M} \frac{2nqc_{mq}}{(n^2 - q^2)^2}\right)\right], \end{aligned} \tag{d}$$

where the summation is taken over all numbers q such that $n \pm q$ is an odd number. Taking $m = 1$ in Eq. (d), we obtain

$$D\left(s^2 + n^2\right)^2 c_{1n} = N_0 \frac{s^2 b^2}{\pi^2}\left[c_{1n}\left(1 - \frac{c_0}{2}\right) + \frac{8c_0}{\pi^2}\sum_{q=1}^{M}\frac{nqc_{1q}}{(n^2 - q^2)^2}\right], \tag{e}$$

where s denotes the aspect ratio $s = b/a$. A nontrivial solution, i.e., for nonzero c_{1i}, the determinant of the linear equations in (e) must be zero if the plate buckles.

For the one-parameter approximation ($N = M = 1$), we obtain

$$N_0 = \frac{\pi^2 D}{s^2 b^2}(s^2 + n^2)^2 \frac{1}{1 - 0.5c_0}, \tag{f}$$

which gives a satisfactory result only for small values of c_0, i.e., in cases where the bending stresses are small compared with the uniform compressive stress. Higher-order approximations yield sufficiently accurate results for the case of pure bending. Table 8.14 contains critical buckling loads $\bar{N} = N_0(b^2/\pi^2 D)$ obtained using

Table 8.14 Nondimensionalized critical buckling loads $\bar{N}$ of simply supported rectangular isotropic plates under combined bending and compression $\hat{N}_{xy} = N_0(1 - c_0 y/b)$ (the Ritz solutions)

c_0 $a/b \rightarrow$	0.4	0.5	0.6	2/3	0.75	0.8	0.9	1.0	1.5
2	29.1	25.6	24.1	23.9	24.1	24.4	25.6	25.6	24.1
4/3	18.7	—	12.9	—	11.5	11.2	—	11.0	11.5
1	15.1	—	9.7	—	8.4	8.1	—	7.8	8.4
4/5	13.3	—	8.3	—	7.1	6.9	—	6.6	7.1
2/3	10.8	—	7.1	—	6.1	6.0	—	5.8	6.1

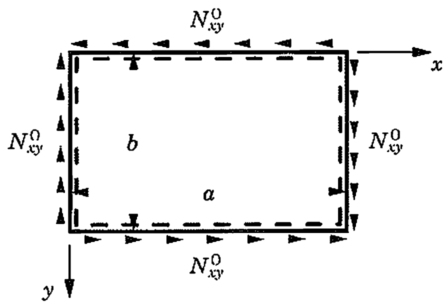


Figure 8.26 Buckling of rectangular plates under the action of shearing stresses.

the two-parameter approximation, except for $c_0 = 2$, where the three-parameter approximation was used (see [30]).

Example 8.24 When the plate is simply supported on all its edges and subjected to uniformly distributed in-plane shear force $\hat{N}_{xy} = N^0_{xy}$ (see Fig. 8.26), the Navier or Lévy solution procedure cannot be used to determine the critical buckling load. Hence we will seek an approximate solution by a variational method.

Let us seek the solution in the form

$$w(x, y) \approx W_{MN} = \sum_{n=1}^{N} \sum_{m=1}^{M} c_{mn} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b}. \tag{a}$$

The approximate solution satisfies the geometric ($w = 0$) as well as the natural ($M_{xx} = 0$ on sides $x = 0, a$ and $M_{yy} = 0$ on sides $y = 0, b$) boundary conditions of the problem. Therefore, both the Ritz and Galerkin methods yield the same solutions.

The Galerkin solution is obtained by substituting Eq. (a) in the weighted-residual statement

$$0 = \int_0^b \int_0^a \left[D\left(\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} \right) - 2N^0_{xy} \frac{\partial^2 w}{\partial x \partial y} \right] \varphi_{pq}\, dx dy. \tag{b}$$

We obtain

$$0 = \frac{Dab}{4}\left[\left(\frac{p\pi}{a}\right)^4 + \left(\frac{p\pi}{a}\right)^2\left(\frac{q\pi}{b}\right)^2 + \left(\frac{q\pi}{b}\right)^4\right]c_{pq}$$
$$- 2N_{xy}^0 \sum_{m=1}^{N}\sum_{n=1}^{M}\left(\frac{m\pi}{a}\right)\left(\frac{n\pi}{b}\right) I_{mp} I_{nq}\, c_{mn}, \tag{c}$$

where

$$I_{mp} = \int_0^a \cos\frac{m\pi x}{a}\sin\frac{p\pi x}{a}\,dx = \left(\frac{2a}{\pi^2}\right)\frac{p}{(p^2 - m^2)} \qquad \text{for } p^2 \neq m^2,$$
$$I_{nq} = \int_0^b \cos\frac{n\pi y}{b}\sin\frac{q\pi y}{b}\,dy = \left(\frac{2b}{\pi^2}\right)\frac{q}{(q^2 - n^2)} \qquad \text{for } q^2 \neq n^2,$$

and the integral I_{mp} is zero when $p = m$ or $p \pm m$ is an even number, and I_{nq} is zero when $q = n$ or $q \pm n$ is an even number. The set of mn homogeneous equations (c) define an eigenvalue problem

$$\sum_{m=1}^{N}\sum_{n=1}^{M}\left(A_{(mn),(pq)} - N_{xy}^0 G_{(mn)(pq)}\right)c_{mn} = 0$$
$$\left([A] - N_{xy}^0[G]\right)\{c\} = \{0\}, \tag{d}$$

where

$$A_{mn,pq} = D\delta_{mp}\delta_{nq}\frac{ab}{4}\left[\alpha_m^2\alpha_p^2 + 2\alpha_m\alpha_p\beta_n\beta_q + \beta_n^2\beta_q^2\right],$$
$$G_{mn,pq} = 2\alpha_m\beta_n\, I_{mp}I_{nq}, \quad \alpha_m = \frac{m\pi}{a}, \quad \beta_n = \frac{n\pi}{b}.$$

Equation (d) has a nontrivial solution (i.e., $c_{mn} \neq 0$) when the determinant of the coefficient matrix is zero. Note that $[A]$ is a diagonal matrix while $[G]$ is a nonpositive-definite matrix; hence the solution of (d) requires an eigenvalue routine that is suitable for nonpositive-definite matrices. It is found that the solution of (d) converges very slowly with increasing values of M and N. We note that $[G]$ does not exist for $M = N = 1$.

For $M = N = 2$, we find from Eq. (d) the result ($G_{11,12} = G_{11,21} = 0$)

$$\left(\begin{bmatrix} A_{11} & 0 \\ 0 & A_{22}\end{bmatrix} - N_{xy}^0\begin{bmatrix} 0 & G_{12} \\ G_{21} & 0\end{bmatrix}\right)\begin{Bmatrix} c_{11} \\ c_{22}\end{Bmatrix} = \begin{Bmatrix} 0 \\ 0\end{Bmatrix}, \tag{e}$$

where the coefficients are given by

$$A_{11} \equiv A_{11,11} = \frac{\pi^4 D}{4sb^2}(s^2+1)^2, \qquad A_{22} \equiv A_{22,22} = 16A_{11,11},$$
$$G_{11} \equiv G_{11,11} = G_{22} \equiv G_{22,22} = 0, \qquad G_{12} \equiv G_{11,22} = G_{21} \equiv G_{22,11} = \tfrac{32}{9},$$

Table 8.15 Nondimensionalized critical buckling loads $\bar{N}$ of rectangular isotropic ($\nu = 0.25$) plates under uniform shear $\hat{N}_{xy} = N_{xy}^0$

a/b	1.0	1.2	1.4	1.5	1.6	1.8	2.0	2.5	3	4
$\bar{N}$	9.34	8.0	7.3	7.1	7.0	6.8	6.6	6.1	5.9	5.7

and $s = b/a$ is the plate aspect ratio. Setting the determinant of the coefficient matrix in Eq. (e) to zero, we obtain

$$N_{xy}^0 = \pm\frac{9D\pi^4}{32sb^2}(s^2+1)^2.$$

The two signs indicate that the value of the critical buckling load does not depend on sign.

Timoshenko and Gere [30] obtained the following equation for short isotropic plates ($a/b < 2$) using a five-term ($c_{11}, c_{22}, c_{13} = c_{31}, c_{33}$ and c_{42}) approximation:

$$\lambda^2 = \frac{s^4}{81(1+s^2)^4}\left[1 + \frac{81}{625} + \frac{81}{25}\left(\frac{1+s^2}{9+s^2}\right)^2 + \frac{81}{25}\left(\frac{1+s^2}{1+9s^2}\right)^2\right],$$

where

$$s = \frac{b}{a}, \qquad \lambda = -\frac{\pi^4 D}{32abN_{xy}^0}.$$

For a square plate, the above equation yields the critical buckling load $(N_{xy}^0)_{cr} = 9.4(\pi^2 D/b^2)$, whereas the value obtained with a larger (than 5) number of equations give 9.34 in place of 9.4. Table 8.15 contains the critical buckling loads $\bar{N} = (N_{xy}^0)_{cr}(b^2/\pi^2 D)$ obtained using a large number of parameters (see [30]).

Example 8.25 Here we consider the buckling of a clamped rectangular plate under in-plane shear. The principle of the minimum total potential energy for this case is

$$0 = D\int_0^b\int_0^a\left[\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 \delta w}{\partial x^2} + \nu\left(\frac{\partial^2 w}{\partial y^2}\frac{\partial^2 \delta w}{\partial x^2} + \frac{\partial^2 w}{\partial x^2}\frac{\partial^2 \delta w}{\partial y^2}\right)\right.$$
$$+ 2(1-\nu)\frac{\partial^2 w}{\partial x\partial y}\frac{\partial^2 \delta w}{\partial x\partial y} + \frac{\partial^2 w}{\partial y^2}\frac{\partial^2 \delta w}{\partial y^2}$$
$$\left. - \frac{N_{xy}^0}{D}\left(\frac{\partial \delta w}{\partial x}\frac{\partial w}{\partial y} + \frac{\partial w}{\partial x}\frac{\partial \delta w}{\partial y}\right)\right]dxdy. \tag{a}$$

We assume a Ritz approximation of the form

$$w(x, y) \approx W_{MN}(x, y) = \sum_{j=1}^{N} \sum_{i=1}^{M} c_{ij} \, X_i(x) \, Y_j(y),$$

and we obtain

$$0 = D \sum_{j=1}^{N} \sum_{i=1}^{M} \left\{ \int_0^b \int_0^a \left[\frac{d^2 X_i}{dx^2} Y_j \frac{d^2 X_p}{dx^2} Y_q + X_i \frac{d^2 Y_j}{dy^2} X_p \frac{d^2 Y_q}{dy^2} \right. \right.$$
$$+ 2 \frac{dX_i}{dx} \frac{dY_j}{dy} \frac{dX_p}{dx} \frac{dY_q}{dy}$$
$$\left. \left. - \frac{N_{xy}^0}{D} \left(\frac{dX_i}{dx} Y_j X_p \frac{dY_q}{dy} + X_i \frac{dY_j}{dy} \frac{dX_p}{dx} Y_q \right) \right] dx dy \right\} c_{ij}. \quad \text{(b)}$$

Using the two-parameter approximation

$$w(x, y) \approx c_{11} X_1(x) Y_1(y) + c_{22} X_2(x) Y_2(y), \quad \text{(c)}$$

with

$$X_1(x) = \sin \frac{4.73x}{a} - \sinh \frac{4.73x}{a} + 1.0178 \left(\cosh \frac{4.73x}{a} - \cos \frac{4.73x}{a} \right),$$
$$X_2(x) = \sin \frac{7.853x}{a} - \sinh \frac{7.853x}{a} + 0.9992 \left(\cosh \frac{7.853x}{a} - \cos \frac{7.853x}{a} \right),$$
$$Y_1(y) = \sin \frac{4.73y}{b} - \sinh \frac{4.73y}{b} + 1.0178 \left(\cosh \frac{4.73y}{b} - \cos \frac{4.73y}{b} \right),$$
$$Y_2(y) = \sin \frac{7.853y}{b} - \sinh \frac{7.853y}{b} + 0.9992 \left(\cosh \frac{7.853y}{b} - \cos \frac{7.853y}{b} \right),$$

we obtain

$$\begin{bmatrix} A_{11} & -N_{xy}^0 A_{12} \\ -N_{xy}^0 A_{12} & A_{22} \end{bmatrix} \begin{Bmatrix} c_{11} \\ c_{22} \end{Bmatrix} = - \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}, \quad \text{(d)}$$

where

$$A_{11} = D \left(\frac{537.181}{a^4} + \frac{324.829}{a^2 b^2} + \frac{537.181}{b^4} \right), \qquad A_{12} = \frac{23.107}{ab},$$
$$A_{22} = D \left(\frac{3791.532}{a^4} + \frac{4227.255}{a^2 b^2} + \frac{3791.532}{b^4} \right).$$

For a nontrivial solution, the determinant of the coefficient matrix in Eq. (d) should be zero, $A_{11}A_{22} - A_{12}A_{12}(N_{xy}^0)^2 = 0$. Solving for the buckling load N_{xy}^0, we obtain

$$N_{xy}^0 = \pm\frac{1}{A_{12}}\sqrt{A_{11}A_{22}}.$$

The $\pm$ sign indicates that the shear buckling load may be either positive or negative. For an isotropic square plate, we have $a = b$, and the shear buckling load becomes

$$N_{xy}^0 = \pm 176\frac{D}{a^2}, \tag{e}$$

whereas the "exact" critical buckling load is

$$N_{xy}^0 = \pm 145\frac{D}{a^2}. \tag{f}$$

The two-term Ritz solution (e) is over 21% in error.

This completes the discussion on the application of the Ritz method to the buckling of rectangular plates.

8.3 SHEAR DEFORMATION PLATE THEORY

8.3.1 Governing Equations of Circular Plates

We begin with the following displacement field:

$$\begin{aligned} u_r(r,\theta,z) &= z\phi_r(r,\theta),\\ u_\theta(r,\theta,z) &= z\phi_\theta(r,\theta),\\ u_z(r,\theta,z) &= w_0(r,\theta), \end{aligned} \tag{8.265}$$

where w_0 is the transverse displacement and (ϕ_r, ϕ_θ) are rotations of a transverse normal line about the (r, θ) coordinates. The quantities $(w_0, \phi_r, \phi_\theta)$ are called the *generalized displacements*. For thin plates, i.e., when the plate in-plane characteristic dimension to thickness ratio is on the order of 50 or greater, the rotation functions ϕ_r and ϕ_θ will approach the respective slopes of the transverse deflection:

$$\phi_r = -\frac{\partial w_0}{\partial r}, \qquad \phi_\theta = -\frac{1}{r}\frac{\partial w_0}{\partial \theta}.$$

The linear strain components referred to the cylindrical coordinate system are given by

$$\varepsilon_{rr} = z\varepsilon_{rr}^{(1)}, \qquad \varepsilon_{\theta\theta} = z\varepsilon_{\theta\theta}^{(1)}, \qquad 2\varepsilon_{r\theta} = z\gamma_{r\theta}^{(1)}, \tag{8.266a}$$

$$2\varepsilon_{z\theta} = \phi_\theta + \frac{1}{r}\frac{\partial w_0}{\partial \theta}, \qquad 2\varepsilon_{zr} = \phi_r + \frac{\partial w_0}{\partial r}, \tag{8.266b}$$

where

$$\varepsilon_{rr}^{(1)} = \frac{\partial \phi_r}{\partial r}, \qquad \varepsilon_{\theta\theta}^{(1)} = \frac{\phi_r}{r} + \frac{1}{r}\frac{\partial \phi_\theta}{\partial \theta},$$
$$\gamma_{r\theta}^{(1)} = \frac{1}{r}\frac{\partial \phi_r}{\partial \theta} + \frac{\partial \phi_\theta}{\partial r} - \frac{\phi_\theta}{r}. \tag{8.266c}$$

Since we are interested in deriving the equations of motion and the form of the boundary conditions, we assume that the plate is subjected to transverse load q. Using the principle of virtual displacements, $\delta W = 0$, we can write

$$0 = \int_{\Omega_0}\int_{-h/2}^{h/2} (\sigma_{rr}\delta\varepsilon_{rr} + \sigma_{\theta\theta}\delta\varepsilon_{\theta\theta} + \sigma_{r\theta}\delta\gamma_{r\theta} + \sigma_{rz}\delta\gamma_{rz} + \sigma_{z\theta}\delta\gamma_{z\theta})\, dz\, r\, dr d\theta$$
$$- \int_{\Omega_0} q\delta w_0\, r\, dr d\theta$$
$$= \int_{\Omega_0}\left[M_{rr}\frac{\partial \delta\phi_r}{\partial r} + M_{r\theta}\left(\frac{1}{r}\frac{\partial \delta\phi_r}{\partial \theta} + \frac{\partial \delta\phi_\theta}{\partial r} - \frac{\delta\phi_\theta}{r}\right) + M_{\theta\theta}\left(\frac{\delta\phi_r}{r} + \frac{1}{r}\frac{\partial \delta\phi_\theta}{\partial \theta}\right)\right.$$
$$\left. + Q_r\left(\delta\phi_r + \frac{\partial \delta w_0}{\partial r}\right) + Q_\theta\left(\delta\phi_\theta + \frac{1}{r}\frac{\partial \delta w_0}{\partial \theta}\right) - q\delta w_0\right] r\, dr d\theta, \tag{8.267a}$$

where [also see Eq. (8.19)]

$$Q_\theta = K_s \int_{-h/2}^{h/2} \sigma_{z\theta}\, dz, \qquad Q_r = K_s \int_{-h/2}^{h/2} \sigma_{zr}\, dz. \tag{8.267b}$$

The Euler equations are

$$\delta\phi_r: \quad -\frac{1}{r}\left[\frac{\partial}{\partial r}(rM_{rr}) + \frac{\partial M_{r\theta}}{\partial \theta} - M_{\theta\theta}\right] + Q_r = 0, \tag{8.268}$$

$$\delta\phi_\theta: \quad -\frac{1}{r}\left[\frac{\partial}{\partial r}(rM_{r\theta}) + \frac{\partial M_{\theta\theta}}{\partial \theta} + M_{r\theta}\right] + Q_\theta = 0, \tag{8.269}$$

$$\delta w_0: \quad -\frac{1}{r}\left[\frac{\partial}{\partial r}(rQ_r) + \frac{\partial Q_\theta}{\partial \theta}\right] - q = 0. \tag{8.270}$$

The natural boundary conditions are [in addition to Eqs. (8.25a–c)]

$$\delta\phi_r: \quad rM_{rr}n_r + M_{r\theta}n_\theta = 0, \tag{8.271}$$

$$\delta\phi_\theta: \quad rM_{r\theta}n_r + M_{\theta\theta}n_\theta = 0, \tag{8.272}$$

$$\delta w_0: \quad rQ_rn_r + Q_\theta n_\theta = 0. \tag{8.273}$$

The bending moments and shear forces are related to the displacements $(w_0, \phi_r, \phi_\theta)$ by

$$M_{rr} = D\left[\frac{\partial \phi_r}{\partial r} + \frac{\nu}{r}\left(\phi_r + \frac{1}{r}\frac{\partial \phi_\theta}{\partial \theta}\right)\right], \tag{8.274a}$$

$$M_{\theta\theta} = D\left[\nu\frac{\partial\phi_r}{\partial r} + \frac{1}{r}\left(\phi_r + \frac{1}{r}\frac{\partial\phi_\theta}{\partial\theta}\right)\right], \tag{8.274b}$$

$$M_{r\theta} = (1-\nu)D\frac{1}{r}\left(\frac{\partial\phi_r}{\partial\theta} + r\frac{\partial\phi_\theta}{\partial r} - \phi_0\right), \tag{8.274c}$$

$$Q_r = K_s\int\sigma_{rz}\,dz = K_sGh\left(\phi_r + \frac{\partial w_0}{\partial r}\right), \tag{8.274d}$$

$$Q_\theta = K_s\int\sigma_{\theta z}\,dz = K_sGh\left(\phi_\theta + \frac{1}{r}\frac{\partial w_0}{\partial\theta}\right), \tag{8.274c}$$

where K_s denotes the shear correction factor, $D = Eh^3/[12(1-\nu^2)]$, E Young's modulus, G the shear modulus, ν Poisson's ratio, and h the thickness of the plate.

8.3.2 Governing Equations in Rectangular Coordinates

We begin with the displacement field for the general case that includes in-plane deformation and time dependency:

$$\begin{aligned} u(x,y,z,t) &= u_0(x,y,t) + z\phi_x(x,y,t),\\ v(x,y,z,t) &= v_0(x,y,t) + z\phi_y(x,y,t),\\ w(x,y,z,t) &= w_0(x,y,t). \end{aligned} \tag{8.275}$$

As before, (u_0, v_0, w_0) denote the displacements of a point on the plane $z = 0$, and ϕ_x and ϕ_y are the rotations of a transverse normal about the y- and x-axes, respectively (see Fig. 8.27).

The linear strains associated with the displacement field (8.275) are

$$\begin{Bmatrix}\varepsilon_{xx}\\ \varepsilon_{yy}\\ \gamma_{yz}\\ \gamma_{xz}\\ \gamma_{xy}\end{Bmatrix} = \begin{Bmatrix}\varepsilon_{xx}^0\\ \varepsilon_{yy}^0\\ \gamma_{yz}^0\\ \gamma_{xz}^0\\ \gamma_{xy}^0\end{Bmatrix} + z\begin{Bmatrix}\varepsilon_{xx}^1\\ \varepsilon_{yy}^1\\ 0\\ 0\\ \gamma_{xy}^1\end{Bmatrix}, \quad \text{or} \quad \{\varepsilon\} = \{\varepsilon^0\} + z\{\varepsilon^1\}, \tag{8.276a}$$

$$\{\varepsilon^0\} = \begin{Bmatrix}\dfrac{\partial u_0}{\partial x}\\ \dfrac{\partial v_0}{\partial y}\\ \dfrac{\partial w_0}{\partial y} + \phi_y\\ \dfrac{\partial w_0}{\partial x} + \phi_x\\ \dfrac{\partial u_0}{\partial y} + \dfrac{\partial v_0}{\partial x}\end{Bmatrix}, \quad \{\varepsilon^1\} = \begin{Bmatrix}\dfrac{\partial\phi_x}{\partial x}\\ \dfrac{\partial\phi_y}{\partial y}\\ 0\\ 0\\ \dfrac{\partial\phi_x}{\partial y} + \dfrac{\partial\phi_y}{\partial x}\end{Bmatrix}, \tag{8.276b}$$

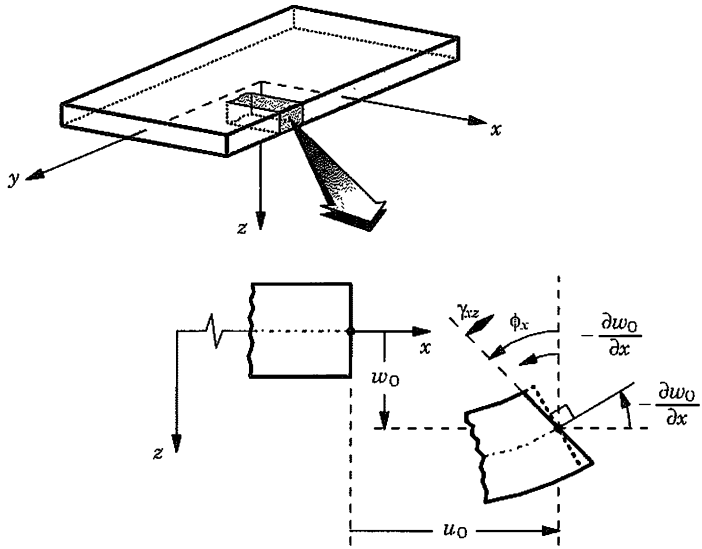


Figure 8.27 Undeformed and deformed geometries of an edge of a plate under the assumptions of the first-order shear deformation plate theory (FSDT).

The governing equations of the first-order plate theory can be derived using the dynamic version of the principle of virtual displacements (or Hamilton's principle of Chapter 6):

$$
\begin{aligned}
0 = \int_0^T \Bigg\{ \int_{\Omega_0} \Bigg[& N_{xx}\delta\varepsilon_{xx}^0 + M_{xx}\delta\varepsilon_{xx}^1 + N_{yy}\delta\varepsilon_{yy}^0 + M_{yy}\delta\varepsilon_{yy}^1 + N_{xy}\delta\gamma_{xy}^0 \\
& + M_{xy}\delta\gamma_{xy}^1 + Q_x\delta\gamma_{xz}^0 + Q_y\delta\gamma_{yz}^0 + kw_0\delta w_0 - q\delta w_0 \\
& - I_0\left(\dot{u}_0\delta\dot{u}_0 + \dot{v}_0\delta\dot{v}_0 + \dot{w}_0\delta\dot{w}_0\right) - I_2\left(\dot{\phi}_x\delta\dot{\phi}_x + \dot{\phi}_y\delta\dot{\phi}_y\right) \\
& - \hat{N}_{xx}\frac{\partial w_0}{\partial x}\frac{\partial \delta w_0}{\partial x} - \hat{N}_{yy}\frac{\partial w_0}{\partial y}\frac{\partial \delta w_0}{\partial y} \\
& - \hat{N}_{xy}\left(\frac{\partial w_0}{\partial x}\frac{\partial \delta w_0}{\partial y} + \frac{\partial w_0}{\partial y}\frac{\partial \delta w_0}{\partial x}\right)\Bigg] dxdy \\
& - \int_{\Gamma_\sigma}\left(\hat{N}_{nn}\delta u_n + \hat{N}_{ns}\delta u_s + \hat{M}_{nn}\delta\phi_n + \hat{M}_{ns}\delta\phi_s + \hat{Q}_n\delta w_0\right)ds \Bigg\} dt,
\end{aligned}
\tag{8.277}
$$

where k denotes the modulus of the elastic foundation, and the inertias (I_0, I_2) were defined in Eqs. (8.129), (M_{nn}, M_{ns}) in Eq. (8.138a,b), the transverse shear forces (Q_x, Q_y) are defined by

$$
\begin{Bmatrix} Q_x \\ Q_y \end{Bmatrix} = K_s \int_{-h/2}^{h/2} \begin{Bmatrix} \sigma_{xz} \\ \sigma_{yz} \end{Bmatrix} dz, \tag{8.278}
$$

and in-plane forces (N_{xx}, N_{xy}, N_{yy}) are defined by

$$N_{xx} = \int_{-h/2}^{h/2} \sigma_{xx}\, dz, \quad N_{xy} = \int_{-h/2}^{h/2} \sigma_{xy}\, dz, \quad N_{yy} = \int_{-h/2}^{h/2} \sigma_{yy}\, dz. \tag{8.279}$$

Here $(\hat{N}_{xx}, \hat{N}_{yy}, \hat{N}_{xy})$ denote the applied in-plane compressive and shear forces on the edges and (N_{nn}, N_{ns}) are defined in a manner similar to the moments in Eq. (8.138a,b).

The Euler–Lagrange equations are

$$\delta u_0: \quad \frac{\partial N_{xx}}{\partial x} + \frac{\partial N_{xy}}{\partial y} = I_0 \frac{\partial^2 u_0}{\partial t^2}, \tag{8.280}$$

$$\delta v_0: \quad \frac{\partial N_{xy}}{\partial x} + \frac{\partial N_{yy}}{\partial y} = I_0 \frac{\partial^2 v_0}{\partial t^2}, \tag{8.281}$$

$$\delta w_0: \quad \frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} - k w_0 + q - \frac{\partial}{\partial x}\left(\hat{N}_{xx}\frac{\partial w_0}{\partial x} + \hat{N}_{xy}\frac{\partial w_0}{\partial y}\right)$$
$$- \frac{\partial}{\partial y}\left(\hat{N}_{xy}\frac{\partial w_0}{\partial x} + \hat{N}_{yy}\frac{\partial w_0}{\partial y}\right) = I_0 \frac{\partial^2 w_0}{\partial t^2}, \tag{8.282}$$

$$\delta \phi_x: \quad \frac{\partial M_{xx}}{\partial x} + \frac{\partial M_{xy}}{\partial y} - Q_x = I_2 \frac{\partial^2 \phi_x}{\partial t^2}, \tag{8.283}$$

$$\delta \phi_y: \quad \frac{\partial M_{xy}}{\partial x} + \frac{\partial M_{yy}}{\partial y} - Q_y = I_2 \frac{\partial^2 \phi_y}{\partial t^2}. \tag{8.284}$$

The natural boundary conditions are

$$\begin{aligned} &N_{nn} - \hat{N}_{nn} = 0, \quad N_{ns} - \hat{N}_{ns} = 0, \quad Q_n - \hat{Q}_n = 0, \\ &M_{nn} - \hat{M}_{nn} = 0, \quad M_{ns} - \hat{M}_{ns} = 0, \end{aligned} \tag{8.285}$$

where

$$Q_n \equiv Q_x n_x + Q_y n_y. \tag{8.286}$$

The stress resultants in an isotropic plate are related to the generalized displacements by

$$\begin{Bmatrix} N_{xx} \\ N_{yy} \\ N_{xy} \end{Bmatrix} = A \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix} \begin{Bmatrix} \frac{\partial u_0}{\partial x} \\ \frac{\partial v_0}{\partial y} \\ \frac{\partial u_0}{\partial y} + \frac{\partial v_0}{\partial x} \end{Bmatrix}, \tag{8.287}$$

$$\begin{Bmatrix} M_{xx} \\ M_{yy} \\ M_{xy} \end{Bmatrix} = D \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \dfrac{1-\nu}{2} \end{bmatrix} \begin{Bmatrix} \dfrac{\partial \phi_x}{\partial x} \\ \dfrac{\partial \phi_y}{\partial y} \\ \dfrac{\partial \phi_x}{\partial y} + \dfrac{\partial \phi_y}{\partial x} \end{Bmatrix}, \tag{8.288}$$

$$\begin{Bmatrix} Q_y \\ Q_x \end{Bmatrix} = K_s G h \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{Bmatrix} \dfrac{\partial w_0}{\partial y} + \phi_y \\ \dfrac{\partial w_0}{\partial x} + \phi_x \end{Bmatrix}, \tag{8.289}$$

where the extensional stiffness A and bending stiffness D are defined as:

$$A = \frac{Eh}{1-\nu^2}, \qquad D = \frac{Eh^3}{12(1-\nu^2)}.$$

In the present study we are interested in the bending deformation of plates, and it is governed by Eqs. (8.282)–(8.284). These equations can be expressed in terms of the generalized displacements (w_0, ϕ_x, ϕ_y) by means of Eqs. (8.288) and (8.289):

$$K_s G h \left(\frac{\partial^2 w_0}{\partial x^2} + \frac{\partial^2 w_0}{\partial y^2} + \frac{\partial \phi_x}{\partial x} + \frac{\partial \phi_y}{\partial y} \right) - k w_0 + q(x, y)$$
$$= I_0 \frac{\partial^2 w_0}{\partial t^2} + \frac{\partial}{\partial x} \left(\hat{N}_{xx} \frac{\partial w_0}{\partial x} + \hat{N}_{xy} \frac{\partial w_0}{\partial y} \right) + \frac{\partial}{\partial y} \left(\hat{N}_{xy} \frac{\partial w_0}{\partial x} + \hat{N}_{yy} \frac{\partial w_0}{\partial y} \right), \tag{8.290}$$

$$D \left[\frac{\partial^2 \phi_x}{\partial x^2} + \nu \frac{\partial^2 \phi_y}{\partial y \partial x} + \frac{(1-\nu)}{2} \left(\frac{\partial^2 \phi_x}{\partial y^2} + \frac{\partial^2 \phi_y}{\partial y \partial x} \right) \right]$$
$$- K_s G h \left(\frac{\partial w_0}{\partial x} + \phi_x \right) = I_2 \frac{\partial^2 \phi_x}{\partial t^2}, \tag{8.291}$$

$$D \left[\frac{(1-\nu)}{2} \left(\frac{\partial^2 \phi_x}{\partial x \partial y} + \frac{\partial^2 \phi_y}{\partial x^2} \right) + \nu \frac{\partial^2 \phi_x}{\partial x \partial y} + \frac{\partial^2 \phi_y}{\partial y^2} \right]$$
$$- K_s G h \left(\frac{\partial w_0}{\partial y} + \phi_y \right) = I_2 \frac{\partial^2 \phi_y}{\partial t^2}. \tag{8.292}$$

8.3.3 Exact Solutions of Axisymmetric Circular Plates

For axisymmetric bending, the equilibrium equations (8.268) and (8.270) become

$$-\frac{1}{r} \left[\frac{d}{dr} (r M_{rr}) - M_{\theta\theta} \right] + Q_r = 0, \tag{8.293}$$

$$-\frac{1}{r} \frac{d}{dr} (r Q_r) - q = 0, \tag{8.294}$$

where [see Eqs. (8.274a–e)]

$$M_{rr} = D\left(\frac{d\phi_r}{dr} + \nu\frac{\phi_r}{r}\right), \tag{8.295}$$

$$M_{\theta\theta} = D\left(\nu\frac{d\phi_r}{dr} + \frac{\phi_r}{r}\right), \tag{8.296}$$

$$Q_r = K_s G h\left(\phi_r + \frac{dw_0}{dr}\right), \tag{8.297}$$

and $D = Eh^3/[12(1-\nu^2)]$.

Integration of Eq. (8.294) gives

$$rQ_r = -\int rq(r)\,dr + c_1. \tag{8.298}$$

Use of Eqs. (8.295), (8.296), and (8.298) in Eq. (8.293) and integration leads to the result

$$D\phi_r(r) = -F'(r) + \frac{r}{4}(2\log r - 1)c_1 + \frac{r}{2}c_2 + \frac{1}{r}c_3, \tag{8.299}$$

where

$$F(r) = \int \frac{1}{r}\int r \int \frac{1}{r}\int rq(r)\,dr dr dr dr. \tag{8.300}$$

Finally, from Eqs. (8.297)–(8.299), we arrive at

$$K_s G h\,\frac{dw_0}{dr} = -\frac{K_s G h}{D}\left[-F' + \frac{r}{4}(2\log r - 1)c_1 + \frac{r}{2}c_2 + \frac{1}{r}c_3\right] - \frac{1}{r}\int rq(r)\,dr + \frac{c_1}{r}, \tag{8.301}$$

$$K_s G h\, w_0(r) = -\frac{K_s G h}{D}\left[-F(r) + \frac{r^2}{4}(\log r - 1)c_1 + \frac{r^2}{4}c_2 + c_3\log r\right] - \int\frac{1}{r}\int rq(r)\,dr + c_1\log r + c_4. \tag{8.302}$$

The constants of integration, c_1, c_2, c_3, and c_4, are determined using the boundary conditions. The derivative of ϕ_r is given by

$$D\frac{d\phi_r}{dr} = -F'' + \frac{r}{4}(2\log r + 1)c_1 + \frac{1}{2}c_2 - \frac{1}{2r^2}c_3. \tag{8.303}$$

For solid circular plates, the condition that the rotation ϕ_r be finite at $r = 0$ requires $c_3 = 0$ [from Eq. (8.299)]. In addition, if the plate is not subjected to a point load at

$r = 0$, the shear force must be zero there. This implies that [from Eq. (8.298)] $c_1 = 0$. Thus, for solid circular plates without a point load at the center, we must have

$$c_1 = 0, \qquad c_3 = 0. \tag{8.304}$$

If a solid circular plate is subjected to a point load Q_0 at the center, we have

$$2\pi(rQ_r) = -Q_0 \text{ at } r = 0 \rightarrow c_1 = -\frac{Q_0}{2\pi}. \tag{8.305}$$

Obviously, conditions (8.304) and (8.305) are not meaningful for annular plates.

Example 8.26 Consider a clamped, solid circular plate of radius a. For this case, $c_3 = 0$. The boundary conditions associated with the clamped edge $r = a$ are

$$w_0(a) = 0, \quad \phi_r(a) = 0.$$

Uniform Load For uniform load of intensity q_0, we have $c_1 = 0$ and

$$c_2 = \frac{q_0 a^2}{8}, \qquad c_4 = \frac{q_0 a^2}{4} + \frac{K_s G h}{D}\frac{q_0 a^4}{64}.$$

Hence the deflection and rotation become

$$w_0(r) = \frac{q_0 a^4}{64D}\left(1 - \frac{r^2}{a^2}\right)^2 + \frac{q_0 a^2}{4K_s G h}\left(1 - \frac{r^2}{a^2}\right), \tag{8.306}$$

$$\phi_r(r) = \frac{q_0 a^3}{16D}\frac{r}{a}\left(1 - \frac{r^2}{a^2}\right). \tag{8.307}$$

Note that the deflection in (8.306) has two parts, one due to bending $w_0^b(r)$ and the other due to shear $w_0^s(r)$:

$$w_0^b(r) = \frac{q_0 a^4}{64D}\left(1 - \frac{r^2}{a^2}\right)^2, \qquad w_0^s(r) = \frac{q_0 a^2}{4K_s G h}\left(1 - \frac{r^2}{a^2}\right). \tag{8.308}$$

The bending deflection is the same as that predicted by the classical plate theory [see Eq. (8.45)]. Thus, the effect of including transverse shear strain in the formulation is to increase the deflection by $w_0^s(r)$. In other words, the classical plate theory underpredicts deflection compared to the first-order shear deformation theory. We also note that the rotation ϕ_r, in the present case, is equal to

$$\phi_r(r) = -\frac{dw_0^b}{dr}. \tag{8.309}$$

That is, the rotation of a transverse normal is not affected by the shear deformation. Also, note that (dw_0^s/dr) is not zero at $r = a$.

The maximum deflection of the plate is

$$w_{max} = \frac{q_0 a^4}{64D} + \frac{q_0 a^2}{4K_s Gh} = \frac{q_0 a^4}{64D}\left(1 + \frac{8}{3(1-\nu)K_s}\frac{h^2}{a^2}\right). \tag{8.310}$$

For $\nu = 0.3$, $K_s = 5/6$, and $h/a = 0.01$ (a thin plate), the difference between the maximum deflections predicted by the classical and first-order shear deformation theories is 4.57×10^{-4}, which is negligible. For $h/a = 0.1$ (a moderately thick plate) it is 4.57×10^{-2}, or 4.57%. Thus, the effect of shear deformation on the deflection is greater for thick plates.

The expressions for the bending moments and stresses are the same as those in Eqs. (8.47a–d) (why so?). The shear force Q_r and shear stress σ_{rz} are given by

$$Q_r(r) = K_s Gh\left(\phi_r + \frac{dw_0}{dr}\right) = K_s Gh\frac{dw_0^s}{dr} = -\frac{q_0 r}{2} \tag{8.311a}$$

$$\sigma_{rz}(r) = G\left(\phi_r + \frac{dw_0}{dr}\right) = -\frac{K_s q_0 r}{2h}. \tag{8.311b}$$

Point Load For point load Q_0 at the center, we have

$$c_1 = -\frac{Q_0}{2\pi}, \qquad c_3 = 0.$$

The boundary conditions of the clamped edge give

$$c_2 = \frac{Q_0}{4\pi}(2\log a - 1), \qquad c_4 = \frac{K_s Gh}{D}\frac{Q_0 a^2}{16\pi} + \frac{Q_0}{2\pi}\log a.$$

Hence the solution becomes

$$w_0(r) = \frac{Q_0 a^2}{16\pi D}\left[1 - \frac{r^2}{a^2} + 2\frac{r^2}{a^2}\log\left(\frac{r}{a}\right)\right] - \frac{Q_0}{2\pi K_s Gh}\log\frac{r}{a}, \tag{8.312}$$

$$\phi_r(r) = -\frac{Q_0 r}{4\pi D}\log\frac{r}{a}. \tag{8.313}$$

As before, the deflection predicted by the first-order shear deformation plate theory has two parts: one due to bending and the other due to shear. The bending part is the same as that predicted by the classical plate theory, and $\phi_r(r) = -(dw_0^b/dr)$. However, the shear part $w_0^s(r)$ is singular at $r = 0$, making it difficult to determine the maximum deflection.

Example 8.27 Here we consider a simply supported solid circular plate under uniformly distributed load of intensity q_0. The boundary conditions are

$$\text{at } r = 0: \quad (rQ_r) = 0; \qquad \text{at } r = a: \quad w_0 = 0, \quad M_{rr} = 0. \tag{8.314}$$

The constants of integration are calculated to be ($c_1 = c_3 = 0$)

$$c_2 = \frac{q_0 a^2}{8}\frac{(3+\nu)}{(1+\nu)}, \qquad c_4 = \frac{q_0 a^2}{4} + \frac{K_s G h}{D}\frac{q_0 a^4}{64}\frac{(5+\nu)}{(1+\nu)}.$$

The deflection and rotation become

$$w_0(r) = \frac{q_0 a^4}{64D}\left(\frac{r^4}{a^4} - 2\frac{3+\nu}{1+\nu}\frac{r^2}{a^2} + \frac{5+\nu}{1+\nu}\right) + \frac{q_0 a^2}{4K_s Gh}\left(1 - \frac{r^2}{a^2}\right), \tag{8.315}$$

$$\phi_r(r) = \frac{q_0 a^3}{16D}\frac{r}{a}\left[\frac{(3+\nu)}{(1+\nu)} - \frac{r^2}{a^2}\right]. \tag{8.316}$$

Note that the deflection is affected by shear deformation [the second term in Eq. (8.315)]; however, for this problem the rotation is not affected by transverse shear deformation.

The maximum deflection is

$$w_{max} = w_0(0) = \left(\frac{5+\nu}{1+\nu}\right)\frac{q_0 a^4}{64D} + \frac{q_0 a^2}{4K_s Gh}, \tag{8.317a}$$

and the maximum rotation is equal to the negative of the slope at $r = a$:

$$\phi_{max} = \phi_r(a) = \frac{q_0 a^3}{8D(1+\nu)}. \tag{8.317b}$$

8.3.4 Exact Solutions of Rectangular Plates

As in the case of the classical plate theory, analytical solutions of the first-order shear deformation plate theory can be developed for simply supported rectangular plates using Navier's method and for rectangular plates with parallel edges simply supported by Lévy's (semidiscretization) method. Here we limit our discussion to the Navier method of solution for the pure bending case (i.e., omit stretching deformation). The Lévy method of analysis for the first-order shear deformation plate theory is more involved than the classical plate theory (as there are three equations to be solved even for pure bending case), and details can be found in the books by Reddy [7,8].

The simply supported boundary conditions for the first-order shear deformation plate theory (FSDT) can be expressed as (Fig. 8.28):

$$w_0(x,0,t) = 0, \quad w_0(x,b,t) = 0, \quad w_0(0,y,t) = 0, \quad w_0(a,y,t) = 0, \tag{8.318a}$$

$$\phi_x(x,0,t) = 0, \quad \phi_x(x,b,t) = 0, \quad \phi_y(0,y,t) = 0, \quad \phi_y(a,y,t) = 0, \tag{8.318b}$$

$$M_{yy}(x,0,t) = 0, \quad M_{yy}(x,b,t) = 0, \quad M_{xx}(0,y,t) = 0, \quad M_{xx}(a,y,t) = 0. \tag{8.318c}$$

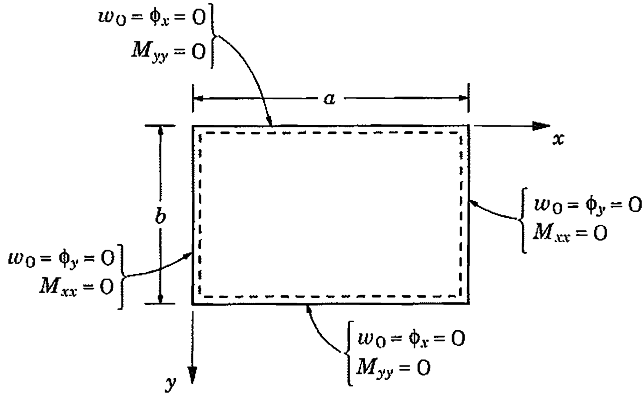


Figure 8.28 The simply supported boundary conditions of the first-order shear deformation theory.

The boundary conditions in Eqs. (8.318a–c) are satisfied by the following expansions:

$$w_0(x, y, t) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} W_{mn}(t) \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b}, \tag{8.319a}$$

$$\phi_x(x, y, t) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} X_{mn}(t) \cos \frac{m\pi x}{a} \sin \frac{n\pi y}{b}, \tag{8.319b}$$

$$\phi_y(x, y, t) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} Y_{mn}(t) \sin \frac{m\pi x}{a} \cos \frac{n\pi y}{b}, \tag{8.319c}$$

where a and b denote the dimensions of the rectangular plate. The mechanical load is also expanded in double sine series:

$$q(x, y, t) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} q_{mn}(t) \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b}, \tag{8.320a}$$

where

$$q_{mn}(t) = \frac{4}{ab} \int_0^a \int_0^b q(x, y, t) \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \, dx dy. \tag{8.320b}$$

Substitution of Eqs. (8.319a–c) into Eqs. (8.290)–(8.292), with $\hat{N}_{xy} = 0$, yields the following equations for the coefficients (W_{mn}, X_{mn}, Y_{mn}):

$$\begin{bmatrix} s_{11} - \bar{s}_{11} & s_{12} & s_{13} \\ s_{12} & s_{22} & s_{23} \\ s_{13} & s_{23} & s_{33} \end{bmatrix} \begin{Bmatrix} W_{mn} \\ X_{mn} \\ Y_{mn} \end{Bmatrix} + \begin{bmatrix} m_{11} & 0 & 0 \\ 0 & m_{22} & 0 \\ 0 & 0 & m_{33} \end{bmatrix} \begin{Bmatrix} \ddot{W}_{mn} \\ \ddot{X}_{mn} \\ \ddot{Y}_{mn} \end{Bmatrix} = \begin{Bmatrix} Q_{mn} \\ 0 \\ 0 \end{Bmatrix}, \tag{8.321}$$

where s_{ij} and $\hat{m}_{ij}$ are defined by (for orthotropic plates)

$$
\begin{aligned}
&s_{11} = K_s\left(A_{55}\alpha_m^2 + A_{44}\beta_n^2\right) + k, \qquad \bar{s}_{11} = \hat{N}_{xx}\alpha_m^2 + \hat{N}_{yy}\beta_n^2,\\
&s_{12} = K_s A_{55}\alpha_m, \qquad s_{13} = K_s A_{44}\beta_n, \qquad s_{22} = \left(D_{11}\alpha_m^2 + D_{66}\beta_n^2 + K_s A_{55}\right),\\
&s_{23} = (D_{12} + D_{66})\alpha_m\beta_n, \qquad s_{33} = \left(D_{66}\alpha_m^2 + D_{22}\beta_n^2 + K_s A_{44}\right),\\
&m_{11} = I_0, \qquad m_{22} = I_2, \qquad m_{33} = I_2,
\end{aligned}
\tag{8.322}
$$

and $\alpha_m = m\pi/a$ and $\beta_n = n\pi/b$.

Bending Analysis The static solution can be obtained from Eq. (8.321) by setting the time derivative terms to zero:

$$
\begin{bmatrix} s_{11} & s_{12} & s_{13} \\ s_{12} & s_{22} & s_{23} \\ s_{13} & s_{23} & s_{33} \end{bmatrix}
\begin{Bmatrix} W_{mn} \\ X_{mn} \\ Y_{mn} \end{Bmatrix} = \begin{Bmatrix} Q_{mn} \\ 0 \\ 0 \end{Bmatrix}. \tag{8.323}
$$

Solution of Eq. (8.323) for each $m, n = 1, 2, \ldots$ gives (W_{mn}, X_{mn}, Y_{mn}), which can then be used to compute the solution (w_0, ϕ_x, ϕ_y) from Eqs. (8.319a–c). We obtain

$$
W_{mn} = \frac{b_0}{b_{mn}} Q_{mn}, \qquad X_{mn} = \frac{b_1}{b_{mn}} Q_{mn}, \qquad Y_{mn} = \frac{b_2}{b_{mn}} Q_{mn}, \tag{8.324a}
$$

where

$$
\begin{aligned}
b_{mn} &= s_{11}b_0 + s_{12}b_1 + s_{13}b_2, \qquad b_0 = s_{22}s_{33} - s_{23}s_{23},\\
b_1 &= s_{23}s_{13} - s_{12}s_{33}, \qquad b_2 = s_{12}s_{23} - s_{22}s_{13}.
\end{aligned}
\tag{8.324b}
$$

The bending moments are given by

$$
\begin{aligned}
M_{xx} &= -D\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}(\alpha_m X_{mn} + \nu\beta_n Y_{mn})\sin\alpha_m x \sin\beta_n y,\\
M_{yy} &= -D\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}(\nu\alpha_m X_{mn} + \beta Y_{mn})\sin\alpha_m x \sin\beta_n y,\\
M_{xy} &= \frac{(1-\nu)D}{2}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}(\beta_n X_{mn} + \alpha_m Y_{mn})\cos\alpha_m x \cos\beta_n y,
\end{aligned}
\tag{8.325}
$$

and the shear forces can be computed from

$$
\begin{aligned}
Q_x &= -K_s G h\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}(\alpha_m W_{mn} + X_{mn})\cos\alpha_m x \sin\beta_n y,\\
Q_y &= -K_s G h\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}(\beta_n W_{mn} + Y_{mn})\sin\alpha_m x \cos\beta_n y.
\end{aligned}
\tag{8.326}
$$

The stresses can be computed using the constitutive equations:

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{Bmatrix} = -\frac{Ez}{(1-\nu^2)} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \begin{Bmatrix} (\alpha_m X_{mn} + \nu\beta_n Y_{mn}) \sin \alpha_m x \sin \beta_n y \\ (\nu\alpha_m X_{mn} + \beta_n Y_{mn}) \sin \alpha_m x \sin \beta_n y \\ -\frac{(1-\nu)}{2}(\beta_n X_{mn} + \alpha_m Y_{mn}) \cos \alpha_m x \cos \beta_n y \end{Bmatrix}, \tag{8.327}$$

$$\begin{Bmatrix} \sigma_{yz} \\ \sigma_{xz} \end{Bmatrix} = Gh \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \begin{Bmatrix} (Y_{mn} + \beta_n W_{mn}) \sin \alpha_m x \cos \beta_n y \\ (X_{mn} + \alpha_m W_{mn}) \cos \alpha_m x \sin \beta_n y \end{Bmatrix}. \tag{8.328}$$

The transverse shear stresses can also be computed using 3D equilibrium equations in terms of stresses. They are given by

$$\begin{aligned} \sigma_{xz} &= -\frac{h^2}{8}\left[1 - \left(\frac{2z}{h}\right)^2\right](T_{11}X_{mn} + T_{12}Y_{mn}) \cos \alpha_m x \sin \beta_n y, \\ \sigma_{yz} &= -\frac{h^2}{8}\left[1 - \left(\frac{2z}{h}\right)^2\right](T_{12}X_{mn} + T_{22}Y_{mn}) \sin \alpha_m x \cos \beta_n y, \\ \sigma_{zz} &= \frac{h^3}{48}\left\{\left[1 + \left(\frac{2z}{h}\right)^3\right] - 3\left[1 + \left(\frac{2z}{h}\right)\right]\right\}(T_{31}X_{mn} + T_{32}Y_{mn}) \\ &\quad \times \sin \alpha_m x \sin \beta_n y, \end{aligned} \tag{8.329a}$$

where

$$\begin{aligned} T_{11} &= \frac{E}{(1-\nu^2)}\left(\alpha_m^2 + \frac{1-\nu}{2}\beta_n^2\right), \\ T_{12} &= \frac{E}{2(1-\nu)}\alpha_m \beta_n, \\ T_{22} &= \frac{E}{(1-\nu^2)}\left(\frac{1-\nu}{2}\alpha_m^2 + \beta_n^2\right), \\ T_{31} &= \frac{E}{(1-\nu^2)}(\alpha_m^3 + \alpha_m \beta_n^2), \\ T_{32} &= \frac{E}{(1-\nu^2)}(\alpha_m^2 \beta_n + \beta_n^3). \end{aligned} \tag{8.329b}$$

Example 8.28 Numerical results for the deflection and stresses of simply supported, isotropic, rectangular plates are presented here. The following nondimensionalizations are used:

$$\bar{w} = w_0\left(\frac{Eh^3}{b^4 q_0}\right) \quad \text{for UL}, \qquad \bar{w} = w_0\left(\frac{D}{b^4 q_0}\right) \times 10^2 \quad \text{for SL},$$

Table 8.16 Effect of the transverse shear deformation on deflections and stresses in isotropic ($\nu = 0.25$) square ($a = b$) plates subjected to distributed loads[a].

Load	b/h	$\bar{w}$	$\bar{\sigma}_{xx}$	$\bar{\sigma}_{yy}$	$\bar{\sigma}_{xy}$	$\bar{\sigma}_{xz}$	$\bar{\sigma}_{yz}$
SL	10	0.2702	0.1900	0.1900	0.1140	0.1910	0.1910
						0.2387[b]	0.2387[b]
	20	0.2600	0.1900	0.1900	0.1140	0.1910	0.1910
	50	0.2572	0.1900	0.1900	0.1140	0.1910	0.1910
	100	0.2568	0.1900	0.1900	0.1140	0.1910	0.1910
	CPT	0.2566	0.1900	0.1900	0.1140	0.2387[b]	0.2387[b]
UL (19)[c]	10	0.4259	0.2762	0.2762	0.2085	0.3927	0.3927
						0.4909[b]	0.4909[b]
	20	0.4111	0.2762	0.2762	0.2085	0.3927	0.3927
	50	0.4070	0.2762	0.2762	0.2085	0.3927	0.3927
	100	0.4060	0.2762	0.2762	0.2085	0.3927	0.3927
	CPT	0.4062	0.2762	0.2762	0.2085	0.4909[b]	0.4909[b]

[a]See Fig. 8.28 for the plate geometry and coordinate system.

[b]Stresses computed using the stress equilibrium equations (8.329a,b); they are the same for all ratios of b/h.

[c]The numbers in parentheses denotes the maximum values of $m = n$ used to evaluate the series.

$$\bar{\sigma}_{xx} = \sigma_{xx}\left(\frac{h^2}{b^2 q_0}\right), \quad \bar{\sigma}_{yy} = \sigma_{yy}\left(\frac{h^2}{b^2 q_0}\right), \quad \bar{\sigma}_{xy} = \sigma_{xy}\left(\frac{h^2}{b^2 q_0}\right),$$
$$\bar{\sigma}_{xz} = \sigma_{xz}\left(\frac{h}{b q_0}\right), \quad \bar{\sigma}_{yz} = \sigma_{yz}\left(\frac{h}{b q_0}\right). \tag{8.330}$$

Table 8.16 contains the maximum nondimensionalized deflections ($\bar{w}$) and stresses of simply supported square plates under sinusoidally distributed load (SL) and uniformly distributed load (UL), and for different side-to-thickness ratios. The stresses were evaluated at the locations indicated below:

$$\bar{\sigma}_{xx}\left(\frac{a}{2}, \frac{b}{2}, \frac{h}{2}\right), \quad \bar{\sigma}_{yy}\left(\frac{a}{2}, \frac{b}{2}, \frac{h}{2}\right), \quad \bar{\sigma}_{xy}\left(a, b, -\frac{h}{2}\right).$$

The transverse shear stresses are calculated using the constitutive equations as well as equilibrium equations. They are the maximum at the locations indicated below:

$$\bar{\sigma}_{xz}\left(0, \frac{b}{2}, \frac{h}{2}\right), \quad \bar{\sigma}_{yz}\left(\frac{a}{2}, 0, \frac{h}{2}\right).$$

Of course, the constitutively derived transverse stresses are independent of the z-coordinate. The nondimensionalized quantities in the classical plate theory are independent of the side-to-thickness ratio. The influence of transverse shear deformation is to increase the transverse deflection. The difference between the deflections predicted by the first-order shear deformation theory and classical plate theory decreases with the increase in the ratio a/h (see Fig. 8.29).

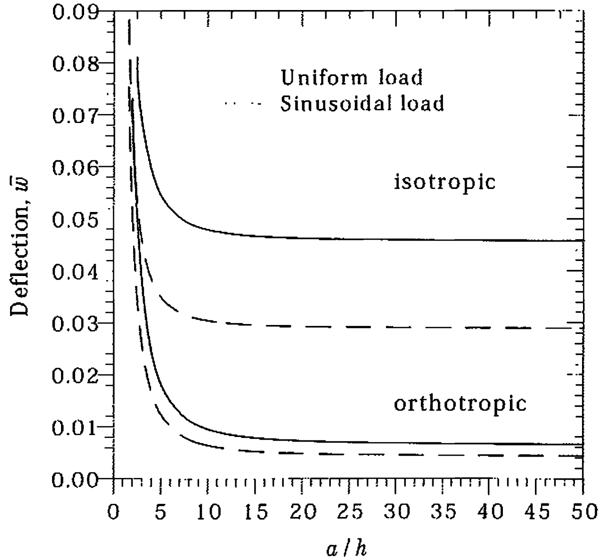


Figure 8.29 Nondimensionalized center transverse deflection ($\bar{w}$) versus side-to-thickness ratio (a/h) for simply supported square plates.

Natural Vibration For natural vibration, we set the mechanical load to zero and seek a periodic solution to Eq. (8.321) in the form

$$W_{mn}(t) = W_{mn}^0 e^{i\omega t}, \qquad X_{mn}(t) = X_{mn}^0 e^{i\omega t}, \qquad Y_{mn}(t) = Y_{mn}^0 e^{i\omega t}, \tag{8.331}$$

and obtain the following 3×3 system of eigenvalue problem:

$$\left(\begin{bmatrix} s_{11} & s_{12} & s_{13} \\ s_{12} & s_{22} & s_{23} \\ s_{13} & s_{23} & s_{33} \end{bmatrix} - \omega^2 \begin{bmatrix} m_{11} & 0 & 0 \\ 0 & m_{22} & 0 \\ 0 & 0 & m_{33} \end{bmatrix} \right) \begin{Bmatrix} W_{mn}^0 \\ X_{mn}^0 \\ Y_{mn}^0 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}, \tag{8.332}$$

where s_{ij} and m_{ij} are defined in Eq. (8.322). Setting the determinant of the coefficient matrix in Eq. (8.332) to zero yields a cubic equation for ω^2.

If the rotatory inertia I_2 is omitted (i.e., $m_{22} = m_{33} = 0$), the frequency equation can be solved for ω^2:

$$\omega^2 = \frac{1}{m_{11}} \left(s_{11} - \frac{s_{13}s_{23} - s_{12}s_{33}}{s_{22}s_{33} - s_{23}s_{23}} s_{12} - \frac{s_{12}s_{23} - s_{13}s_{22}}{s_{22}s_{33} - s_{23}s_{23}} s_{13} \right). \tag{8.333}$$

Example 8.29 Consider a simply supported rectangular plate. We wish to determine the natural frequencies using the first-order shear deformation plate

Table 8.17 Effect of the shear deformation, rotatory inertia, and shear correction coefficient on nondimensionalized natural frequencies of simply supported plates ($\bar{\omega} = \omega(a^2/h)\sqrt{\rho/E}$; $\nu = 0.3$, $a/h = 10$)

m	n	CPT[a] w/o RI	CPT with RI	K_s	FSDT w/o RI	FSDT with RI
1	1	5.973	5.925	1.0	5.838	5.794
				5/6	5.812	5.769
				2/3	5.773	5.732
2	1	14.933	14.635	1.0	14.127	13.899
				5/6	13.980	13.764
				2/3	13.769	13.568
2	2	23.893	23.144	1.0	21.922	21.424
				5/6	21.582	21.121
				2/3	21.103	20.688

[a]w/o RI means without rotary inertia.

Table 8.18 Effect of shear deformation, material orthotropy, and rotatory inertia on dimensionless fundamental frequencies of simply supported square plates [$\bar{\omega} = \omega(a^2/h)\sqrt{\rho/E}$, $K = 5/6$, $\nu = 0.25$]

Theory	$a/h \rightarrow$	5	10	20	25	50	100
FSDT	w-RI[a]	5.232	5.694	5.835	5.853	5.877	5.883
	w/o-RI	5.349	5.736	5.847	5.860	5.879	5.883
CPT	(5.885)[b]	5.700	5.837	5.873	5.877	5.883	5.885

[a]w-RI = with rotatory inertia; w/o-RI = without rotatory inertia.
[b]Value in parentheses is the frequency without rotatory inertia.

theory. Using Eq. (8.333), we can compute the frequencies. Table 8.17 contains the first four natural frequencies of square isotropic plates. The rotary inertia (RI) has the effect of decreasing the frequencies. Table 8.18 contains fundamental natural frequencies of square plates for various values of side-to-thickness ratio a/b.

Buckling Analysis For buckling analysis, we assume that the only applied loads are the in-plane compressive forces

$$\hat{N}_{xx} = N_0, \qquad \hat{N}_{yy} = \gamma N_0, \qquad \gamma = \frac{\hat{N}_{yy}}{\hat{N}_{xx}}, \tag{8.334}$$

and all other loads are zero (and $k = 0$). From Eq. (8.231) we have

$$\begin{bmatrix} s_{11} - N_0(\alpha_m^2 + \gamma\beta_n^2) & s_{12} & s_{13} \\ s_{12} & s_{22} & s_{23} \\ s_{13} & s_{23} & s_{33} \end{bmatrix} \begin{Bmatrix} W_{mn} \\ X_{mn} \\ Y_{mn} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}. \tag{8.335}$$

For a nontrivial solution, the determinant of the coefficient matrix in Eq. (8.335) must be zero. This gives the following expression for the buckling load:

$$N_0 = \left(\frac{1}{\alpha_m^2 + \gamma\beta_n^2}\right)\left[\frac{c_0 + \left(\dfrac{\alpha_m^2}{K_s A_{44}} + \dfrac{\beta_n^2}{K_s A_{55}}\right)c_1}{1 + \dfrac{c_1}{K_s^2 A_{44} A_{55}} + \dfrac{c_2}{K_s A_{55}} + \dfrac{c_3}{K_s A_{44}}}\right], \tag{8.336}$$

$$c_0 = D_{11}\alpha_m^4 + 2\left(D_{12} + 2D_{66}\right)\alpha_m^2\beta_n^2 + D_{22}\beta_n^4, \qquad c_1 = c_2 c_3 - (c_4)^2 > 0,$$
$$c_2 = D_{11}\alpha_m^2 + D_{66}\beta_n^2, \qquad c_3 = D_{66}\alpha_m^2 + D_{22}\beta_n^2, \qquad c_4 = \left(D_{12} + D_{66}\right)\alpha_m\beta_n. \tag{8.337}$$

The expression in (8.336) is of the form

$$c\left(\frac{1+k_1}{1+k_2}\right) \quad \text{with} \quad k_1 < k_2 \quad \text{which implies} \quad 1 \geq \frac{1+k_1}{1+k_2},$$

indicating that transverse shear deformation has the effect of *reducing* the buckling load.

Table 8.19 contains the critical buckling loads $\bar{N} = N_{cr}b^2/(\pi^2 D_{22})$ as a function of the plate aspect ratio a/b, side-to-thickness ratio b/h, and modulus ratio E_1/E_2 for uniaxial ($\gamma = 0$) and biaxial ($\gamma = 1$) compression. The classical plate theory (CPT) results are also included for comparison. The effect of transverse shear deformation is significant for lower aspect ratios, thick plates, and larger modular ratios. For thin plates, irrespective of the aspect ratio and modular ratio, the buckling loads predicted by the shear deformation plate theory are very close to those of the classical plate theory.

8.3.5 Relationships Between Bending Solutions of Classical and Shear Deformation Theories

From the analytical solutions of axisymmetric bending of circular plates, it is clear that there exists a relationship between the solutions of the two theories, namely, the classical (CPT) and first-order shear deformation (FSDT) theories. Indeed, Reddy and Wang [31] developed such relationships using the similarity of the equations of the two theories and the load equivalence (also see Wang et al. [32]). We present a brief discussion of the relationships for the case of bending.

First, we consider axisymmetric bending of circular plates. The governing equations of the two theories are summarized below. The superscripts on variables refer to the theory (C for CPT and F for FSDT).

CPT

$$\frac{1}{r}\frac{d}{dr}\left(rQ_r^C\right) = -q, \tag{8.338}$$

Table 8.19 Buckling loads $\bar{N}$ of simply supported plates under in-plane uniform uniaxial ($\gamma = 0$) and biaxial ($\gamma = 1$) compression (K_s =5/6)

γ	$\frac{a}{b}$	$\frac{h}{b}$	$\frac{E_1}{E_2} = 1$	$\frac{E_1}{E_2} = 3$	$\frac{E_1}{E_2} = 10$	$\frac{E_1}{E_2} = 25$
0	0.5	10	5.523	11.583	23.781	34.701
		20	6.051	13.779	35.615	68.798
		100	6.242	14.669	42.398	100.750
		CPT	6.250	14.708	42.737	102.750
	1.0	10	3.800	5.901	11.205	19.252
		20	3.948	6.309	12.832	25.412
		100	3.998	6.452	13.460	28.357
		CPT	4.000	6.458	13.488	28.495
	1.5	10	4.045(2,1)[a]	5.664	8.354	13.166
		20	4.262(2,1)	5.942	8.959	15.077
		100	4.337(2,1)	6.037	9.173	15.823
		CPT	4.340(2,1)	6.042	9.182	15.856
	3.0	10	3.800(3,1)	5.664(2,1)	8.354(2,1)	13.166(2,1)
		20	3.948(3,1)	5.942(2,1)	8.959(2,1)	14.052
		100	3.998(3,1)	6.037(2,1)	9.173(2,1)	14.264
		CPT	4.000(3,1)	6.042(2,1)	9.182(2,1)	14.273
1	0.5	10	4.418	9.405	15.191(1,3)	17.773(1,3)
		20	4.841	11.070	21.565(1,3)	30.073(1,4)
		100	4.993	11.737	25.241(1,3)	40.157(1,4)
		CPT	5.000	11.767	25.427(1,3)	40.784(1,4)
	1.0	10	1.900	3.015	5.662	7.518(1,2)
		20	1.974	3.173	6.433	9.308(1,2)
		100	1.999	3.227	6.731	10.156(1,2)
		CPT	2.000	3.229	6.744	10.196(1,2)
	1.5	10	1.391	1.788	2.614	4.093
		20	1.431	1.841	2.769	4.651
		100	1.444	1.858	2.823	4.869
		CPT	1.444	1.859	2.825	4.879
	3.0	10	1.079	1.151	1.227	1.375
		20	1.103	1.172	1.251	1.414
		100	1.111	1.179	1.259	1.426
		CPT	1.111	1.179	1.260	1.427

[a]Denotes mode numbers (m, n) at which the critical buckling load occurred; $(m, n) = (1, 1)$ for all other cases.

where

$$rQ_r^C \equiv \frac{d}{dr}\left(rM_{rr}^C\right) - M_{\theta\theta}^C, \tag{8.339a}$$

$$M_{rr}^C = -D\left(\frac{d^2 w_0^C}{dr^2} + \frac{\nu}{r}\frac{dw_0^C}{dr}\right), \tag{8.339b}$$

$$M_{\theta\theta}^C = -D\left(\nu\frac{d^2w_0^C}{dr^2} + \frac{1}{r}\frac{dw_0^C}{dr}\right). \tag{8.339c}$$

FSDT

$$\frac{1}{r}\frac{d}{dr}\left(rQ_r^F\right) = -q, \qquad rQ_r^F = \frac{d}{dr}\left(rM_{rr}^F\right) - M_{\theta\theta}^F \tag{8.340a,b}$$

where

$$M_{rr}^F = D\left(\frac{d\phi_r^F}{dr} + \frac{\nu}{r}\phi_r^F\right), \tag{8.341a}$$

$$M_{\theta\theta}^F = D\left(\nu\frac{d\phi_r^F}{dr} + \frac{1}{r}\phi_r^F\right), \tag{8.341b}$$

$$Q_r^F = K_sGh\left(\phi_r^F + \frac{dw_0^F}{dr}\right). \tag{8.341c}$$

Next, we introduce the moment sum

$$\mathcal{M} = \frac{M_{rr} + M_{\theta\theta}}{1+\nu}. \tag{8.342}$$

Using Eqs. (8.338)–(8.342), we can show that

$$\mathcal{M}^C = -D\left(\frac{d^2w_0^C}{dr^2} + \frac{1}{r}\frac{dw_0^C}{dr}\right) = -D\frac{1}{r}\frac{d}{dr}\left(r\frac{dw_0^C}{dr}\right), \tag{8.343}$$

$$\mathcal{M}^F = D\left(\frac{d\phi_r}{dr} + \frac{1}{r}\phi_r\right) = D\frac{1}{r}\frac{d}{dr}\left(r\phi_r\right), \tag{8.344}$$

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{d\mathcal{M}^C}{dr}\right) = -q, \qquad \frac{1}{r}\frac{d}{dr}\left(r\frac{d\mathcal{M}^F}{dr}\right) = -q, \tag{8.345}$$

and

$$r\frac{d\mathcal{M}^C}{dr} = \frac{d}{dr}(rM_{rr}^C) - M_{\theta\theta}^C = rQ_r^C, \tag{8.346}$$

$$r\frac{d\mathcal{M}^F}{dr} = \frac{d}{dr}(rM_{rr}^F) - M_{00}^F = rQ_r^F. \tag{8.347}$$

From Eqs. (8.338) and (8.340a), because the load q is the same, it follows that

$$rQ_r^F = rQ_r^C + C_1, \tag{8.348}$$

and from Eqs. (8.346)–(8.348) we have

$$r\frac{d\mathcal{M}^F}{dr} = r\frac{d\mathcal{M}^C}{dr} + C_1 \rightarrow \mathcal{M}^F = \mathcal{M}^C + C_1\log r + C_2, \tag{8.349}$$

where C_1 and C_2 are constants of integration. Next, from Eqs. (8.343), (8.344), and (8.349), we have

$$\phi_r = -\frac{dw_0^C}{dr} + \frac{C_1 r}{4D}(2\log r - 1) + \frac{C_2 r}{2D} + \frac{C_3}{rD}. \tag{8.350}$$

Finally, from Eqs. (8.341c), (8.346), (8.347), (8.349), and (8.350), we obtain

$$\frac{dw_0^F}{dr} = -\phi_r^F + \frac{1}{K_s Gh}\left(Q_r^C + \frac{C_1}{r}\right), \tag{8.351}$$

and noting that $Q_r^C = d\mathcal{M}^C/dr$, we have

$$w_0^F = w_0^C + \frac{\mathcal{M}^C}{K_s Gh} + \frac{C_1 r^2}{4D}(1 - \log r) + \frac{C_1}{K_s Gh}\log r - \frac{C_2 r^2}{4D} - \frac{C_3 \log r}{D} + \frac{C_4}{D}. \tag{8.352}$$

The four constants of integration are determined using the boundary conditions. The boundary conditions for various cases are given below.

Free Edge

$$rQ_r^F = rQ_r^C = 0, \qquad rM_{rr}^F = rM_{rr}^C = 0. \tag{8.353}$$

Simply Supported Edge

$$w_0^F = w_0^C = 0, \qquad rM_{rr}^F = rM_{rr}^C = 0. \tag{8.354}$$

Clamped Edge

$$w_0^F = w_0^C = 0, \qquad \phi_r^F = \frac{dw_0^C}{dr} = 0. \tag{8.355}$$

Solid Circular Plate at $r = 0$ (i.e. at the plate center):

$$\phi_r^F = \frac{dw_0^C}{dr} = 0, \qquad C_1 = 0. \tag{8.356}$$

Example 8.30 Consider a circular plate of radius a and subjected to a linearly varying axisymmetric load $q = q_0(1 - r/a)$ (set $q_1 = 0$ in Fig. 8.30). For simply supported as well as clamped (at $r = a$) circular plates, the boundary conditions (8.354) and (8.355) give

$$C_1 = C_2 = C_3 = 0 \qquad \text{and} \qquad C_4 = -\frac{D\mathcal{M}_a^C}{K_s Gh}, \tag{8.357}$$

where $\mathcal{M}_a^C$ is the moment sum at the simply supported or clamped edge ($r = a$) of the classical plate theory, and it is given as follows.

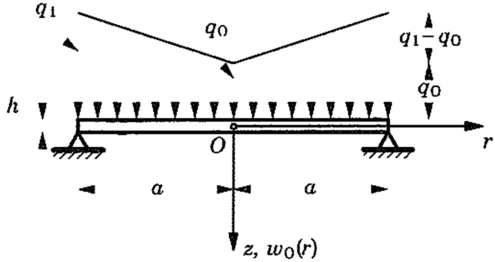


Figure 8.30 Simply supported solid circular plate under linearly varying axisymmetric load.

Simply Supported Edge

$$\mathcal{M}_a^C = \frac{M_{\theta\theta}^C(a)}{1+\nu} = -\frac{D(1-\nu)}{a}\frac{dw_0^C}{dr}\bigg|_{r=a} = \frac{D(1-\nu)}{\nu}\frac{d^2w_0^C}{dr^2}\bigg|_{r=a}. \tag{8.358a}$$

Clamped Edge

$$\mathcal{M}_a^C = -D\frac{d^2w_0^C}{dr^2}\bigg|_{r=a}. \tag{8.358b}$$

Hence, the relationships (8.350) and (8.352) become

$$\phi_r(r) = -\frac{dw_0^C}{dr}, \tag{8.359a}$$

$$w_0^F(r) = w_0^C + \frac{\mathcal{M}^C(r) - \mathcal{M}_a^C}{K_s G h}. \tag{8.359b}$$

The classical plate theory solution for such plates is given by (see Reddy [8])

$$w_0^C = \frac{q_0 a^4}{14400D}\left[\frac{3(183+43\nu)}{1+\nu} - \frac{10(71+29\nu)}{1+\nu}\left(\frac{r}{a}\right)^2 + 255\left(\frac{r}{a}\right)^4 - 64\left(\frac{r}{a}\right)^5\right] \tag{8.360a}$$

for a simply supported plate, and

$$w_0^C = \frac{q_0 a^4}{14400D}\left[129 - 290\left(\frac{r}{a}\right)^2 + 225\left(\frac{r}{a}\right)^4 - 64\left(\frac{r}{a}\right)^5\right] \tag{8.360b}$$

for a clamped plate. Substitution of Eqs. (8.360a,b) into Eqs. (8.358a), and the result into Eq. (8.359b), yields the deflection of the simply supported plate according to the first-order plate theory:

$$w_0^F = w_0^C + \frac{q_0 a^4}{36 K_s G h}\left[5 - 9\left(\frac{r}{a}\right)^2 + 4\left(\frac{r}{a}\right)^3\right]. \tag{8.361}$$

The maximum deflection is (occurs at $r = 0$)

$$w_{max}^F = w_{max}^C + \frac{5q_0a^4}{36K_sGh}. \tag{8.362}$$

Next, we consider the relationships between the bending solutions of the classical plate theory and the first-order shear deformation plate theory for polygonal plates (i.e., plates with multiple straight edges). The equations of equilibrium of plates according to the classical [Eq. (8.195)] and the first-order [Eqs. (8.290)–(8.292)] plate theories can be expressed in terms of the deflection w_0 and the moment sum (or Marcus moment) $\mathcal{M}$ of each theory,

$$\mathcal{M}^C = \frac{M_{xx}^C + M_{yy}^C}{1+\nu}, \qquad \mathcal{M}^F = \frac{M_{xx}^F + M_{yy}^F}{1+\nu}, \tag{8.363}$$

as

$$\nabla^2\mathcal{M}^C = -q, \qquad \nabla^2 w_0^C = -\frac{\mathcal{M}^C}{D}, \tag{8.364a,b}$$

$$\nabla^2\mathcal{M}^F = -q, \qquad \nabla^2\left(w_0^F - \frac{\mathcal{M}^F}{K_sGh}\right) = -\frac{\mathcal{M}^F}{D}, \tag{8.365a,b}$$

where the superscripts C and F refer to quantities of the classical and first-order plate theories, respectively, D is the flexural rigidity, and ν Poisson's ratio. The moment sum in each theory is related to the generalized displacements of the theory by the relations

$$\mathcal{M}^C = -D\left(\frac{\partial^2 w_0^C}{\partial x^2} + \frac{\partial^2 w_0^C}{\partial y^2}\right), \tag{8.366a}$$

$$\mathcal{M}^F = D\left(\frac{\partial\phi_x}{\partial x} + \frac{\partial\phi_y}{\partial y}\right). \tag{8.366b}$$

From Eqs. (8.364a) and (8.365a), since q is the same in both equations, it follows that

$$\mathcal{M}^F = \mathcal{M}^C + D\nabla^2\Phi, \tag{8.367}$$

where Φ is a function such that it satisfies the biharmonic equation

$$\nabla^4\Phi = 0. \tag{8.368}$$

Using this result in Eqs. (8.364b) and (8.365b), we arrive at the relationship

$$w_0^F = w_0^C + \frac{\mathcal{M}^C}{K_sGh} + \Psi - \Phi \tag{8.369a}$$

$$= w_0^C + \frac{h^2}{6K_s(1-\nu)}\nabla^2 w_0^C + \Psi - \Phi, \tag{8.369b}$$

where Ψ is a harmonic function that satisfies the Laplace equation

$$\nabla^2\Psi = 0. \tag{8.370}$$

Note that the relationship (8.369b) is valid for all plates with arbitrary boundary conditions and transverse load. One must determine Φ and Ψ from Eqs. (8.368) and (8.370), respectively, subject to the boundary conditions of the plate. It is difficult to determine these functions for arbitrary geometries and boundary conditions.

In cases where $w_0^F = w_0^C$ on the boundary of the polygonal plates and $\mathcal{M}^C$ is either zero or equal to a constant $\mathcal{M}^{*C}$ (which can be zero) over the boundary, $\Psi - \Phi$ simply takes on the value of $-\mathcal{M}^{*C}/(K_sGh)$. However, if $\mathcal{M}^C$ varies over the boundary, the functions Ψ and Φ must be determined separately. Restricting our analysis to the former case, Eq. (8.369a) can be written as

$$w_0^F(x, y) = w_0^C(x, y) + \frac{\mathcal{M}^C - \mathcal{M}^{*C}}{K_sGh}. \tag{8.371}$$

Equation (8.371) can be used to establish relationships between deflection gradients, bending moments, twisting moment, and shear forces of the classical and first-order shear deformation plate theories, as given below:

$$\frac{\partial w_0^F}{\partial x} = \frac{\partial w_0^C}{\partial x} + \frac{Q_x^C}{K_sGh}, \tag{8.372a}$$

$$\frac{\partial w_0^F}{\partial y} = \frac{\partial w_0^C}{\partial y} + \frac{Q_y^C}{K_sGh}, \tag{8.372b}$$

$$\begin{aligned} M_{xx}^F &= M_{xx}^C + \frac{D(1-\nu)}{2K_sGh}\left[\frac{\partial}{\partial x}(Q_x^F - Q_x^C) - \frac{\partial}{\partial y}(Q_y^F - Q_y^C)\right] \\ &= M_{xx}^C + \frac{D}{K_sGh}\left[\frac{\partial}{\partial x}(Q_x^F - Q_x^C) + \nu\frac{\partial}{\partial y}(Q_y^F - Q_y^C)\right] \\ &= M_{xx}^C + \frac{D(1-\nu)}{K_sGh}\frac{\partial}{\partial x}(Q_x^F - Q_x^C), \end{aligned} \tag{8.373a}$$

$$\begin{aligned} M_{yy}^F &= M_{yy}^C + \frac{D(1-\nu)}{2K_sGh}\left[\frac{\partial}{\partial y}(Q_y^F - Q_y^C) - \frac{\partial}{\partial y}(Q_x^F - Q_x^C)\right] \\ &= M_{yy}^C + \frac{D}{K_sGh}\left[\frac{\partial}{\partial y}(Q_y^F - Q_y^C) + \nu\frac{\partial}{\partial x}(Q_x^F - Q_x^C)\right] \\ &= M_{yy}^C + \frac{D(1-\nu)}{K_sGh}\frac{\partial}{\partial y}(Q_y^F - Q_y^C), \end{aligned} \tag{8.373b}$$

$$M_{xy}^F = M_{xy}^C + \frac{D(1-\nu)}{2K_sGh}\left[\frac{\partial}{\partial y}(Q_x^F - Q_x^C) + \frac{\partial}{\partial x}(Q_y^F - Q_y^C)\right]. \tag{8.373c}$$

The above relationships are exact if $w_0^F = w_0^C$ at the boundaries and the Marcus moments in both theories at the boundary are equal to the same constant.

In the case of simply supported polygonal plates, it can be shown that the Marcus moment (or moment sum) in the classical plate theory is zero along with the deflection:

$$w_0^C = \mathcal{M}^C = 0 \text{ along the straight simply supported edges.} \tag{8.374}$$

In the first-order plate theory, the simply supported boundary condition involves specifying, in addition to the deflection and normal bending moment, the tangential rotation on the edge. This boundary condition is known as the simple support of the "hard" type:

$$w_0^F = 0, \qquad M_{nn}^F = 0, \qquad \phi_s = 0, \tag{8.375}$$

where n is the direction normal to the simply supported edge and s is the direction tangential to the edge. Owing to these conditions, $\partial\phi_s/\partial s = 0$ and the Marcus moment $\mathcal{M}^F$ is thus equal to zero. The boundary conditions of the FSDT for the simply supported plate are therefore

$$w_0^F = \mathcal{M}^F = 0 \text{ along the straight simply supported edges.} \tag{8.376}$$

Since the Marcus moments at the boundaries of plates with any polygonal shape and simply supported edges are equal to zero, Eq. (8.371) applies to such plates with $\mathcal{M}^{*C} = 0$. We have

$$w_0^F = w_0^C + \frac{\mathcal{M}^C}{K_s Gh}. \tag{8.377}$$

Equation (8.373) is an important relationship between the deflections w_0^F and w_0^C of a simply supported polygonal plate. That is, if deflection of a simply supported plate using the classical plate theory is available, we can immediately determine the deflection according to the first-order theory for the same problem from Eq. (8.377), thus bypassing the task of solving the equations of the first-order shear deformation plate theory. Using the same reasoning, one may readily deduce that Eq. (8.371) holds for simply supported polygonal plates under a constant distributed moment $\mathcal{M}^{*K}$ along their edges. For additional results on this topic, the reader may consult the book by Wang et al. [32] and references therein.

Example 8.31 Consider the simply supported equilateral triangular plate of Example 8.17. Equation (e) of that example contains the classical plate theory analytical solution for the deflection of the plate under uniform load q_0:

$$w_0^C = \frac{q_0 a^4}{64D}\left[\bar{x}^3 - 3\bar{y}^2\bar{x} - (\bar{x}^2 + \bar{y}^2) + \frac{4}{27}\right]\left(\frac{4}{9} - \bar{x}^2 - \bar{y}^2\right), \tag{8.378}$$

where $\bar{x} = x/a$ and $\bar{y} = y/a$ (see Fig. 8.22). In view of Eq. (8.378), the Marcus moment is given by

$$\mathcal{M}^C = -D\nabla^2 w_0^C = \frac{q_0 L^2}{4}\left[\bar{x}^3 - 3\bar{x}\bar{y}^2 - (\bar{x}^2 - \bar{y}^2) + \frac{4}{27}\right]. \tag{8.379}$$

Therefore, the deflection w_0^F according to the first-order plate theory of the same problem is given by Eq. (8.377) as

$$w_0^F = \frac{q_0 a^4}{4D}\left[\bar{x}^3 - 3\bar{y}^2 - (\bar{x}^2 + \bar{y}^2) + \frac{4}{27}\right]\left[\frac{\frac{4}{9} - \bar{x}^2 - \bar{y}^2}{16} + \frac{D}{K_s G a^2}\right]. \tag{8.380}$$

Example 8.32 Consider a simply supported rectangular plate of side lengths $a \times b$, as shown in Fig. 8.13 for the classical plate theory and in Fig. 8.28 for the first-order plate theory. The plate is subjected to a distributed load $q(x, y)$. The deflection of this problem according to the classical plate theory is given by Eq. (8.150b):

$$w_0^C = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\left[\frac{q_{mn}}{\pi^4 D\left(\frac{m^2}{a^2} + \frac{n^2}{b^2}\right)^2}\right]\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}, \tag{8.381a}$$

where

$$q_{mn} = \frac{4}{ab}\int_0^b\int_0^a q(x, y)\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}\,dx\,dy. \tag{8.381b}$$

In view of Eq. (8.366a), the Marcus moment is given by

$$\mathcal{M}^C = -D\nabla^2 w_0^C = -\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\left[\frac{q_{mn}}{\pi^2\left(\frac{m^2}{a^2} + \frac{n^2}{b^2}\right)}\right]\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}. \tag{8.382}$$

Using Eq. (8.377), the deflection w_0^F according to the first-order plate theory applied to the simply supported rectangular plate under distributed load is given by

$$w_0^F = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\left[\frac{Q_{mn}}{\pi^4\left(\frac{m^2}{a^2} + \frac{n^2}{b^2}\right)^2}\right]\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b} \tag{8.383a}$$

where

$$Q_{mn} = q_{mn}\left[1 + \frac{\pi^2\left(\frac{m^2}{a^2} + \frac{n^2}{b^2}\right)}{6K_s G}\right]. \tag{8.383b}$$

This solution can be verified to be the same as that in Eq. (8.319a) with W_{mn} defined in Eq. (8.324a). In particular, for a sinusoidally distributed load,

$$q(x, y) = q_0 \sin \frac{\pi x}{a} \sin \frac{\pi y}{b}. \tag{8.384}$$

The deflection becomes ($q_{mn} = q_0$)

$$w_0^F = \frac{q_0 b^4}{\pi^4(1+s^2)^2}\left[1 + \frac{\pi^2(1+s^2)}{6K_S G b^2}\right] \sin \frac{\pi x}{a} \sin \frac{\pi y}{b}, \tag{8.385}$$

where s is the plate aspect ratio, $s = b/a$.

8.3.6 Variational Solutions of Circular and Rectangular Plates

In this section we consider variational formulation of axisymmetric bending of circular plates and plates in rectangular Cartesian coordinates using the first-order shear deformation plate theory. The variational formulation of the shear deformation plate theory mirrors that of the classical plate theory. Note that one may use either the Ritz method or the Galerkin method to solve the problem.

First, we consider axisymmetric circular plates. If we wish to use the Ritz method, we must construct the weak form. The weak form based on the virtual work statement (8.267a) is given by

$$\begin{aligned} 0 = 2\pi \int_0^a \Bigg[& D\left(\frac{d\phi_r}{dr} + \nu\frac{\phi_r}{r}\right)\frac{d\delta\phi_r}{dr} + D\left(\nu\frac{d\phi_r}{dr} + \frac{\phi_r}{r}\right)\frac{\delta\phi_r}{r} \\ & + K_s G h\left(\phi_r + \frac{dw_0}{dr}\right)\left(\delta\phi_r + \frac{d\delta w_0}{dr}\right) - q\delta w_0\Bigg] r\,dr. \end{aligned} \tag{8.386}$$

One must add additional terms corresponding to any applied loads and moments to the above expression.

Next we seek Ritz approximation of the form

$$w_0(r) \approx \sum_{i=1}^{M} a_i \psi_i(r), \qquad \phi_r(r) \approx \sum_{j=1}^{N} b_j \varphi_j(r), \tag{8.387}$$

where we assumed that all specified essential boundary conditions are homogeneous. Substituting Eq. (8.387) into the virtual work statement (8.386) and collecting the coefficients of δa_i and δb_i, we obtain

$$\begin{aligned} 0 = & \sum_{j=1}^{M}\left[K_s G h \int_0^a \frac{d\psi_i}{dr}\frac{d\psi_j}{dr}\, r\,dr\right] a_j + \sum_{j=1}^{N}\left[K_s G h \int_0^a \frac{d\psi_i}{dr}\varphi_j\, r\,dr\right] b_j \\ & - \int_0^a q\psi_i\, r\,dr, \end{aligned} \tag{8.388}$$

$$0 = K_sGh \sum_{j=1}^{M} \left[\int_0^a \varphi_k \frac{d\psi_j}{dr} \, r\, dr \right] a_j + \sum_{j=1}^{N} \left\{ K_sGh \int_0^a \varphi_k \varphi_j \, r\, dr \right.$$

$$\left. + D \int_0^a \left[\frac{d\varphi_k}{dr}\frac{d\varphi_j}{dr} + \frac{\nu}{r}\left(\varphi_k \frac{d\varphi_j}{dr} + \frac{d\varphi_k}{dr}\varphi_j \right) + \frac{1}{r^2}\varphi_k\varphi_j \right] r\, dr \right\} b_j, \quad (8.389)$$

for $k = 1, 2, \ldots, N$. In matrix form, these equations can be written as

$$\begin{bmatrix} [A] & [B] \\ [B]^{\mathrm{T}} & [C] \end{bmatrix} \begin{Bmatrix} \{a\} \\ \{b\} \end{Bmatrix} = \begin{Bmatrix} \{F\} \\ \{0\} \end{Bmatrix}, \quad (8.390)$$

where

$$A_{ij} = K_sGh \int_0^a \frac{d\psi_i}{dr}\frac{d\psi_j}{dr} \, r\, dr, \qquad B_{ij} = K_sGh \int_0^a \frac{d\psi_i}{dr} \varphi_j \, r\, dr,$$

$$C_{kj} = D \int_0^a \left[\frac{d\varphi_k}{dr}\frac{d\varphi_j}{dr} + \frac{\nu}{r}\left(\varphi_k \frac{d\varphi_j}{dr} + \frac{d\varphi_k}{dr}\varphi_j \right) + \frac{1}{r^2}\varphi_k\varphi_j \right] r\, dr$$

$$+ K_sGh \int_0^a \varphi_k\varphi_j \, r\, dr,$$

$$F_i = \int_0^a q\psi_i \, r\, dr. \quad (8.391)$$

Example 8.33 Consider a simply supported solid circular plate. We wish to seek the Ritz approximation (8.387) with $M = 2$ and $N = 1$, and

$$\psi_1(r) = 1 - \frac{r}{a}, \qquad \psi_2(r) = 1 - \frac{r^2}{a^2}, \qquad \varphi_1(r) = \frac{r}{a},$$

which meet the essential boundary conditions of the problem: $w_0(a) = 0$ and $\phi_r(0) = 0$. Evaluating the integrals in Eq. (8.391), we obtain ($\Lambda = D/K_sGh$)

$$\frac{K_sGh}{12} \begin{bmatrix} 6 & 8 & -4a \\ 8 & 12 & -6a \\ -4a & -6a & 3a^2 + 12(1+\nu)\Lambda \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \\ b_1 \end{Bmatrix} = \frac{q_0a^2}{12} \begin{Bmatrix} 2 \\ 3 \\ 0 \end{Bmatrix},$$

whose solution is

$$a_1 = 0, \qquad a_2 = \frac{q_0a^2}{24}\left[\frac{a^2}{(1+\nu)D} + \frac{4}{K_sGh} \right], \qquad b_1 = \frac{2q_0a^3}{3(1+\nu)D}.$$

The Ritz solution becomes

$$W_2(r) = \frac{q_0a^4}{24D}\frac{1}{(1+\nu)}\left(1 - \frac{r^2}{a^2}\right) + \frac{q_0a^2}{6K_sGh}\left(1 - \frac{r^2}{a^2}\right), \quad (8.392)$$

$$\Phi_1(r) = \frac{2q_0a^3}{3(1+\nu)D}\left(\frac{r}{a}\right). \quad (8.393)$$

A close examination of the problem indicates that the deflection is symmetric and the rotation is antisymmetric about $r = 0$. Hence, $w_0(r)$ is a symmetric function and ϕ_r is an antisymmetric function of r. This also explains why $a_1 = 0$ (only when $M > 1$). This understanding helps in selecting proper functions and reducing the effort. Thus, the solution in Eq. (8.393) amounts to using one-parameter ($M = 1$ and $N = 1$) approximations with

$$\psi_1(r) = 1 - \frac{r^2}{a^2}, \qquad \varphi_1(r) = \frac{r}{a}.$$

If a two-parameter approximation for w_0 and a two-parameter approximation for ϕ_r are used, with the following choice of approximation functions:

$$\psi_1(r) = 1 - \frac{r^2}{a^2}, \quad \psi_2(r) = 1 - \frac{r^4}{a^4}, \quad \varphi_1(r) = \frac{r}{a}, \quad \varphi_2(r) = \frac{r^3}{a^3}, \tag{8.394}$$

the exact solution (see Example 8.27) would be obtained.

Next, we consider the variational formulation of plates in rectangular Cartesian coordinates. While no numerical examples are included, the formulation should prove useful in obtaining Ritz solutions of plates with arbitrary boundary conditions.

We begin with the weak form [which is the same as the virtual work statement (8.277) specialized to the static, pure bending case]:

$$\begin{aligned}
0 &= \int_{\Omega_0} \left[M_{xx}\delta\varepsilon_{xx}^1 + M_{yy}\delta\varepsilon_{yy}^1 + M_{xy}\delta\gamma_{xy}^1 + Q_x\delta\gamma_{xz}^0 + Q_y\delta\gamma_{yz}^0 - q\delta w_0\right] dxdy \\
&\quad - \oint_\Gamma \left(\hat{M}_{nn}\delta\phi_n + \hat{M}_{ns}\delta\phi_s + \hat{Q}_n\delta w_0\right) ds, \\
0 &= \int_{\Omega_0} \left\{D\left(\frac{\partial\phi_x}{\partial x} + \nu\frac{\partial\phi_y}{\partial y}\right)\frac{\partial\delta\phi_x}{\partial x} + D\left(\nu\frac{\partial\phi_x}{\partial x} + \frac{\partial\phi_y}{\partial y}\right)\frac{\partial\delta\phi_y}{\partial y}\right. \\
&\quad + D\frac{(1-\nu)}{2}\left(\frac{\partial\phi_x}{\partial y} + \frac{\partial\phi_y}{\partial x}\right)\left(\frac{\partial\delta\phi_x}{\partial y} + \frac{\partial\delta\phi_y}{\partial x}\right) \\
&\quad + K_sGh\left[\left(\frac{\partial w_0}{\partial x} + \phi_x\right)\left(\frac{\partial\delta w_0}{\partial x} + \delta\phi_x\right) + \left(\frac{\partial w_0}{\partial y} + \phi_y\right)\left(\frac{\partial\delta w_0}{\partial y} + \delta\phi_y\right)\right] \\
&\quad \left. - q\delta w_0\right\} dxdy - \oint_\Gamma \left(\hat{M}_{nn}\delta\phi_n + \hat{M}_{ns}\delta\phi_s + \hat{Q}_n\delta w_0\right) ds.
\end{aligned} \tag{8.395}$$

Assuming Ritz approximation of the form

$$w_0 \approx W_N(x, y) = \sum_{i=1}^{N} a_i\psi_i(x, y),$$

$$\phi_x \approx \Phi_x^M(x, y) = \sum_{i=1}^{M} b_i \varphi_i^{(1)}(x, y), \tag{8.396}$$

$$\phi_x \approx \Phi_y^M(x, y) = \sum_{i=1}^{M} c_i \varphi_i^{(2)}(x, y),$$

and substituting into the weak form (8.395), we obtain the following Ritz equations:

$$\begin{bmatrix} [A] & [B] & [C] \\ [B]^{\mathrm{T}} & [D] & [E] \\ [C]^{\mathrm{T}} & [E]^{\mathrm{T}} & [G] \end{bmatrix} \begin{Bmatrix} \{a\} \\ \{b\} \\ \{c\} \end{Bmatrix} = \begin{Bmatrix} \{F^1\} \\ \{F^2\} \\ \{F^3\} \end{Bmatrix}, \tag{8.397a}$$

where

$$\begin{aligned}
A_{ij} &= \int_{\Omega_0} K_s G h \left(\frac{\partial \psi_i}{\partial x} \frac{\partial \psi_j}{\partial x} + \frac{\partial \psi_i}{\partial y} \frac{\partial \psi_j}{\partial y} \right) dxdy, \\
B_{ij} &= \int_{\Omega_0} K_s G h \frac{\partial \psi_i}{\partial x} \varphi_j^{(1)} \, dxdy, \\
C_{ij} &= \int_{\Omega_0} K_s G h \frac{\partial \psi_i}{\partial y} \varphi_j^{(2)} \, dxdy, \\
D_{ij} &= \int_{\Omega_0} \left[D \left(\frac{\partial \varphi_i^{(1)}}{\partial x} \frac{\partial \varphi_j^{(1)}}{\partial x} + \frac{1-\nu}{2} \frac{\partial \varphi_i^{(1)}}{\partial y} \frac{\partial \varphi_j^{(1)}}{\partial y} \right) + K_s G h \, \varphi_i^{(1)} \varphi_j^{(1)} \right] dxdy, \\
E_{ij} &= \int_{\Omega_0} D \left(\nu \frac{\partial \varphi_i^{(1)}}{\partial x} \frac{\partial \varphi_j^{(2)}}{\partial y} + \frac{1-\nu}{2} \frac{\partial \varphi_i^{(1)}}{\partial y} \frac{\partial \varphi_j^{(2)}}{\partial x} \right) dxdy, \qquad (8.397b) \\
G_{ij} &= \int_{\Omega_0} \left[D \left(\frac{\partial \varphi_i^{(2)}}{\partial y} \frac{\partial \varphi_j^{(2)}}{\partial y} + \frac{1-\nu}{2} \frac{\partial \varphi_i^{(2)}}{\partial x} \frac{\partial \varphi_j^{(2)}}{\partial x} \right) + K_s G h \, \varphi_i^{(2)} \varphi_j^{(2)} \right] dxdy, \\
F_i^1 &= \int_{\Omega_0} q \psi_i \, dxdy + \oint_{\Gamma} Q_n \psi_i \, ds, \\
F_i^2 &= \oint_{\Gamma} \left(M_{xx} n_x + M_{xy} n_y \right) \varphi_i^{(1)} \, ds, \\
F_i^3 &= \oint_{\Gamma} \left(M_{xy} n_x + M_{yy} n_y \right) \varphi_i^{(2)} \, ds.
\end{aligned}$$

This completes the variational formulation of the first-order shear deformation plate theory in rectangular Cartesian coordinates. Note that the geometric boundary conditions involve specifying only (w_0, ϕ_x, ϕ_y).

EXERCISES

8.1 Obtain the Euler equations and the natural boundary conditions associated with the functional

$$\Pi(w_0)=\frac{\pi}{2}\int_b^a\left\{D_{11}\left(\frac{d^2w_0}{dr^2}\right)^2+\frac{2D_{12}}{r}\frac{dw_0}{dr}\frac{d^2w_0}{dr^2}+D_{22}\left(\frac{1}{r}\frac{dw_0}{dr}\right)^2\right\}r\,dr,$$

which arises in connection with axisymmetric bending of polar orthotropic annular plates with inner radius b and outer radius a. Assume that the deflection $w_0 = 0$ at $r = a$, the outer radius of the plate.

8.2 Use the virtual work statement in Eq. (8.23) for the pure bending case to show that the total potential energy functional for the bending of circular plates is given by

$$\begin{aligned}\Pi(w_0) = \frac{D}{2}\int_{\Omega_0}\Bigg[&\left(\frac{\partial^2 w_0}{\partial r^2}+\frac{1}{r}\frac{\partial w_0}{\partial r}+\frac{1}{r^2}\frac{\partial^2 w_0}{\partial \theta^2}\right)^2\\ &-2(1-\nu)\frac{\partial^2 w_0}{\partial r^2}\left(\frac{1}{r}\frac{\partial w_0}{\partial r}+\frac{1}{r^2}\frac{\partial^2 w_0}{\partial \theta^2}\right)\\ &+2(1-\nu)\left(\frac{1}{r}\frac{\partial^2 w_0}{\partial r\partial\theta}-\frac{1}{r^2}\frac{\partial w_0}{\partial\theta}\right)^2\Bigg]r\,dr d\theta\\ &+\frac{1}{2}\int_{\Omega_0}kw_0^2\,r\,drd\theta-\int_{\Omega_0}qw_0\,r\,drd\theta.\end{aligned}$$

8.3 Derive the Euler equation and natural boundary conditions of the functional in Exercise 8.2.

8.4 Show that the exact deflection of a *simply supported* circular plate subjected to applied bending moment $M_{rr} = M_a$ at $r = a$ is

$$w_0(r) = \frac{M_a a^2}{2D(1+\nu)}\left(1-\frac{r^2}{a^2}\right).$$

8.5 Determine the deflection and bending moments of a circular plate under uniformly distributed transverse load q_0 when the edge $r = a$ is *elastically built-in*. The boundary conditions are

$$w_0(a) = 0, \qquad M_{rr} = \beta\frac{dw_0}{dr} \quad \text{at } r = a,$$

where β denotes rotational stiffness constant.

8.6 Show that the maximum deflection and bending moment of a *simply supported* circular plate under linearly varying load $q = q_0(1 - r/a)$ are

$$w_{max} = \frac{q_0 a^4}{4800D}\left(\frac{183 + 43\nu}{1 + \nu}\right), \qquad M_{max} = q_0 a^2\left(\frac{71 + 29\nu}{720}\right).$$

8.7 Show that the maximum deflection and bending moment of a *simply supported* circular plate under linearly varying load $q = q_1(r/a)$ are

$$w_{max} = \frac{q_1 a^4}{150D}\left(\frac{6 + \nu}{1 + \nu}\right), \qquad M_{max} = q_1 a^2\left(\frac{4 + \nu}{45}\right).$$

8.8 Show that the deflection of a *clamped* circular plate under the load $q = q_0(r^2/a^2)$ is given by

$$w_0(r) = \frac{q_0 a^4}{576D}\left[2 - 3\left(\frac{r}{a}\right)^2 + \left(\frac{r}{a}\right)^6\right].$$

8.9 Show that the expression for the deflection of a *clamped* (at the outer edge) circular plate under linearly varying load $q = q_0(1 - r/a)$ is

$$w_0(r) = \frac{q_0 a^4}{14400D}\left(129 - 290\frac{r^2}{a^2} + 225\frac{r^4}{a^4} - 64\frac{r^5}{a^5}\right).$$

8.10 Show that the expressions for the deflection and bending moments of a *clamped* (at the outer edge) circular plate under linearly varying load $q = q_1(r/a)$ are

$$w_0(r) = \frac{q_1 a^4}{450D}\left(3 - 5\frac{r^2}{a^2} + 2\frac{r^5}{a^5}\right),$$

$$M_{rr}(r) = \frac{q_1 a^2}{45}\left[(1 + \nu) - (4 + \nu)\frac{r^3}{a^3}\right],$$

$$M_{\theta\theta}(r) = \frac{q_1 a^2}{45}\left[(1 + \nu) - (1 + 4\nu)\frac{r^3}{a^3}\right].$$

8.11 Solve the problem in Exercise 8.7 by one-parameter Ritz approximation.

8.12 Solve the problem in Exercise 8.10 by one-parameter Ritz approximation.

8.13 Determine the fundamental frequency of a *simply supported* circular plate using a one-parameter Ritz approximation.

8.14 Determine the fundamental frequency of a *clamped* circular plate using a two-parameter Ritz approximation. Determine the eigenvector.

8.15 Determine the deflection at the center of a *simply supported* circular plate under asymmetric loading given in Eq. (8.93) using the reciprocity theorem.

8.16 Determine the center deflection of a *simply supported* circular plate under hydrostatic loading $q(r) = q_0(1 - r/a)$ using the reciprocity theorem.

8.17 Repeat Exercise 8.16 for a *clamped* circular plate.

8.18 Determine the center deflection of a *clamped* circular plate subjected to a point load Q_0 at a distance b from the center (and for some θ) using the reciprocity theorem.

8.19 Repeat Exercise 8.18 for a *simply supported* circular plate.

8.20 The total potential energy functional for the axisymmetric bending of a circular plate on an elastic foundation and subjected to concentrated load Q_0 at the center is given by

$$\Pi(w_0) = \frac{\pi}{2}\int^{a}\left[D\left(\frac{d^2w_0}{dr^2}\right)^2 + 2D\nu\frac{1}{r}\frac{dw_0}{dr}\frac{d^2w_0}{dr^2}\right.$$
$$\left. + D\left(\frac{1}{r}\frac{dw_0}{dr}\right)^2 + kw_0^2\right] r\,dr - Q_0 w_0(0),$$

where w_0 is the transverse deflection, a is the radius, D the flexural rigidity of the plate, and k is the foundation modulus. Determine the two-parameter Ritz solution in the form $w_0(r) = c_1 + c_2r^2$.

8.21 Consider an isotropic, *simply supported* rectangular plate subjected to hydrostatic load that varies linearly along the y-axis, as shown in Fig. E8.21 see also Table 8.3). Using the Navier solution method, determine the expressions for deflection $w_0(x, y)$ and bending moment M_{xx}. What are the maximum values of these quantities for a square plate?

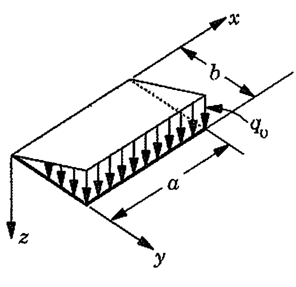


Figure E8.21

8.22 Consider an isotropic rectangular plate, clamped at $y = 0$, simply supported at $x = 0, a$ and $y = b$ (see Fig. E8.22), and subjected to uniformly distributed load q_0. Using the Lévy method of solution, determine the expression for deflection $w_0(x, y)$. What are the maximum values of the deflection and bending moment for a square plate?

8.23 Consider a square isotropic plate, clamped at $y = \pm a/2$, simply supported at $x = 0, a$ (see Fig. 8.17), and subjected to sinusoidal load

$$q(x, y) = q_0 \sin\frac{\pi x}{a}.$$

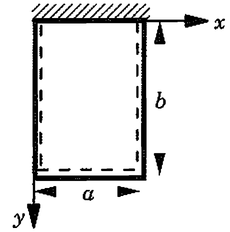


Figure E8.22

Use the Lévy method of solution and determine the expression for deflection $w_0(x, y)$. What are the maximum values of the deflection and bending moment?

Give the algebraic functions X_i and Y_j required in the Ritz approximation (8.222) for the following rectangular plates:

8.24 Plate with edges $x = 0, a$ and $y = 0$ clamped and edge $y = b$ simply supported.

8.25 Plate with edge $x = 0$ clamped and the remaining edges free.

8.26 Plate with edges $x = 0, a$ and $y = 0$ clamped and edge $y = b$ free.

8.27 Plate with edges $x = 0$ and $y = 0$ clamped and edges $x = a$ and $y = b$ free.

8.28 Determine a one-parameter Ritz solution of an isotropic rectangular plate with edges $x = 0, a$ clamped and edges $y = 0, b$ simply supported, and subjected to a uniformly distributed transverse load q_0. Use the approximation

$$W(x, y) = c_{11}\left(1 - \cos\frac{2\pi x}{a}\right)\sin\frac{\pi y}{b} \tag{a}$$

to determine the maximum deflection of a square plate.

8.29 Determine a one-parameter Ritz solution of an isotropic rectangular plate with edges $x = 0, a$ clamped and edges $y = 0, b$ simply supported, and subjected to a center point load Q_0. Use the approximation in Eq. (a) of Exercise 8.28. Determine the maximum deflection of a square plate.

8.30 Determine a one-parameter Ritz solution of an isotropic rectangular plate with all edges clamped and subjected to uniform load q_0. Use the approximation

$$W(x, y) = c_{11}\left(1 - \cos\frac{2\pi x}{a}\right)\left(1 - \cos\frac{2\pi y}{b}\right), \tag{a}$$

where the origin of the coordinate system is taken at the top left corner of the plate with the y-axis down (i.e., $0 \le x \le a$ and $0 \le y \le b$). Determine the maximum deflection of a square plate.

8.31 Repeat Exercise 8.30 for a point load Q_0 at the center of the plate.

8.32 Repeat Exercise 8.30 for an orthotropic plate, and determine the maximum deflection for the case $D_{11} = D_0$, $D_{22} = 0.5D_0$, and $D_{12} + D_{66} = 1.248D_0$.

8.33 Determine the deflection surface $w_0(x, y)$ of the simply supported, isotropic, equilateral triangular plate of Fig. 8.22 when the plate is subjected to a point load Q_0 at the centroid $(x, y) = (0, 0)$. Use the one-parameter Ritz approximation of Example 8.17.

8.34 Consider the equilateral triangular plate of Fig. 8.22. Suppose that all edges are simply supported and loaded by a uniform moment $\hat{M}_{nn} = M_0$ along its edges. Use the one-parameter Ritz solution of Example 8.17 and determine F_1. The exact solution is

$$w_0(x, y) = \frac{M_0 a^2}{4D}\left[\frac{4}{27} - \left(\frac{x}{a}\right)^2 - \left(\frac{y}{a}\right)^2 - 3\left(\frac{x}{a}\right)\left(\frac{y}{a}\right)^2 + \left(\frac{x}{a}\right)^3\right]. \quad \text{(a)}$$

Hint: Evaluate the line integral

$$F_1 = \oint_\Gamma M_0 \left(n_x \frac{\partial \phi_1}{\partial x} + n_y \frac{\partial \phi_1}{\partial y}\right) ds,$$

where Γ denotes the boundary of the triangle and (n_x, n_y) are the direction cosines of the line segment.

8.35 Consider a clamped, isotropic elliptic plate with major and minor axes $2a$ and $2b$, respectively. Obtain a one-parameter Galerkin solution for the case in which the plate is subjected to distributed load $q = q_0(x/a)$.

8.36 Determine a one-parameter Ritz solution of a simply supported, isotropic elliptic plate under uniformly distributed load.

Hint: Use

$$\phi_1(x, y) = \left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}\right), \quad \text{(a)}$$

which satisfies the geometric boundary condition $w_0 = 0$.

8.37 Show that substitution of Eqs. (8.295) and (8.296) for M_{rr} and $M_{\theta\theta}$ into Eq. (8.293) yields the result

$$\begin{aligned} rQ_r &= D\left(r\frac{d^2\phi_r}{dr^2} + \frac{d\phi_r}{dr} - \frac{1}{r}\phi_r\right) \\ &= Dr\left[\frac{d^2\phi_r}{dr^2} + \frac{d}{dr}\left(\frac{1}{r}\phi_r\right)\right] \\ &= Dr\frac{d}{dr}\left[\frac{1}{r}\frac{d}{dr}(r\phi_r)\right]. \end{aligned}$$

8.38 Show that the one-parameter Ritz approximation (8.263) with X_i and Y_j defined by Eq. (a) of Example 8.22 yields the following critical buckling load for a simply supported plate under combined bending and compression:

$$N_{cr} = D\left(\frac{12b}{a^3} + \frac{20}{ab} + \frac{12a}{b^3}\right)\frac{1}{(1 - 0.5c_0)}.$$

8.39 Show that the two-parameter Ritz approximation in Eq. (b) of Example 8.23 yields the equations [see Eq. (e) of Example 8.23]

$$\left[\left(1 + \frac{a^2}{b^2}\right)^2 - N_0\frac{a^2}{\pi^2 D}(1 - 0.5c_0)\right]c_{11} - N_0 c_0 \frac{16a^2}{9\pi^4 D}c_{12} = 0$$

$$-N_0 c_0 \frac{16a^2}{9\pi^4 D}c_{11} + \left[\left(1 + 4\frac{a^2}{b^2}\right)^2 - N_0\frac{a^2}{\pi^2 D}(1 - 0.5c_0)\right]c_{12} = 0, \quad \text{(a)}$$

and show that the associated characteristic equation is

$$A\lambda^2 - B\lambda + C = 0, \quad \text{(b)}$$

where $\lambda = N_0(a^2/\pi^2 D)$ and

$$A = (1 - 0.5c_0)^2 - \left(\frac{16a^2}{9\pi^4 D}\right)^2,$$

$$B = (1 - 0.5c_0)\left[\left(1 + \frac{a^2}{b^2}\right)^2 + \left(1 + 4\frac{a^2}{b^2}\right)^2\right], \quad \text{(c)}$$

$$C = \left(1 + \frac{a^2}{b^2}\right)^2\left(1 + 4\frac{a^2}{b^2}\right)^2.$$

8.40 Eliminate ϕ_x and ϕ_y from Eqs. (8.290)–(8.292) and express them in terms of w_0 alone.

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9

THE FINITE ELEMENT METHOD

9.1 INTRODUCTION

As was pointed out in Chapter 7, the traditional variational methods are ineffective in solving problems that are geometrically complex, have discontinuous loads, or involve discontinuous material or geometric properties. In such cases, the selection of the coordinate functions is a formidable task. Even in cases where the coordinate functions are available, the computation of associated coefficient matrices cannot be automated for a fixed class problems (e.g., bars, beams, or plates) because the coordinate functions are not always algebraic polynomials and they depend on the boundary conditions of the specific problem. Consequently, each time the essential boundary conditions are changed for the same differential equation (i.e., the class of problems is the same), the approximation functions are also changed and the coefficient matrices have to be recalculated. This process is not readily adaptable to computer programming.

The finite element method is a procedure that uses the philosophy of the traditional variational methods to derive the equations relating undetermined coefficients. However, the method differs in two ways from the traditional variational methods in generating the equations of the problem. First, the approximation functions are often algebraic polynomials that are developed using ideas from the interpolation theory; second, the approximation functions are developed for subdomains into which a given domain is divided. The subdomains, called *finite elements*, are geometrically simple shapes that permit a systematic construction of the approximation functions over the element. The division of the whole domain into finite elements not only simplifies the task of generating the approximation functions, but also allows representation of the solution over individual elements. Thus, geometric and/or material discontinuities can be naturally included. Further, since the approximation functions are algebraic polynomials, the computation of the coefficient matrices of the approximation can be automated on a computer. As will be shown shortly, the construction of the approximation functions is systematic, and the process is independent of the boundary conditions and data of the problem. In short, the finite element method is a piecewise application of classical variational methods. The undetermined parameters often, but not always, represent the values of the dependent variables at a finite number of preselected points, whose number and location dictate the degree and form of the approximation functions used. The method is modular and therefore well suited for electronic computation and the development of general-purpose computer programs.

The present study is intended to expose the reader to some of the basic concepts of the finite element method. Therefore, the material presented is introductory in nature, and confined to the basic finite elements of one- and two-dimensional problems of solid mechanics, although the developments presented for these problems apply to all field problems that are governed by the same differential equations (but may have been arrived at using different physical principles). The reader may consult references on the finite element method listed at the end of the chapter for further details (see, for example, [1–5]).

The method is described and illustrated here for (1) one-dimensional second-order equations governing bars, (2) one-dimensional fourth-order equations governing Euler–Bernoulli beam theory, (3) pairs of coupled one-dimensional equations governing the Timoshenko beam theory, (4) two-dimensional fourth-order equations governing the classical plate theory, and (5) sets of coupled two-dimensional equations governing the first-order plate theory. While the general procedure is the same for all five classes of problems, the details differ from each other and therefore it is best to consider the five classes of problems separately. The equations governing these problems closely resemble the equations governing a variety of field problems, e.g., heat transfer, ground water flow, electrostatics, and so on, and thus the developments are readily applicable to those problems as well.

9.2 FINITE ELEMENT ANALYSIS OF BARS

9.2.1 Governing Equation

From earlier discussions, the governing equation of a bar is

$$-\frac{d}{dx}\left(a\frac{du}{dx}\right) + cu = f(x), \qquad 0 < x < L, \tag{9.1}$$

where $u(x)$ is the axial displacement, $a(x) = E(x)A(x)$ the axial stiffness (E is Young's modulus and A is the area of cross section), $c = c(x)$ the surface resistance (zero in most problems), and $f = f(x)$ is the body force per unit length. We are interested in determining an approximate solution that satisfies Eq. (9.1) and appropriate boundary conditions in some sense, say in an integral sense as in the variational methods. In developing the finite element model, we wish to consider all possible boundary conditions that a bar can be subjected to.

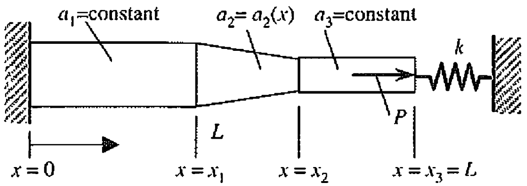


Figure 9.1 A stepped composite bar with an end spring.

In order to account for all possible problem cases, we allow the data, a, c, and f, to be continuous or discontinuous in the domain $(0, L)$. For example, a tapered, stepped composite shaft has discontinuities in its area of cross section as well as material properties at a finite number of points along the length (see Fig. 9.1):

$$a(x) = \begin{cases} a_1(x), & 0 < x < x_1, \\ a_2(x), & x_1 < x < x_2, \\ \vdots & \\ a_n(x), & x_{n-1} < x < x_n = L. \end{cases} \tag{9.2}$$

Also, we must allow for various combinations of the two possible boundary conditions that we derived for bar problems in Chapters 5–7:

$$u = \text{specified} \qquad \text{or} \qquad a\frac{du}{dx} + ku = \text{specified}, \tag{9.3}$$

where k is the spring constant.

Thus, in a general case, we must seek approximate solution to Eq. (9.1) in each of the subintervals of $(0, L)$ in which the equation has continuous coefficients (i.e., a, c, and f are constant or continuous functions of x). A step-by-step procedure for the finite element analysis of Eq. (9.1) for arbitrary variation of a, c, and f, and arbitrary boundary conditions, is given below.

9.2.2 Representation of the Domain by Finite Elements

In the finite element method, the given domain $(0, L)$ is divided into a number of subdomains or intervals, called finite elements. This is necessitated by one or both of the following reasons: (1) It is easier to represent the solution $u(x)$, irrespective of the degree of its variation with x, by a polynomial of a desired degree over each element than to use a single polynomial to approximate $u(x)$ over the entire domain. For example, a linear polynomial over each element may be used to adequately represent a cubic or higher-degree variation of u with x (see Fig. 9.2). Obviously, the greater the number of elements, the smaller is the error between the true solution and the piecewise linear solution. (2) The actual solution is defined piecewise because of the geometric and/or material discontinuities.

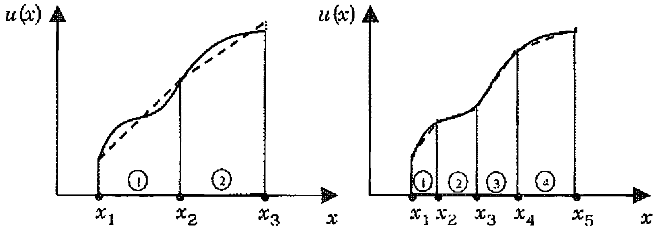
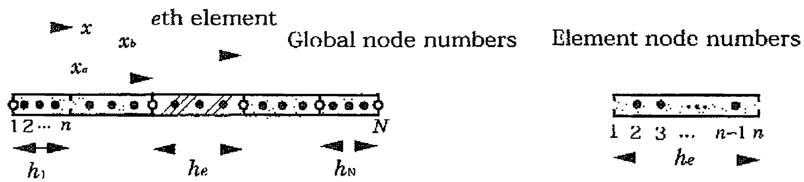


Figure 9.2 Piecewise linear approximation of the solution u that is cubic.



(a) Finite element mesh (b) A typical finite element

Figure 9.3 Subdivision of the domain into finite elements, and a typical finite element.

The collection of elements is called the *finite element mesh* (see Fig. 9.3). The number of divisions is analogous to the number of parameters in the traditional variational methods. Therefore, the larger the number of elements, the more accurate the solution will be. The minimum number of subdivisions is equal to the number of subdivisions created by the discontinuities in the data (i.e., loads, material properties, and geometry) of the problem. An element or interval in one-dimensional problems is a line of finite length.

The elements in a finite element mesh are connected to neighboring elements at a finite number of points, called *global nodes* (see Fig. 9.3). The word "global" refers to the whole problem as opposed to an element. The end points of individual elements are called *element nodes*, and they match with a pair of global nodes. The relationship between the element nodes and global nodes will be discussed below.

9.2.3 Approximation over an Element

Consider an arbitrary element, $\Omega^e = (x_a, x_b)$, located between points $x = x_a \equiv x_1^e$ and $x_b \equiv x_2^e$ in the domain. We isolate the element and study a variational approximation of Eq. (9.1) using any one of the variational methods presented in Chapter 7. In the classical variational methods, we seek approximation of the form

$$u(x) \approx \sum_{i=1}^{N} c_i \phi_i(x) \tag{9.4}$$

over the whole domain and determine relations among c_i such that Eq. (9.1) is satisfied in the weak form or weighted-integral sense *over the entire domain*. In the finite element method we seek a solution of the form

$$u(x) \approx u_e(x) = \sum_{i=1}^{n} c_i^e \phi_i^e(x), \qquad \text{with } \phi_i^e(x) = x^{i-1} \tag{9.5}$$

over an element, and determine relations among c_i by satisfying Eq. (9.1) in a variational (or weak form) sense *over each element*. Since the elements are connected to each other at the nodes, the solutions from various elements connected at a node must have the same value at that node. In order to enforce this continuity (or uniqueness) of solution, it is convenient to express the solution over each element in terms of its values at n nodes of the element:

$$u_e(x) = \sum_{i=1}^{n} c_i^e \phi_i^e(x) = \sum_{i=1}^{n} u_i^e \psi_i^e(x), \tag{9.6}$$

where u_i^e is the value of $u_e(x)$ at the ith node (i.e., $u_e(x_i) = u_i^e$). Hence, by definition, the functions ψ_i^e satisfy the interpolation property

$$\psi_i^e(x_j^e) = \delta_{ij}. \tag{9.7}$$

That is, ψ_i is unity at its own node (i.e., the ith node) and zero at all other nodes. If there are n nodes in the element, a polynomial with n terms is required to fit the solution at the n points. Thus, each ψ_i^e is a $(n-1)$ degree polynomial.

For example, when $n = 2$, we have

$$u_e(x) = c_1^e + c_2^e x. \tag{9.8}$$

Using the definition $u_e(x_i^e) = u_i^e$, we obtain

$$u_e\left(x_1^e\right) = u_1^e = c_1^e + c_2^e x_1^e, \qquad u_e\left(x_2^e\right) = u_2^e = c_1^e + c_2^e x_2^e.$$

Solving for (c_1^e, c_2^e) in terms of (u_1^e, u_2^e), we obtain

$$\begin{Bmatrix} u_1^e \\ u_2^e \end{Bmatrix} = \begin{bmatrix} 1 & x_1^e \\ 1 & x_2^e \end{bmatrix} \begin{Bmatrix} c_1^e \\ c_2^e \end{Bmatrix} \rightarrow c_1^e = \frac{u_1^e x_2^e - u_2^e x_1^e}{x_2^e - x_1^e}, \qquad c_2^e = \frac{u_2^e - u_1^e}{x_2^e - x_1^e}.$$

Substituting for c_1^e and c_2^e into Eq. (9.8), we obtain

$$u_e(x) = c_1^e + c_2^e x = \frac{u_1^e x_2^e - u_2^e x_1^e}{x_2^e - x_1^e} + \left(\frac{u_2^e - u_1^e}{x_2^e - x_1^e}\right) x$$

$$= \left(\frac{x_2^e - x}{x_2^e - x_1^e}\right) u_1^e + \left(\frac{x - x_1^e}{x_2^e - x_1^e}\right) u_2^e \equiv \sum_{i=1}^{2} u_i^e \psi_i^e(x), \tag{9.9}$$

where

$$\psi_1^e(x) = \left(\frac{x_2^e - x}{x_2^e - x_1^e}\right), \qquad \psi_2^e(x) = \left(\frac{x - x_1^e}{x_2^e - x_1^e}\right). \tag{9.10}$$

Clearly, $\psi_i^e(x)$ satisfy the interpolation property in Eq. (9.7) (see Fig. 9.4). Thus the nodal values u_i^e take the place of the undetermined parameters while ψ_i^e take the role of the approximation functions in the traditional variational methods.

Alternatively, we can determine $\psi_i^e(x)$ using the interpolation property (9.7). For example, for linear interpolation, the functions have the properties

$$\psi_1^e(x_1^e) = 1, \qquad \psi_1^e(x_2^e) = 0, \qquad \psi_2^e(x_1^e) = 0, \qquad \psi_2^e(x_2^e) = 1.$$

Since ψ_1^e and ψ_2^e must vanish at x_2^e and x_1^e, respectively, we may write them as

$$\psi_1^e(x) = A\left(x - x_2^e\right), \qquad \psi_2^e(x) = B\left(x - x_1^e\right),$$

where the constants A and B are determined so that ψ_1^e and ψ_2^e are unity at x_1^e and x_2^e, respectively:

$$\psi_1^e(x_1^e) = 1 = A\left(x_1^e - x_2^e\right), \qquad \psi_2^e(x_2^e) = 1 = B\left(x_2^e - x_1^e\right).$$

The above equations give $A = -1/h_e$ and $B = 1/h_e$, where h_e denotes the length of the element, $h_e = x_2^e - x_1^e = x_b^e - x_a^e$. Hence the approximation functions $\psi_i^e(x)$ are given by

$$\psi_1^e(x) = \frac{x_2^e - x}{h_e}, \qquad \psi_2^e(x) = \frac{x - x_1^e}{h_e},$$

which are the same as those listed in Eq. (9.10).

The interpolation functions ψ_i^e are defined over the element Ω^e, and they are zero outside the element. Hence they are said to have compact support. The interpolation in which only the function alone is interpolated—and not its derivatives—is known as the Lagrange interpolation, and the corresponding functions are termed *the Lagrange interpolation functions*. It is convenient to express the interpolation

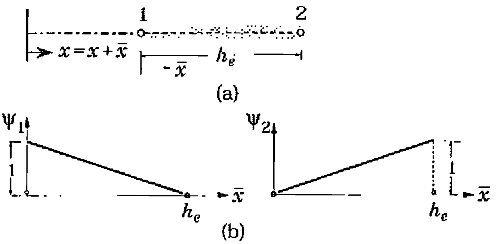


Figure 9.4 Linear interpolation functions.

functions in terms of a coordinate system fixed in the element. For example, we may take the origin of the element coordinate, $\bar{x}$, at node 1 such that

$$x = \bar{x} + x_1^e. \tag{9.11}$$

Then the interpolation functions (9.10) can be expressed in terms of $\bar{x}$ as

$$\psi_1^e(\bar{x}) = 1 - \frac{\bar{x}}{h_e}, \qquad \psi_2^e(x) = \frac{\bar{x}}{h_e}. \tag{9.12}$$

On the other hand, if ξ is the element coordinate such that $\xi = -1$ when $\bar{x} = 0$ and $\xi = 1$ when $\bar{x} = h_e$, i.e.,

$$\bar{x} = \frac{h_e}{2}(1 + \xi), \tag{9.13}$$

the $\psi_i(\bar{x})$ of Eq. (9.12) take the form

$$\psi_1^e(\xi) = \frac{1}{2}(1 - \xi), \qquad \psi_2^e(\xi) = \frac{1}{2}(1 + \xi), \qquad -1 \leq \xi \leq 1. \tag{9.14}$$

The coordinate ξ is known as the *normalized* or *natural coordinate*.

For values $n > 2$ we must identify additional points or element nodes in the interior of the element. These points, in principle, can be any distinctly different points of the domain. For example, when $n = 3$, the third node can be placed anywhere other than at the ends. The optimal location is the one that is equidistant from both end nodes, i.e., the midpoint of the element. When $n = 4$, the one-third points are selected for the two interior nodes. The $(n - 1)$-degree Lagrange interpolation functions (for an element with n nodes) are defined, for arbitrary location of the nodes, by

$$\psi_j^e = \Pi_{k=1,k\neq j}^{m} \frac{(x - x_k^e)}{(x_j^e - x_k^e)}. \tag{9.15}$$

As specific examples, the quadratic and cubic Lagrange interpolation functions are given below.

Quadratic ($n = 3$)

$$\psi_1^e = \frac{x - x_2^e}{x_1^e - x_2^e}\frac{x - x_3^e}{x_1^e - x_3^e}, \qquad \psi_2^e = \frac{x - x_1^e}{x_2^e - x_1^e}\frac{x - x_3^e}{x_2^e - x_3^e}, \qquad \psi_3^e = \frac{x - x_1^e}{x_3^e - x_1^e}\frac{x - x_2^e}{x_3^e - x_2^e}. \tag{9.16a}$$

Cubic ($n = 4$)

$$\begin{aligned} \psi_1^e &= \frac{x - x_2^e}{x_1^e - x_2^e}\frac{x - x_3^e}{x_1^e - x_3^e}\frac{x - x_4^e}{x_1^e - x_4^e}, \qquad & \psi_2^e &= \frac{x - x_1^e}{x_2^e - x_1^e}\frac{x - x_3^e}{x_2^e - x_3^e}\frac{x - x_4^e}{x_2^e - x_4^e}, \\ \psi_3^e &= \frac{x - x_1^e}{x_3^e - x_1^e}\frac{x - x_2^e}{x_3^e - x_2^e}\frac{x - x_4^e}{x_3^e - x_4^e}, \qquad & \psi_4^e &= \frac{x - x_1^e}{x_4^e - x_1^e}\frac{x - x_2^e}{x_4^e - x_2^e}\frac{x - x_3^e}{x_4^e - x_3^e}. \end{aligned} \tag{9.16b}$$

For equally spaced nodal points, these functions have the following simpler form, especially when expressed in terms of the local coordinate $\bar{x}$ (see Fig. 9.5):

Quadratic Functions

$$\begin{aligned} \psi_1^e(\bar{x}) &= \left(1 - \frac{2\bar{x}}{h_e}\right)\left(1 - \frac{\bar{x}}{h_e}\right), \\ \psi_2^e(\bar{x}) &= 4\frac{\bar{x}}{h_e}\left(1 - \frac{\bar{x}}{h_e}\right), \\ \psi_3^e(x) &= -\frac{\bar{x}}{h_e}\left(1 - \frac{2\bar{x}}{h_e}\right). \end{aligned} \tag{9.17}$$

Cubic Functions

$$\begin{aligned} \psi_1^e(\bar{x}) &= \left(1 - \frac{3\bar{x}}{h_e}\right)\left(1 - \frac{3\bar{x}}{2h_e}\right)\left(1 - \frac{\bar{x}}{h_e}\right), \\ \psi_2^e(\bar{x}) &= 9\frac{\bar{x}}{h_e}\left(1 - \frac{3\bar{x}}{2h_e}\right)\left(1 - \frac{\bar{x}}{h_e}\right), \\ \psi_3^e(\bar{x}) &= -9\frac{\bar{x}}{h_e}\left(1 - \frac{3\bar{x}}{h_e}\right)\left(1 - \frac{\bar{x}}{h_e}\right), \\ \psi_4^e(\bar{x}) &= \frac{\bar{x}}{h_e}\left(1 - \frac{3\bar{x}}{h_e}\right)\left(1 - \frac{3\bar{x}}{2h_e}\right). \end{aligned} \tag{9.18}$$

Note that the interpolation functions $\psi_i^e(x)$ $(i = 1, 2, \ldots, n)$ depend only on the geometry and position of the nodes in element $\Omega^e = (x_a, x_b) = (x_1^e, x_n^e)$; they do not depend on the solution or the boundary conditions of the problem. The functions ψ_i^e satisfy the interpolation property (9.7). In addition, we note that

$$\psi_1^e(x) + \psi_2^e(x) + \cdots + \psi_n^e(x) = 1 \qquad \text{or} \qquad \sum_{i=1}^{n} \psi_i^e(x) = 1, \tag{9.19}$$

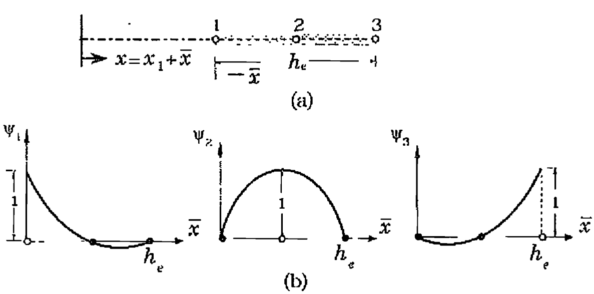
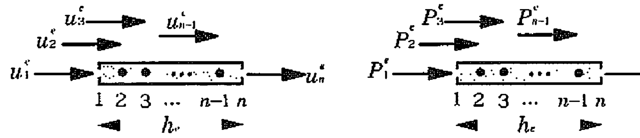


Figure 9.5 Quadratic interpolation functions.



(a) Element nodal forces (b) Element nodal forces

Figure 9.6 (a) A typical finite element with *n* nodes and nodal displacements. (b) A typical finite element with *n* nodes and nodal forces.

which is sometimes referred to as "the partition of unity." This property is the direct result of including the constant term in the approximation (9.8). For example, if the solution is constant, say c^e, over the entire element, then $u_1^e = u_2^e = \cdots = u_n^e = c^e$, and we have

$$c^e = \sum_{i=1}^{n} u_i^e \psi_i^e(x) = c^e \sum_{i=1}^{n} \psi_i^e(x) \quad \rightarrow \quad 1 = \sum_{i=1}^{n} \psi_i^e(x).$$

9.2.4 Weak Form

Having established a systematic way of deriving the approximation functions needed for the Ritz solution over an element, we now turn our attention to developing the weak form of the governing equation (9.1) over the domain $\Omega^e = (x_a, x_b) = (x_1^e, x_n^e)$ of a typical element. A typical element with n nodes and nodal displacements is shown in Fig. 9.6a, while Fig. 9.6b contains the n-node element with forces (i.e., a free-body diagram of the element). The forces and displacements at the end nodes are defined by

$$\begin{aligned} u(x_1^e) &= u_1^e, & u(x_n^e) &= u_n^e, \\ \left(-a\frac{du}{dx}\right)\bigg|_{x=x_1^e} &= P_1^e, & \left(a\frac{du}{dx}\right)\bigg|_{x=x_n^e} &= P_n^e. \end{aligned} \tag{9.20}$$

The nodal forces $P_2^e, P_3^e, \ldots, P_{n-1}^e$ are externally applied (i.e., known) forces, if any.

The variational statement for the bar element in Fig. 9.6 is provided by the principle of minimum total potential energy, $\delta\Pi^e = 0$:

$$\begin{aligned} 0 &= \delta\left\{\int_{x_a}^{x_b}\left[\frac{E_eA_e}{2}\left(\frac{du}{dx}\right)^2 + \frac{c_e}{2}(u)^2 - fu\right]dx \right. \\ &\quad \left. - P_1^e u_1^e - P_2^e u_2^e - \cdots - P_n^e u_n^e\right\} \\ &= \int_{x_a}^{x_b}\left(E_eA_e\frac{d\delta u}{dx}\frac{du}{dx} + c_e\delta u\, u - f\delta u\right)dx - \sum_{k=1}^{n} P_k^e \delta u_k^e, \end{aligned} \tag{9.21}$$

where δ is the variational symbol, and the subscript e on the variables indicates that the variables are defined in element Ω^e.

Since the model equation (9.1) also arises in fields other than solid and structural mechanics, it is informative to discuss the procedure by which we can obtain the weak form (9.21) from Eq. (9.1) directly. As discussed in Chapter 7, we use the three-step procedure to construct the weak form. We have

$$
\begin{aligned}
0 &= \int_{x_a}^{x_b} w\left[-\frac{d}{dx}\left(a\frac{du}{dx}\right) + cu - f\right]dx \\
&= \int_{x_a}^{x_b}\left[a\frac{dw}{dx}\frac{du}{dx} + cwu - wf\right]dx - \left[w\cdot\left(a\frac{du}{dx}\right)\right]_{x_a}^{x_b} \\
&= \int_{x_a}^{x_b}\left[a\frac{dw}{dx}\frac{du}{dx} + cwu - wf\right]dx - \sum_{k=1}^{n-1}\left[w\cdot\left(a\frac{du}{dx}\right)\right]_{x_k^e}^{x_{k+1}^e} \\
&= \int_{x_a}^{x_b}\left[a\frac{dw}{dx}\frac{du}{dx} + cwu - wf\right]dx - \sum_{k=1}^{n} w(x_k^e)P_k^e,
\end{aligned}
\tag{9.22a}
$$

where $x_1^e = x_a$, $x_n^e = x_b$, and

$$
\left(-a\frac{du}{dx}\right)_{x_a} = P_1^e, \qquad \left[a\frac{du}{dx}\right]_{x_k^e} = P_k^e \quad (k=2,\ldots,n-1), \qquad \left(a\frac{du}{dx}\right)_{x_b} = P_n^e,
\tag{9.22b}
$$

and $[\cdot]_P$ denotes the jump in the enclosed quantity at point P. Equation (9.22b) is the same as Eq. (9.21) with the weight function w replaced by δu and $a = EA$.

9.2.5 Finite Element Equations

We seek approximation of $u(x)$ in the form

$$
u(x) \approx u_e(x) = \sum_{j=1}^{n} u_j^e \psi_j^e(x), \qquad w = \delta u(x) \approx \sum_{i=1}^{n} \delta u_i^e \psi_i^e(x),
\tag{9.23}
$$

where ψ_j^e are the approximation functions derived earlier; they can be linear ($n = 2$), quadratic ($n = 3$), or higher ($n > 3$).

Substitution of Eq. (9.23) into Eq. (9.21) yields

$$
\begin{aligned}
0 = \int_{x_a}^{x_b}\Bigg[& E_eA_e\left(\sum_{i=1}^{n}\delta u_i^e\frac{d\psi_i}{dx}\right)\left(\sum_{j=1}^{n}u_j^e\frac{d\psi_j}{dx}\right) + c_e\left(\sum_{i=1}^{n}\delta u_i^e\psi_i\right)\left(\sum_{j=1}^{n}u_j^e\psi_j\right) \\
& - f\left(\sum_{i=1}^{n}\delta u_i^e\psi_i\right)\Bigg]dx - \sum_{i=1}^{n}P_i^e\delta u_i^e
\end{aligned}
$$

$$= \sum_{i=1}^{n} \delta u_i^e \left\{ \sum_{j=1}^{n} \left[\int_{x_a}^{x_b} \left(E_e A_e \frac{d\psi_i}{dx} \frac{d\psi_j}{dx} + c_e \psi_i^e \psi_j^e \right) dx \right] u_j^e - \int_{x_a}^{x_b} f \psi_i \, dx - P_i^e \right\}$$

$$= \sum_{i=1}^{n} \delta u_i^e \left[\sum_{j=1}^{n} K_{ij}^e u_j^e - f_i^e - P_i^e \right]. \tag{9.24}$$

Since $\delta u_1^e, \delta u_2^e, \ldots, \delta u_n^e$ are arbitrary and the above equation must hold for all $i = 1, 2, \ldots, n$, we obtain

$$0 = \sum_{j=1}^{n} K_{ij}^e u_j^e - f_i^e - P_i^e \equiv \sum_{j=1}^{n} K_{ij}^e u_j^e - F_i^e, \tag{9.25a}$$

where

$$K_{ij}^e = \int_{x_a}^{x_b} \left(E_e A_e \frac{d\psi_i^e}{dx} \frac{d\psi_j^e}{dx} + c_e \psi_i^e \psi_j^e \right) dx$$
$$= \int_0^{h_e} \left(E_e(\bar{x}) A_e(\bar{x}) \frac{d\psi_i^e}{d\bar{x}} \frac{d\psi_j^e}{d\bar{x}} + c_e \psi_i^e \psi_j^e \right) d\bar{x},$$
$$f_i^e = \int_{x_a}^{x_b} f_e(x) \psi_i^e \, dx = \int_0^{h_e} f_e(\bar{x}) \psi_i^e(\bar{x}) \, d\bar{x}. \tag{9.25b}$$

The coefficient matrix $[K^e]$ is called the *stiffness matrix*, and $\{F^e\} \equiv \{f_i^e\} + \{P_i^e\}$ is the *force vector*. Equation (9.25a) is often referred to as the *finite element model* of the differential equation (9.1), and it provides n linear algebraic equations relating n nodal values u_j^e, $(j = 1, 2, \ldots, n)$.

The coefficient matrix $[K^e]$, which is symmetric, and the source vector $\{f^e\}$ can be evaluated for a given element type (i.e., linear, quadratic, etc.) and element data (a_e, c_e, and f_e). For element-wise constant values of a_e, c_e, and f_e, the coefficients K_{ij}^e and f_i^e can easily be evaluated. For linear and quadratic elements, these matrices are presented below.

Linear Element For a typical linear element of length $h_e = x_b - x_a$, we have

$$[K^e] = \frac{a_e}{h_e} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} + \frac{c_e h_e}{6} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}, \tag{9.26a}$$

$$\{f^e\} = \frac{f_e h_e}{2} \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}, \qquad a_e = E_e A_e. \tag{9.26b}$$

If $a = a_e \cdot x$ and $c = c_e$, the coefficient matrix $[K^e]$ for a linear element can be evaluated as

$$[K^e] = \frac{a_e}{h_e} \left(\frac{x_1^e + x_2^e}{2} \right) \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} + \frac{c_e h_e}{6} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}, \tag{9.27}$$

where (x_1^e, x_2^e) are global coordinates of node 1 and node 2 of the element $\Omega^e = (x_a, x_b) = (x_1^e, x_2^e)$. The reader should verify this. Note that $[K^e]$ in Eq. (9.27) is the same as that in Eq. (9.26a) with a_e replaced by the average value

$$a_{\text{avg}} = \frac{1}{2}(x_1^e + x_2^e)a_e.$$

For example, in the study of bars with linearly varying cross section A but constant modulus of elasticity E, we have

$$a(x) = EA(x) = E\left(A_1^e + \frac{A_2^e - A_1^e}{h_e}\bar{x}\right),$$

where $\bar{x}$ is the local or element coordinate with origin at node 1, and A_1^e and A_2^e are areas of cross section at nodes 1 and 2, respectively. Then the element stiffness matrix is the same as that of a constant cross-section bar with the cross-sectional area being the average of the two ends, $A_{\text{avg}} = (A_1^e + A_2^e)/2$.

When $a(x) = a_e =$ constant, and $f(x) = f_e =$ constant, and $c_e = 0$, the finite element equations corresponding to the linear element reduce to

$$\frac{a_e}{h_e}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} u_1^e \\ u_2^e \end{Bmatrix} = \frac{f_e h_e}{2}\begin{Bmatrix} 1 \\ 1 \end{Bmatrix} + \begin{Bmatrix} P_1^e \\ P_2^e \end{Bmatrix} \tag{9.28a}$$

or

$$\begin{aligned} \frac{a_e}{h_e}u_1^e - \frac{a_e}{h_e}u_2^e &= \frac{1}{2}f_e h_e + P_1^e, \\ -\frac{a_e}{h_e}u_1^e + \frac{a_e}{h_e}u_2^e &= \frac{1}{2}f_e h_e + P_2^e. \end{aligned} \tag{9.28b}$$

Quadratic Element For a quadratic element $\Omega^e = (x_a, x_b) = (x_1^e, x_3^e)$, $h_e = x_b - x_a = x_3^e - x_1^e$, we have

$$[K^e] = \frac{a_e}{3h_e}\begin{bmatrix} 7 & -8 & 1 \\ -8 & 16 & -8 \\ 1 & -8 & 7 \end{bmatrix} + \frac{c_e h_e}{30}\begin{bmatrix} 4 & 2 & -1 \\ 2 & 16 & 2 \\ -1 & 2 & 4 \end{bmatrix}, \tag{9.29a}$$

$$\{f^e\} = \frac{f_e h_e}{6}\begin{Bmatrix} 1 \\ 4 \\ 1 \end{Bmatrix}. \tag{9.29b}$$

For arbitrary variation of the data a_e, c_e, and f_e, numerical integration may be used to evaluate the coefficients K_{ij}^e and f_i^e (see Reddy [4]).

9.2.6 Assembly (or Connectivity) of Elements

The finite element equations (9.25a) can be specialized to each one of the elements in the mesh by assigning the values of x_a, x_b, a_e, c_e, etc. Because each of the elements in the mesh is connected to its neighboring elements at the global nodes, and the displacement is continuous from one element to the next, one can relate the nodal values of displacements at the interelement connecting nodes. To this end, let U_I denote the value of the displacement $u(x)$ at the Ith global node. Then we have the following correspondence between U_I and u_j^e (see Fig. 9.7a) of a linear element mesh:

$$u_1^1 = U_1, \qquad u_2^1 = u_1^2 \equiv U_2, \ldots, \qquad u_2^e = u_1^{e+1} \equiv U_{e+1}, \ldots, u_2^N \equiv U_{N+1}, \tag{9.30}$$

where N is the total number of linear elements connected in series. Equation (9.30) relates the global displacements to local displacements and enforces interelement continuity of the displacements (see Fig. 9.7a).

The assembly of element equations is based on the satisfaction of the principle of minimum total potential energy for the whole system:

$$\delta\Pi = \sum_{I=1}^{N+1} \frac{\partial \Pi}{\partial U_I}\delta U_I = 0 \qquad \text{or} \qquad \frac{\partial \Pi}{\partial U_I} = 0, \qquad I = 1, 2, \ldots, N+1, \tag{9.31}$$

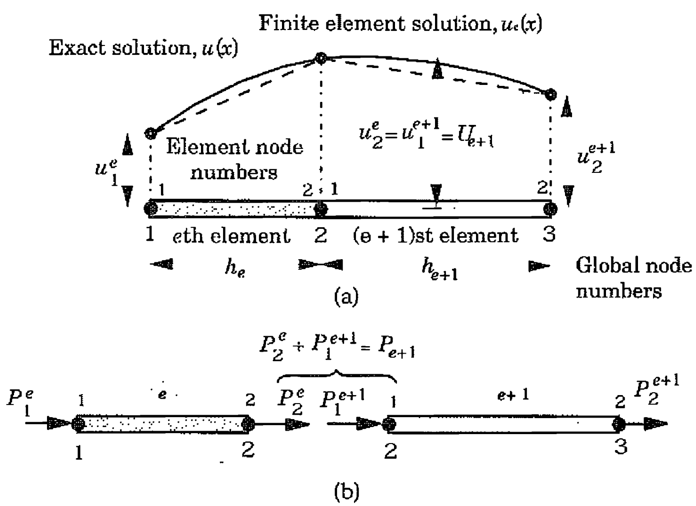


Figure 9.7 Connectivity of elements in one dimension. (a) Element-wise linear approximation of the displacement and interelement continuity of the displacements. (b) Balance of element forces at common nodes.

where Π is the sum of Π^e for $e = 1, 2, \ldots, N$:

$$\Pi = \sum_{e=1}^{N} \Pi^e = \sum_{e=1}^{N} \sum_{i=1}^{2} u_i^e \left(\frac{1}{2} \sum_{j=1}^{2} K_{ij}^e u_j^e - F_i^e \right)$$

$$= \tfrac{1}{2} \sum_{e=1}^{N} \left[u_1^e (K_{11}^e u_1^e + K_{12}^e u_2^e - 2F_1^e) + u_2^e (K_{21}^e u_1^e + K_{22}^e u_2^e - 2F_2^e) \right]$$

$$= \tfrac{1}{2} \left[u_1^1 (K_{11}^1 u_1^1 + K_{12}^1 u_2^1 - 2F_1^1) + u_1^2 (K_{11}^2 u_1^2 + K_{12}^2 u_2^2 - 2F_1^2) + \cdots \right.$$
$$\left. + u_2^1 (K_{21}^1 u_1^1 + K_{22}^1 u_2^1 - 2F_2^1) + u_2^2 (K_{21}^2 u_1^2 + K_{22}^2 u_2^2 - 2F_2^2) + \cdots \right]$$

$$= \tfrac{1}{2} \left[U_1 (K_{11}^1 U_1 + K_{12}^1 U_2 - 2F_1^1) + U_2 (K_{11}^2 U_2 + K_{12}^2 U_3 - 2F_1^2) + \cdots \right.$$
$$\left. + U_2 (K_{21}^1 U_1 + K_{22}^2 U_2 - 2F_2^1) + U_3 (K_{21}^2 U_2 + K_{22}^2 U_3 - 2F_2^2) + \cdots \right], \tag{9.32}$$

where, in arriving at the last line, the correspondence in Eq. (9.30) is used to replace the element nodal displacements by the global nodal displacements. Using Eq. (9.32) in Eq. (9.31), we obtain

$$\begin{aligned}
&K_{11}^1 U_1 + \tfrac{1}{2}(K_{12}^1 + K_{21}^1) U_2 + \cdots - F_1^1 = 0, \\
&\tfrac{1}{2}(K_{12}^1 + K_{21}^1) U_1 + (K_{22}^1 + K_{11}^2) U_2 \\
&\quad + \tfrac{1}{2}(K_{12}^2 + K_{21}^2) U_3 + \cdots - F_2^1 - F_1^2 = 0, \\
&\tfrac{1}{2}(K_{12}^2 + K_{21}^2) U_2 + (K_{22}^2 + K_{11}^3) U_3 + \cdots - F_2^2 - F_1^3 = 0, \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
&\tfrac{1}{2}(K_{12}^{N-1} + K_{21}^{N-1}) U_{N-1} + (K_{22}^{N-1} + K_{11}^N) U_N \\
&\quad + \tfrac{1}{2}(K_{12}^N + K_{21}^N) U_{N+1} - F_2^{N-1} - F_1^N = 0, \\
&\tfrac{1}{2}(K_{12}^N + K_{21}^N) U_N + K_{22}^N U_{N+1} - F_2^N = 0.
\end{aligned} \tag{9.33a}$$

Using the symmetry of the element stiffness matrices, $K_{ji}^e = K_{ij}^e$, and the fact that $\{F^e\} = \{f^e\} + \{P^e\}$, we can write the above equations in the matrix form

$$\begin{bmatrix}
K_{11}^1 & K_{12}^1 & 0 & \ldots & 0 \\
K_{12}^1 & K_{22}^1 + K_{11}^2 & K_{12}^2 & 0 & 0 \\
0 & K_{12}^2 & K_{22}^2 + K_{11}^3 & K_{12}^3 & 0 \\
\vdots & \vdots & & \vdots & \vdots \\
0 & 0 & & K_{22}^{N-1} + K_{11}^N & K_{12}^N \\
0 & 0 & \ldots & K_{12}^N & K_{22}^N
\end{bmatrix}
\begin{Bmatrix} U_1 \\ U_2 \\ U_3 \\ \vdots \\ U_N \\ U_{N+1} \end{Bmatrix}$$

$$= \begin{Bmatrix} f_1^1 \\ f_2^1 + f_1^2 \\ \vdots \\ f_2^{N-1} + f_1^N \\ f_2^N \end{Bmatrix} + \begin{Bmatrix} P_1^1 \\ P_2^1 + P_1^2 \\ \vdots \\ P_2^{N-1} + P_1^N \\ P_2^N \end{Bmatrix}. \tag{9.33b}$$

It is clear from Eq. (9.33b) that the diagonal elements of stiffness matrices of elements Ω^e and Ω^{e+1} add up at the common global node $I = e + 1$, and the global stiffness coefficient K_{IJ} is zero if global nodes I and J do not belong to the same element. Thus the resulting global stiffness matrix is not only symmetric, but is also banded, i.e., all entries beyond a diagonal parallel to the main diagonal, below and above, are zero. This is a feature of all finite element equations, irrespective of the differential equation being solved, and is a result of the piecewise definition of the coordinate functions. Keeping in mind the general pattern of the assembled stiffness matrix and force column, one can routinely assemble the element matrices for any number of elements.

The assembly procedure for a general case is based on the following two requirements:

1. Continuity of the primary variable(s) at the interelement boundary, as expressed by Eq. (9.30).
2. Balance of secondary variables; i.e., the secondary variables from the elements connected at a global node should add up to the value of the externally applied secondary variable at the node.

The second condition for the mesh of two linear elements shown in Fig. 9.7b requires

$$P_2^e + P_1^{e+1} = P_{e+1}, \tag{9.34}$$

where P_{e+1} is the value of externally applied force at node $e + 1$ (see Fig. 9.7b). These conditions require the addition of the second equation of element Ω^e to the first equation of element Ω^{e+1} so that we can replace $P_2^e + P_1^{e+1}$ with P_{e+1}. This reduces $2N$ equations to $N + 1$ equations, where N is the number of linear elements connected in series, as shown in Fig. 9.7a. In general, if the ith node of element Ω^e is connected to the jth node of element Ω^f, the balance of secondary variables requires

$$P_i^e + P_j^f = F_K, \tag{9.35}$$

where K is the global node number of the ith node of element Ω^e, which is the same as the jth node of element Ω^f.

9.2.7 Imposition of the Boundary Conditions

Equations (9.33b) contain more variables than the number of equations, prior to applying the boundary conditions of the problem. In general, for the model problem in Eq. (9.1), we must know the value of either the primary variable u or the secondary variable $a(du/dx)$ at each node, including the boundary nodes. In all bar problems, we have either a displacement u or a force $a(du/dx)$ specified at a point. If the displacement is specified at a global node, the corresponding nodal displacement should be replaced by the specified value. To impose a specified force, recall that F_i^e consists of contributions due to the distributed force $f(x)$ and internal force $a(du/dx)$ at the nodes [see Eq. (9.25a)], $F_i^e = f_i^e + P_i^e$, where P_i^e and f_i^e are the components defined in Eqs. (9.22b) and (9.25b), respectively. At any node where the displacement is unknown, the externally applied point force at the node should be equal to the sum of internal forces from all elements connected at the node. To illustrate this point further, consider a bar fixed at the left end, $u_0 = 0$, and subjected to distributed force f throughout the length of the bar and a point load P_0 at the right end. The fact that no point loads are specified at the intermediate nodes implies that

$$P_2^1 + P_1^2 = 0, \qquad P_2^2 + P_1^3 = 0, \ldots, P_2^{N-1} + P_1^N = 0. \tag{9.36}$$

Then Eq. (9.33b) becomes

$$\begin{bmatrix} K_{22}^1 + K_{11}^2 & K_{12}^2 & 0 & \cdots & 0 \\ K_{12}^2 & K_{22}^2 + K_{11}^3 & K_{12}^3 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & K_{22}^{N-1} + K_{11}^N & K_{12}^N \\ 0 & 0 & \cdots & K_{12}^N & K_{22}^N \end{bmatrix} \begin{Bmatrix} U_2 \\ U_3 \\ \vdots \\ U_n \\ U_{N+1} \end{Bmatrix}$$

$$= \begin{Bmatrix} f_2^1 + f_1^2 \\ f_2^2 + f_1^3 \\ \vdots \\ f_2^{N-1} + f_1^N \\ f_2^N + P_0 \end{Bmatrix} - \begin{Bmatrix} K_{12}^1 u_0 \\ 0 \\ \vdots \\ 0 \end{Bmatrix}, \tag{9.37}$$

which consists of N equations for the N unknowns, U_I ($I = 2, \ldots, N+1$). Since the right-hand column is completely known, one can use any standard solution procedure to solve the linear algebraic equations.

9.2.8 Calculation of Reactions and Derivatives of Solution: Postprocessing

The displacement at any point can be computed using the equation

$$u_e(x) = \begin{cases} u_1(x), & 0 \le x \le h_1, \\ u_2(x), & h_1 \le x \le h_2, \\ \vdots & \\ u_N(x), & h_{N-1} \le x \le h_N, \end{cases} \tag{9.38}$$

where $u_e(x)$ is the finite element solution of (9.23) in element Ω^e. For example, if $u(x_0)$, $h_1 \le x_0 \le h_2$, is desired, for a linear element we have

$$u_2(x_0) = u_1^{(2)}\psi_1^{(2)}(x_0) + u_2^{(2)}\psi_2^{(2)}(x_0) = U_2\psi_1^{(2)}(x_0) + U_3\psi_2^{(2)}(x_0). \tag{9.39}$$

Similarly, the derivatives of $u_e(x)$ can be computed from

$$\frac{du_e}{dx} = \begin{cases} \sum_{j=1}^{n} u_j^1 \frac{d\psi_j^1}{dx}, & 0 < x < h_1, \\ \sum_{j=1}^{n} u_j^2 \frac{d\psi_j^2}{dx}, & h_1 < x < h_2, \\ & \\ \sum_{j=1}^{n} u_j^N \frac{d\psi_j^N}{dx}, & h_{N-1} < x < h_N. \end{cases} \tag{9.40}$$

It should be noted that the nodal values of the derivative computed from the two elements sharing that node do not coincide, irrespective of the order of the elements, because the continuity of the derivative is not imposed in the procedure. Therefore, one must further process these values to assign a single value at the node (e.g., the weighted average of the values of the derivative from all elements connected at the node).

The unknown reaction forces at any global node can be computed after the nodal displacements U_i are computed. They can be computed from either (a) the definition (9.22b) or (a) the element equations (9.25a). For the problem at hand, the first equation of element 1 can be used to determine the reaction force at node 1 ($U_1 = u_0$):

$$P_1^1 = -f_1^1 + K_{11}^1 u_0 + K_{12}^1 U_2. \tag{9.41a}$$

Alternatively, P_1^1 can be computed using the definition

$$\begin{aligned} P_1^1 &\equiv \left(-a\frac{du}{dx}\right)_{x=0} \\ &= \sum_{j=1}^{2} u_j^1 \left(\frac{d\psi_j^1}{dx}\right)_{x=0} = \frac{U_2 - U_1}{h_1}. \end{aligned} \tag{9.41b}$$

The two values for P_1^1 from Eqs. (9.41a) and (9.41b) are the same only when the distributed force $f(x)$ is zero. The difference reduces as the number of elements is increased (i.e., f_i^e gets smaller in magnitude as the number of elements is increased).

This completes the description of the basic steps in the finite element analysis of the model problem. The derivation of the coordinate functions is not a usual step in the finite element formulation because these functions are already derived and are available in textbooks. One should only select appropriate coordinate functions for the problem at hand. The derivation is included here to illustrate the procedure. Although we used the terminology of solid mechanics in the development of the finite element equations, all the concepts are applicable to any physical problem described by Eq. (9.1). In other words, if a computer program is developed to solve Eq. (9.1) for any boundary conditions, then the program can be used to analyze not only bars but all other physical problems described by the equation (e.g., transverse deflection of cables, heat transfer in fins, flow through pipes, etc.; see Table 8.2). For additional details, the reader may consult the textbook by Reddy [4]. In the following example, we illustrate the steps involved in the finite element analysis of a composite bar.

Example 9.1 Consider the composite bar consisting of a tapered steel bar fastened to an aluminum rod of uniform cross section, and subjected to loads as shown in Fig. 9.8. We wish to determine the deformation in the bar using the linear elements. The governing equations are given by

$$
\begin{aligned}
-\frac{d}{dx}\left(E_s A_s \frac{du}{dx}\right) &= 0, \qquad 0 < x < h_1, \\
-\frac{d}{dx}\left(E_a A_a \frac{du}{dx}\right) &= 0, \qquad h_1 < x < h_1 + h_2 = L,
\end{aligned}
\tag{9.42a}
$$

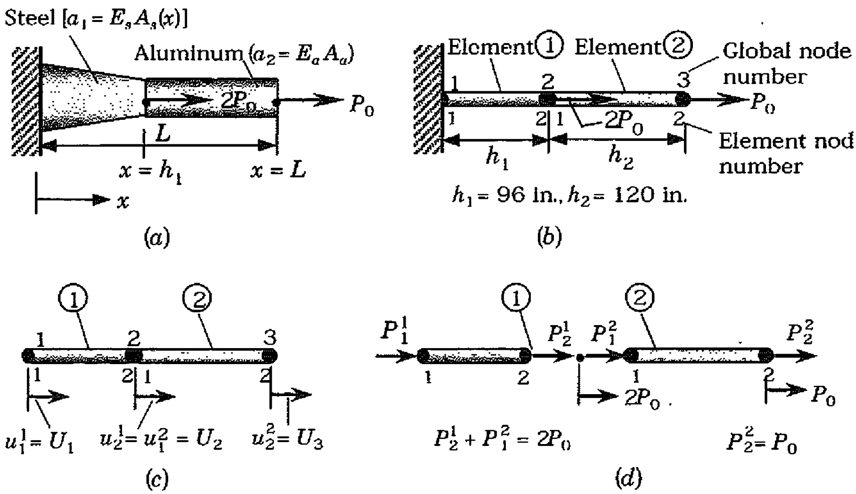


Figure 9.8 Axial deformation of a composite member. (a) Geometry and loading. (b) Finite element representation. (c) Continuity of displacements. (d) Balance of forces.

where the subscript s refers to steel and a to aluminum. We need not worry about the boundary conditions until step 4 of the analysis.

We take the following values of the data:

$$\begin{aligned} &E_s = 30 \times 10^6 \text{ psi}, \quad A_s = (c_1 + c_2 x)^2, \quad E_a = 10^7 \text{ psi}, \\ &A_a = 1 \text{ in.}^2, \quad h_1 = 96 \text{ in.}, \quad L = 216 \text{ in.}, \quad P_0 = 10{,}000 \text{ lb}. \end{aligned} \tag{9.42b}$$

We will follow the steps in the finite element analysis of the problem. In actual practice, all the steps can be carried on a digital computer.

1. *Finite Element Mesh.* From the discontinuity in the material property, area of cross section, and loading at $x = h_1$, we are required to divide the domain $\Omega = (0, L)$ into at least two elements: $\Omega^1 = (0, h_1)$ and $\Omega^2 = (h_1, L)$. For the mesh of two linear elements, there will be three global nodes; if quadratic elements are used, then there will be five global nodes.

2. *Element Equations.* For each element, whether linear or quadratic, the element stiffness matrix and force vector are given by Eq. (9.25b), with

$$a_e = E_e\left(a_1^e + a_2^e x\right)^2, \qquad c_e = 0, \qquad f_e = 0.$$

We now compute $[K^e]$ using the linear interpolation functions.

The element stiffness matrix and force vector can be computed using either the local coordinate $\bar{x}$ or the global coordinate x. If the local coordinate is used to evaluate the coefficients, then the transformation from x to $\bar{x}$ ($x = \bar{x} + x_e$) should be used to convert all functions of x to functions of $\bar{x}$. We have

$$\begin{aligned} K_{ij}^e &= \int_{x_a = x_e}^{x_b = x_{e+1}} E_e\left(a_1^e + a_2^e x\right)^2 \frac{d\psi_i^e}{dx}\frac{d\psi_j^e}{dx}\, dx \\ &= \int_0^{h_e} E_e\left[a_1^e + a_2^e(\bar{x} + x_e)\right]^2 \frac{d\psi_i^e}{dx}\frac{d\psi_j^e}{dx}\, d\bar{x}, \\ K_{11}^e &= \frac{E_e}{h_e^2}\int_{x_e}^{x_{e+1}} \left(a_1^e + a_2^e x\right)^2 dx \\ &= \frac{E_e}{h_e^2}\left[\left(a_1^e\right)^2 h_e + \frac{1}{3}\left(a_2^e\right)^2\left(x_{e+1}^3 - x_e^3\right) + a_1^e a_2^e\left(x_{e+1}^2 - x_e^2\right)\right], \\ K_{12}^e &= K_{21}^e = -K_{11}^e, \qquad K_{22}^e = K_{11}^e, \end{aligned}$$

or

$$[K^e] = \frac{E_e A_e}{h_e}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}, \qquad \{F^e\} = \begin{Bmatrix} P_1^e \\ P_2^e \end{Bmatrix}, \tag{9.43a}$$

where [we have used the algebraic equalities, $a^2 - b^2 = (a-b)(a+b)$ and $a^3 - b^3 = (a-b)(a^2 + b^2 + ab)$]

$$A_e = \left(a_1^e\right)^2 + \frac{1}{3}\left(a_2^e\right)^2\left(x_{e+1}^2 + x_e^2 + x_e x_{e+1}\right) + a_1^e a_2^e (x_{e+1} + x_e). \tag{9.43b}$$

More specifically, we have ($a_1^1 = 1.5$, $a_2^1 = -0.5/96$, $a_1^2 = 1$, $a_2^2 = 0$, $h_1 = 96$ in., $h_2 = 120$ in.):

$$\begin{aligned} [K^1] &= \frac{4.75}{96} \times 10^7 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}, \qquad \{F^1\} = \begin{Bmatrix} P_1^1 \\ P_2^1 \end{Bmatrix}, \\ [K^2] &= \frac{1}{120} \times 10^7 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}, \qquad \{F^2\} = \begin{Bmatrix} P_1^2 \\ P_2^2 \end{Bmatrix}. \end{aligned} \tag{9.44}$$

3. *Assembly of Element Equations.* For the two-element mesh we have from Eq. (9.33b) the assembled set of equations

$$\begin{bmatrix} K_{11}^1 & K_{12}^1 & 0 \\ K_{21}^1 & K_{22}^1 + K_{11}^2 & K_{12}^2 \\ 0 & K_{21}^2 & K_{22}^2 \end{bmatrix} \begin{Bmatrix} U_1 \\ U_2 \\ U_3 \end{Bmatrix} = \begin{Bmatrix} P_1^{(1)} \\ P_2^{(1)} + P_1^{(2)} \\ P_2^{(2)} \end{Bmatrix}. \tag{9.45}$$

For the problem at hand, these have the explicit form

$$10^7 \begin{bmatrix} \frac{4.75}{96} & -\frac{4.75}{96} & 0 \\ -\frac{4.75}{96} & \frac{4.75}{96} + \frac{1}{120} & -\frac{1}{120} \\ 0 & -\frac{1}{120} & \frac{1}{120} \end{bmatrix} \begin{Bmatrix} U_1 \\ U_2 \\ U_3 \end{Bmatrix} = \begin{Bmatrix} P_1^{(1)} \\ P_2^{(1)} + P_1^{(2)} \\ P_2^{(2)} \end{Bmatrix}. \tag{9.46}$$

4. *Imposition of Boundary Conditions.* From Fig. 9.8, we have

$$u(0) \equiv U_1 = 0, \qquad P_2^{(1)} + P_1^{(2)} = 2P_0, \qquad P_2^{(2)} = P_0. \tag{9.47}$$

Using the boundary conditions, Eq. (9.46) can be written as

$$10^4 \begin{bmatrix} 49.479 & -49.479 & 0.000 \\ -49.479 & 57.812 & -8.333 \\ 0.000 & -8.333 & 8.333 \end{bmatrix} \begin{Bmatrix} 0 \\ U_2 \\ U_3 \end{Bmatrix} = \begin{Bmatrix} P_1^{(2)} \\ 2P_0 \\ P_0 \end{Bmatrix}.$$

5. *Solution of Equations.* From Eq. (9.47), we have

$$10^4 \begin{bmatrix} 57.812 & -8.333 \\ -8.333 & 8.333 \end{bmatrix} \begin{Bmatrix} U_2 \\ U_3 \end{Bmatrix} = \begin{Bmatrix} 2P_0 \\ P_0 \end{Bmatrix} = \begin{Bmatrix} 20{,}000 \\ 10{,}000 \end{Bmatrix}$$

or

$$U_2 = 0.06063 \text{ in.}, \qquad U_3 = 0.18063 \text{ in.} \tag{9.48}$$

6. *Postprocessing of Results.* The reaction force at the left end of the bar is given by [either by definition (9.41b) or from the first equation of (9.46)]:

$$P_1^{(1)} \equiv 10^4\left(49.479U_1 - 49.479U_2\right) = -30{,}000 \text{ lb.}$$

The negative sign indicates that $P_1^{(1)}$ is acting opposite to the convention used in Fig. 9.8, and therefore the reaction is acting away from the end (i.e., tensile force). The magnitude of $P_1^{(1)}$ is consistent with the static equilibrium of the forces:

$$P_1^{(1)} + 2P_0 + P_0 = 0 \qquad \text{or} \qquad P_1^{(1)} = -3P_0 = -30{,}000 \text{ lb.}$$

The axial displacement at any point x along the bar is given by

$$u_e(x) = \begin{cases} u_1^{(1)}\psi_1^{(1)} + u_2^{(1)}\psi_2^{(1)} = 0.06063x/96, & 0 \le x \le 96, \\ u_1^{(2)}\psi_1^{(2)} + u_2^{(2)}\psi_2^{(2)} = -0.03537 + 0.001x, & 96 \le x \le 216, \end{cases} \tag{9.49a}$$

and its first derivative is given by

$$\frac{du_e}{dx} = \begin{cases} 0.06063/96, & 0 \le x \le 96, \\ 0.001, & 96 \le x \le 216. \end{cases} \tag{9.49b}$$

The exact solution of Eq. (9.42a) subject to the boundary conditions

$$u(0) = 0, \qquad \left[\left(a\frac{du}{dx}\right)_{x=96+} - \left(a\frac{du}{dx}\right)_{x=96-}\right] = 2P_0, \qquad \left(a\frac{du}{dx}\right)_{x=216} = P_0,$$

is given by

$$u(x) = \begin{cases} 0.128x/(288 - x), & 0 \le x \le 96, \\ 0.001(x - 32), & 96 \le x \le 216, \end{cases} \tag{9.50a}$$

$$\frac{du}{dx} = \begin{cases} 36.864/(288 - x)^2, & 0 \le x \le 96, \\ 0.001, & 96 \le x \le 216. \end{cases} \tag{9.50b}$$

In particular, the exact solution at nodes 2 and 3 is given by

$$u(96) = 0.064 \text{ in.}, \qquad u(216) = 0.1840 \text{ in.}$$

Thus the two-element solution is about 1.8% off from the maximum displacement.

Next consider a two-element mesh of quadratic elements. The element matrix and force vector for element 1 are

$$\left[K^1\right] = 10^4 \begin{bmatrix} 142.19 & -159.37 & 17.18 \\ -159.37 & 266.67 & -107.29 \\ 17.18 & -107.29 & 90.10 \end{bmatrix}, \qquad \left\{F^1\right\} = \begin{Bmatrix} P_1^{(1)} \\ 0 \\ P_3^{(1)} \end{Bmatrix}.$$

Table 9.1 Comparison of the finite element solutions with the exact solution of the bar problem in Example 9.1

x		Linear					Quadratic	
(in.)	Exact	1 + 1[a]	2 + 1	3 + 2	4 + 2	6 + 2	1 + 1	2 + 1
16	0.00753	—	—	—	—	0.00752	—	—
24	0.01600	—	—	—	0.01161	—	—	0.01164
32	0.01600	—	—	0.01593	—	0.01598	—	—
48	0.02560	—	0.02532	—	0.02553	0.02557	0.02572	0.02560
64	0.03657	—	—	0.03638	—	0.03652	—	—
72	0.04267	—	—	—	0.04253	—	—	0.04268
80	0.04923	—	—	—	—	0.04916	—	—
96	0.06400	0.06063	0.06309	0.06359	0.06377	0.06390	0.06392	0.06399
156	0.12392	—	—	—	0.12377	0.12390	0.12392	0.12399
216	0.18400	0.18063	0.18309	0.18359	0.18377	0.18390	0.18392	0.18399

[a] $m + n$ means m elements in the interval (0, 96) and n elements in the interval (90, 216); all elements in each interval are of the same size.

The assembled stiffness matrix is of order 5 × 5, and it is of the form

$$[K] = \begin{bmatrix} K_{11}^1 & K_{12}^2 & K_{13}^1 & 0 & 0 \\ K_{21}^1 & K_{22}^1 & K_{23}^1 & 0 & 0 \\ K_{31}^1 & K_{32}^1 & K_{33}^1 + K_{11}^2 & K_{12}^2 & K_{13}^2 \\ 0 & 0 & K_{21}^2 & K_{22}^2 & K_{23}^2 \\ 0 & 0 & K_{31}^2 & K_{32}^2 & K_{33}^2 \end{bmatrix}.$$

After imposing boundary conditions ($U_1 = 0$, $P_3^1 + P_1^2 = 20{,}000$, $P_3^2 = 10{,}000$), and solving the resulting 4 × 4 equations, we obtain

$$U_2 = 0.02572\,\text{in.}, \quad U_3 = 0.06392\,\text{in.}, \quad U_4 = 0.12392\,\text{in.}, \quad U_5 = 0.18392\,\text{in.}$$

Obviously, the two-element solution obtained using the quadratic element is very accurate.

A comparison of the finite element solution obtained by various meshes of linear and quadratic elements with the exact solution is presented in Table 9.1.

9.3 FINITE ELEMENT ANALYSIS OF THE EULER–BERNOULLI BEAM THEORY

9.3.1 Governing Equation

Here we consider the finite element formulation of the fourth-order equation governing the bending of elastic beams according to the Euler–Bernoulli beam theory

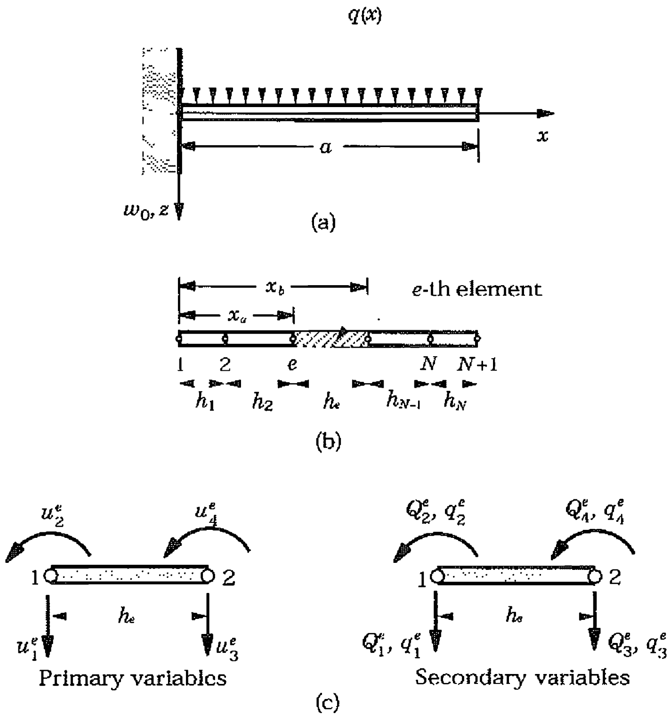


Figure 9.9 Finite element representation of a beam. (a) Geometry and loading on a straight beam. (b) Finite element mesh. (c) Beam element with primary (displacement) and secondary (force) degrees of freedom.

(see Examples 4.3 and 5.1). The equation governing the transverse deflection is given by [see Eq. (5.35)]

$$\frac{d^2}{dx^2}\left(EI\frac{d^2w_0}{dx^2}\right) - q = 0, \qquad 0 < x < L, \tag{9.51}$$

where w_0 denotes the transverse deflection, EI the bending stiffness, and q the distributed transverse load (see Fig. 9.9a). A finite element mesh of the beam is shown in Fig. 9.9b. The basic steps in the formulation are the same as those discussed for bars, but the specific mathematical details differ from those of second-order equations. Since the basic terminology of the finite element method has already been introduced in the preceding section, we go straight to the approximation over an element (see Example 5.6).

9.3.2 Weak Form over an Element

A finite element for a beam is again a free-body diagram of a portion, $\Omega^e = (x_a, x_b)$, of a typical beam, as shown in Fig. 9.9c, with nodal displacements (primary variables) and forces (secondary variables). As discussed earlier, the choice of the nodal variables is dictated by the variational formulation. From the total potential energy principle for beams, we know that the essential boundary conditions involve the specification of w_0 and dw_0/dx:

$$w_0(x_a) \equiv u_1^e, \qquad \frac{dw_0}{dx}(x_a) \equiv -u_2^e, \qquad w_0(x_b) \equiv u_3^e, \qquad \frac{dw_0}{dx}(x_b) = -u_4^e, \tag{9.52}$$

and the natural boundary conditions involve the specification of bending moments and shear forces [see Eq. (4.33b) for the moment–deflection relation and Eq. (4.30c) for the moment–shear force relation; see also Fig. 9.9c for the sign convention]:

$$\left[\frac{d}{dx}\left(EI\frac{d^2w_0}{dx^2}\right)\right]_{x=x_a} \equiv Q_1^e, \qquad \left(EI\frac{d^2w_0}{dx^2}\right)\Big|_{x=x_a} \equiv Q_2^e, \tag{9.53a}$$

$$\left[\frac{d}{dx}\left(EI\frac{d^2w_0}{dx^2}\right)\right]_{x=x_b} \equiv -Q_3^e, \qquad \left(EI\frac{d^2w_0}{dx^2}\right)\Big|_{x=x_b} \equiv -Q_4^e. \tag{9.53b}$$

The derivation of the weak form of Eq. (9.51) was discussed in Chapter 7. Following the same procedure, we obtain the following weak form over an element:

$$0 = \int_{x_a}^{x_b}\left(E_eI_e\frac{d^2v}{dx^2}\frac{d^2w_0}{dx^2} - vq_e\right)dx - Q_1^e v(x_a) - Q_2^e\left(-\frac{dv}{dx}\right)_{x_a}$$
$$- Q_3^e v(x_b) - Q_4^e\left(-\frac{dv}{dx}\right)_{x_b}, \tag{9.54a}$$

where v is the weight function. The total potential energy functional for the beam element is given by

$$\Pi^e(w_0) = \int_{x_a}^{x_b}\left[\frac{E_eI_e}{2}\left(\frac{d^2w_0}{dx^2}\right)^2 - w_0q_e\right]dx - Q_1^eu_1^e - Q_2^eu_2^e - Q_3^eu_3^e - Q_4^eu_4^e, \tag{9.54b}$$

where the sum $\sum_{i=1}^4 Q_i^e u_i^e$ represents the work done by the forces Q_1^e and Q_3^e and moments Q_2^e and Q_4^e in moving and rotating the ends of the element through displacement u_1^e and u_3^e, and slopes u_2^e and u_4^e. The weak form (9.54a) is the same as the statement of the principle of minimum total potential energy, $\delta\Pi_e = 0$, with $v = \delta w_0$.

9.3.3 Derivation of the Approximation Functions

To derive the coordinate functions for the Ritz approximation of the function w_0, we must first select a polynomial that is continuous and complete up to the degree required. It is clear from the functional Π^e that the function we select should be twice-differentiable to yield nonzero strain energy, and it should be cubic to yield nonzero shear forces Q_1^e and Q_2^e. Another reason for selecting w_0 to be a cubic polynomial is that we need four parameters in the polynomial in order to satisfy all four essential boundary conditions in Eq. (9.52). Thus, a minimum degree polynomial is cubic (see also, Example 5.6):

$$w_0(\bar{x}) \approx \alpha_0 + \alpha_1 \bar{x} + \alpha_2 \bar{x}^2 + \alpha_3 \bar{x}^3, \tag{9.55}$$

where $\bar{x}$ denotes the local coordinate. Next, select the parameters α_0, α_1, α_2, and α_3 such that w_0 satisfies the essential boundary conditions, Eq. (9.52):

$$\begin{aligned}
u_1^e &\equiv w_0(0) = \alpha_0,\\
u_2^e &\equiv -\frac{dw_0}{dx}(0) = -\alpha_1,\\
u_3^e &\equiv w_0(h_e) = \alpha_0 + \alpha_1 h_e + \alpha_2 h_e^2 + \alpha_3 h_e^3,\\
u_4^e &\equiv -\frac{dw_0}{dx}(h_e) = -\alpha_1 - 2\alpha_2 h_e - 3\alpha_3 h_e^2,
\end{aligned} \tag{9.56}$$

where $h_e = x_b - x_a$. Solving Eq. (9.56) for α_0, α_1, α_2, and α_3 in terms of $u_i^e (i = 1, 2, 3, 4)$, and substituting the result into Eq. (9.55), we obtain

$$w_0(x) \approx \sum_{j=1}^{4} u_j^e \phi_j^e(x), \tag{9.57}$$

where the approximation functions $\phi_j^e(\bar{x})$, $\bar{x} = x - x_e$, for the beam element are given by

$$\begin{aligned}
\phi_1^e(\bar{x}) &= 1 - 3\left(\frac{\bar{x}}{h_e}\right)^2 + 2\left(\frac{\bar{x}}{h_e}\right)^3,\\
\phi_2^e(\bar{x}) &= -\bar{x}\left[1 - \left(\frac{\bar{x}}{h_e}\right)\right]^2,\\
\phi_3^e(\bar{x}) &= 3\left(\frac{\bar{x}}{h_e}\right)^2 - 2\left(\frac{\bar{x}}{h_e}\right)^3,\\
\phi_4^e(\bar{x}) &= -\bar{x}\left[\left(\frac{\bar{x}}{h_e}\right)^2 - \left(\frac{\bar{x}}{h_e}\right)\right].
\end{aligned} \tag{9.58}$$

The approximate functions ϕ_i^e are called the *Hermite cubic interpolation functions*. We note the following properties of ϕ_i^e, $i = 1, 2, 3, 4$ (see Fig. 9.10):

$$\begin{aligned}
&\phi_{2\alpha-1}^e(\bar{x}_\beta) = \delta_{\alpha\beta}, \qquad \phi_{2\alpha}^e(\bar{x}_\beta) = 0, \qquad \sum_{\alpha=1}^{2} \phi_{2\alpha-1}^e = 0,\\
&\frac{d\phi_{2\alpha-1}^e}{dx}(\bar{x}_\beta) = 0, \qquad -\frac{d\phi_{2\alpha}^e}{dx}(\bar{x}_\beta) = \delta_{\alpha\beta}, \qquad (\alpha, \beta = 1, 2),
\end{aligned} \tag{9.59}$$

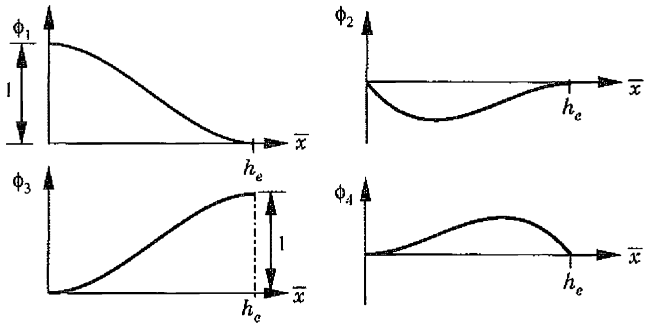


Figure 9.10 Hermite cubic interpolation functions for the beam element.

where $\bar{x}_1 = 0$ and $\bar{x}_2 = h_e$ are the local coordinates of nodes 1 and 2 of element $\Omega^e = (x_e, x_{e+1})$, and x_e is the global coordinate of global node e.

It should be noted that by including the slopes u_2^e and u_4^e as nodal quantities, we will be able to enforce the slope continuity at the interelement boundaries during the assembly process. Of course, the total potential energy formulation of the governing equation yields slopes as part of the essential boundary conditions. All dependent variables and their derivatives that enter the specification of the essential boundary conditions of a problem always end up as the nodal variables, and they take the role of the undetermined coefficients of the Ritz method.

9.3.4 Finite Element Model

To derive the Ritz equations corresponding to the functional in Eq. (9.54), we substitute Eq. (9.57) for w_e into Π^e, and set its derivatives with respect to each u_j^e, $j = 1, 2, 3, 4$, to zero:

$$\begin{aligned}
\Pi^e(u_1^e, u_2^e, u_3^e, u_4^e) &= \int_{x_a=x_e}^{x_b=x_{e+1}} \left[\frac{E_e I_e}{2} \left(\sum_{j=1}^{4} u_j^e \frac{d^2\phi_j^e}{dx^2} \right)^2 - \left(\sum_{i=1}^{4} u_i^e \phi_i^e \right) q \right] dx \\
&\quad - \sum_{i=1}^{4} Q_i^e u_i^e \\
&= \frac{1}{2} \sum_{i=1}^{4} \sum_{j=1}^{4} u_i^e \left(\int_{x_e}^{x_{e+1}} E_e I_e \frac{d^2\phi_i^e}{dx^2} \frac{d^2\phi_j^e}{dx^2} dx \right) u_j^e \\
&\quad - \sum_{i=1}^{4} u_i^e \left(\int_{x_e}^{x_{e+1}} q_e \phi_i^e \, dx + Q_i^e \right) \\
&= \frac{1}{2} \sum_{i=1}^{4} \left(\sum_{j=1}^{4} u_i^e K_{ij}^e u_j^e - 2u_i^e F_i^e \right),
\end{aligned} \tag{9.60a}$$

$$\frac{\partial \Pi^e}{\partial u_i^e} = 0 = \sum_{j=1}^{4} K_{ij}^e u_j^e - F_i^e, \tag{9.60b}$$

where

$$K_{ij}^e = \int_{x_e}^{x_{e+1}} E_e I_e \frac{d^2\phi_i^e}{dx^2}\frac{d^2\phi_j^e}{dx^2}\, dx, \qquad F_i^e = \int_{x_e}^{x_{e+1}} q_e \phi_i^e \, dx + Q_i^e. \tag{9.60c}$$

In Eq. (9.60c) it is understood that ϕ_i^e are expressed in terms of the global coordinate x. In the element coordinate $\bar{x}$, Eq. (9.60c) takes the simple form,

$$K_{ij}^e = \int_0^{h_e} \bar{E}_e \bar{I}_e \frac{d^2\phi_i^e}{d\bar{x}^2}\frac{d^2\phi_j^e}{d\bar{x}^2}\, d\bar{x}, \qquad F_i^e = \int_0^{h_e} \bar{q}_e \phi_i^e \, d\bar{x} + Q_i^e, \tag{9.60d}$$

where $\bar{E}_e$, $\bar{I}_e$, and $\bar{q}_e$ are the transformed functions,

$$\bar{E}_e = E_e(\bar{x}), \quad \bar{I}_e = I_e(\bar{x}), \quad \bar{q}_e = q(\bar{x}).$$

For the case in which E_e, I_e, and q_e are constant over the element Ω^e, the stiffness matrix $[K^e]$ and force vector $\{F^e\}$ are given by

$$[K^e] = \frac{2E_e I_e}{h_e^3}\begin{bmatrix} 6 & -3h_e & -6 & -3h_e \\ -3h_e & 2h_e^2 & 3h_e & h_e^2 \\ -6 & 3h_e & 6 & 3h_e \\ -3h_e & h_e^2 & 3h_e & 2h_e^2 \end{bmatrix}; \qquad \{F^e\} = \frac{q_0 h_e}{12}\begin{Bmatrix} 6 \\ -h_e \\ 6 \\ h_e \end{Bmatrix} + \begin{Bmatrix} Q_1^e \\ Q_2^e \\ Q_3^e \\ Q_4^e \end{Bmatrix}. \tag{9.61}$$

where the label e on the variables is omitted in the interest of brevity. One can show that the first column of the force vector represents the "statically equivalent" forces and moments at the nodes due to uniformly distributed load q_0 over the element. This element gives, for element-wise constant values of EI, the exact values of w_0 and dw_0/dx at the nodes for any load $q(x)$, and thus it is called a *superconvergent element*.

9.3.5 Assembly of Element Equations

The assembly procedure for the beam element is analogous to that described for the bar element. The difference is that in the beam element there are two unknowns, called *degrees of freedom*, per node. Consequently, at every global node that is shared by two elements, the elements in the last two rows and columns of Ith element overlap with the elements in the first two rows and columns of $(I + 1)$th element. For a

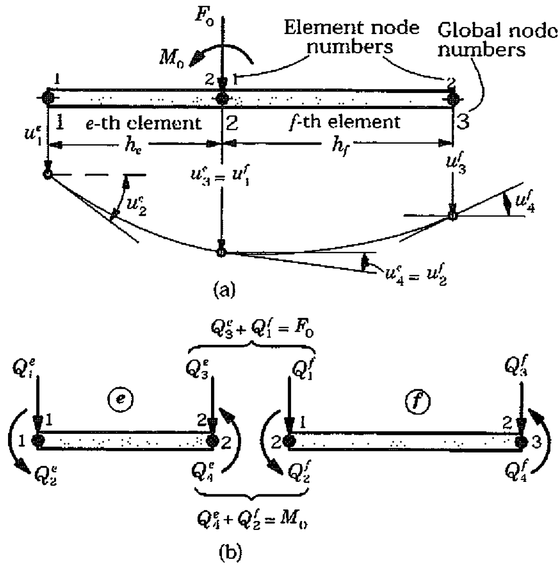


Figure 9.11 Assembly of two beam finite elements. (a) Continuity of generalized displacements. (b) Balance of generalized displacements.

three-element case, the assembled force vector and stiffness matrix are shown below (see Fig. 9.11):

$$\{F\} = \begin{Bmatrix} q_1^{(1)} \\ q_2^{(1)} \\ q_3^{(1)} + q_1^{(2)} \\ q_4^{(1)} + q_2^{(2)} \\ q_3^{(2)} + q_1^{(3)} \\ q_4^{(2)} + q_2^{(3)} \\ q_3^{(3)} \\ q_4^{(3)} \end{Bmatrix} + \begin{Bmatrix} Q_1^{(1)} \\ Q_2^{(1)} \\ Q_3^{(1)} + Q_1^{(2)} \\ Q_4^{(1)} + Q_2^{(2)} \\ Q_3^{(2)} + Q_1^{(3)} \\ Q_4^{(2)} + Q_2^{(3)} \\ Q_3^{(3)} \\ Q_4^{(3)} \end{Bmatrix}, \tag{9.62a}$$

$$[K] = \begin{bmatrix} K_{11}^1 & K_{12}^1 & K_{13}^1 & K_{14}^1 & 0 & 0 & 0 & 0 \\ K_{21}^1 & K_{22}^1 & K_{23}^1 & K_{24}^1 & 0 & 0 & 0 & 0 \\ K_{31}^1 & K_{32}^1 & K_{33}^1 + K_{11}^2 & K_{34}^1 + K_{12}^2 & K_{13}^2 & K_{14}^2 & 0 & 0 \\ K_{41}^1 & K_{42}^1 & K_{43}^1 + K_{21}^2 & K_{44}^1 + K_{22}^2 & K_{23}^2 & K_{24}^2 & 0 & 0 \\ 0 & 0 & K_{31}^2 & K_{32}^2 & K_{33}^2 + K_{11}^3 & K_{34}^2 + K_{12}^3 & K_{13}^3 & K_{14}^3 \\ 0 & 0 & K_{41}^2 & K_{42}^2 & K_{43}^2 + K_{21}^3 & K_{44}^2 + K_{22}^3 & K_{23}^3 & K_{24}^3 \\ 0 & 0 & 0 & 0 & K_{31}^3 & K_{32}^3 & K_{33}^3 & K_{34}^3 \\ 0 & 0 & 0 & 0 & K_{41}^3 & K_{42}^3 & K_{43}^3 & K_{44}^3 \end{bmatrix}. \tag{9.62b}$$

9.3.6 Imposition of Boundary Conditions

The boundary conditions on the generalized displacements can be imposed in the manner described earlier for bars. The boundary (or equilibrium) conditions on forces are imposed by modifying the second column of the assembled force vector [see Eq. (9.62a)]. If, for example, the force at the Ith global node is specified to be F_0, and the moment at Kth global node is specified to be M_0, then the following assembled coefficients get modified:

$$Q_3^{I-1} + Q_1^I = F_0, \qquad Q_4^{K-1} + Q_2^K = M_0. \tag{9.63}$$

If no force or moment is specified at a nodal point that is unconstrained, it is understood that the force and moment are specified to be zero there.

The solution and postcomputation of the displacements and their derivatives at various points of the beam follow along the same lines as that described for bar problems. We now consider a specific example to illustrate the steps involved in the finite element analysis of beams.

Example 9.2 Consider the indeterminate beam shown in Fig. 9.12. The beam is made of steel ($E = 30 \times 10^6$ psi) and the cross-sectional dimensions are 2×3 in. ($I = 4.5$ in.^{4}). We are interested in finding the deflection w_0 and its derivatives using the Euler–Bernoulli beam finite element.

Because of the discontinuity in loading, the beam should be divided into three elements: $\Omega^1 = (0, 16)$, $\Omega^2 = (16, 36)$, and $\Omega^3 = (36, 48)$; the element lengths are; $h_1 = 16$ in., $h_2 = 20$ in., and $h_3 = 12$ in.

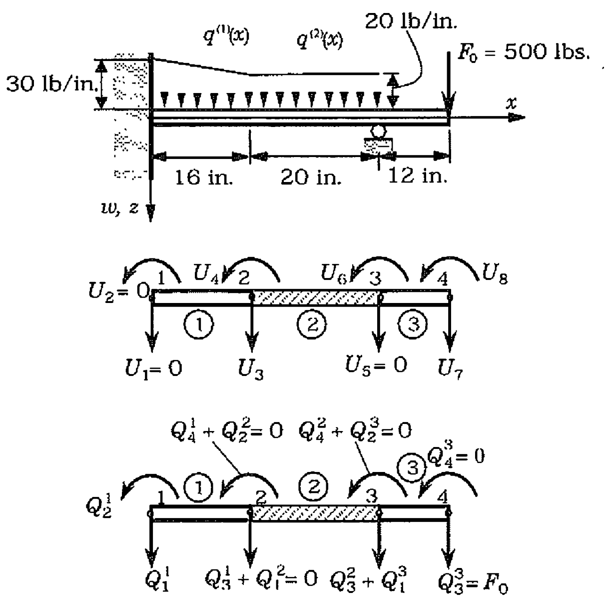


Figure 9.12 Finite element modeling of an indeterminate beam.

The element equations for all three elements are readily available from Eq. (9.61). The only exception is that the load vector for element 1, which has linearly varying distributed load, must be computed. The load variation is given by

$$q^{(1)}(x) = \left(30 - \frac{10}{16}x\right), \qquad q^{(2)}(x) = 20, \qquad q^{(3)}(x) = 0,$$

and their contribution to the element nodes can be computed using Eq. (9.60d). For element 1, we have

$$q_i^1 = \int_0^{16} \left(30 - \frac{10}{16}x\right) \phi_i^1(x)\, dx \qquad (i = 1, 2, 3, 4).$$

We have

$$\{q^1\} = \begin{Bmatrix} 216.00 \\ -554.67 \\ 184.00 \\ 512.00 \end{Bmatrix}, \qquad \{q^2\} = \begin{Bmatrix} 200.00 \\ -666.77 \\ 200.00 \\ 666.67 \end{Bmatrix}, \qquad \{q^3\} = \{0\}. \tag{9.64}$$

The element stiffness matrices can be computed from Eq. (9.61) by substituting appropriate values of h, E, and I. For example, for element 1 we have ($h_1 = 16$ in., $EI = 135 \times 10^6$ lb.-in.^{2}):

$$[K^1] = 10^6 \begin{bmatrix} 0.3955 & -3.1641 & -0.3955 & -3.1641 \\ -3.1641 & 33.7500 & 3.1641 & 16.8750 \\ -0.3955 & 3.1641 & 0.3955 & 3.1641 \\ -3.1641 & 16.8750 & 3.1641 & 33.750 \end{bmatrix}.$$

The assembled matrix for this mesh is of the same form as given in Eq. (9.62b). The boundary conditions for the problem are

$$w_0(0) = 0 \rightarrow U_1 = 0, \qquad \frac{dw_0}{dx}(0) = 0 \rightarrow U_2 = 0, \qquad w_0(36) = 0 \rightarrow U_5 = 0,$$

$$Q_3^1 + Q_1^2 = 0, \quad Q_4^1 + Q_2^2 = 0, \quad Q_4^2 + Q_2^3 = 0, \quad Q_3^3 = 500, \quad Q_4^3 = 0. \tag{9.65}$$

Note that Q_1^1, Q_2^1, and $Q_3^2 + Q_1^3$ are the reactions that are not known, and are to be calculated in the postcomputation. Since the specified essential boundary conditions are homogeneous, one can delete the rows and columns corresponding to the specified displacements (i.e., delete rows and columns 1, 2, and 5) and solve the remaining five equations for U_3, U_4, U_6, U_7, and U_8:

$$U_3 = -0.000322 \text{ in.}, \qquad U_4 = 0.0000593 \text{ rad}, \qquad U_6 = -0.0002513 \text{ rad},$$

$$U_7 = 0.00515 \text{ in.}, \qquad U_8 = -0.000518 \text{ rad}. \tag{9.66}$$

As stated before, these nodal values are exact for any number of elements.

The exact solution to the problem can be obtained by integrating the moment–deflection expression $(d^2 w_0/dx^2) = -M/EI$ [see Eq. (4.33b)], in which the bending moment can be readily obtained in terms of the reactions at the supports and applied loads. The deflection in the three intervals of the beam is given by (large numbers were rounded to whole numbers):

$$w_0(x) = \begin{cases} \dfrac{1}{EI}\left(268.543x^2 - \dfrac{691}{15}x^3 + \dfrac{5}{4}x^4 - \dfrac{1}{192}x^5\right), & 0 \le x \le 16, \\ \dfrac{1}{EI}\left(-\dfrac{16384}{3} + \dfrac{5120}{3}x + 55.2097x^2 + \dfrac{491}{15}x^3 + \dfrac{5}{6}x^4\right), & 16 \le x \le 36, \\ \dfrac{1}{EI}\left(6554368 - 506066x + 12000x^2 - \dfrac{250}{3}x^3\right), & 36 \le x \le 48. \end{cases}$$

It can be verified that the nodal values in Eq. (9.66) are exact.

A comparison of the finite element results for deflections, slopes, and bending moments (calculated in the postcomputation) *at points other than the nodes* are compared with the exact values in Table 9.2 for three different meshes. The values of deflections, slopes, and bending moments were computed using the finite element solution (9.57) and its derivatives.

9.4 FINITE ELEMENT MODELS OF THE TIMOSHENKO BEAM THEORY

9.4.1 Governing Equations

The displacement field of the Timoshenko beam theory (see Problems 4.26, 5.4, and 6.15, and Examples 6.4 and 7.10; see also Reddy [5–7]), for the pure bending case, is

$$u(x, z) = z\phi(x), \qquad v = 0, \qquad w(x, z) = w_0(x), \tag{9.67}$$

where w_0 is the transverse deflection and ϕ the rotation of a transverse normal line about the y-axis. The strains and stresses of the Timoshenko beam theory are

$$\varepsilon_{xx} = z\frac{d\phi}{dx} \equiv z\kappa_{xx}, \qquad \gamma_{xz} = \phi + \frac{dw_0}{dx}, \qquad \sigma_{xx} = E\varepsilon_{xx}, \qquad \sigma_{xz} = G\gamma_{xz}. \tag{9.68}$$

The balance of internal moments and transverse forces give the relations

$$M(x) = \int_A z\sigma_{xx}\, dA = EI\frac{d\phi}{dx}, \qquad Q(x) = K_s \int_A \sigma_{xz}\, dA = GAK_s\left(\phi + \frac{dw_0}{dx}\right), \tag{9.69}$$

and the equilibrium of moments and transverse forces give

$$Q - \frac{dM}{dx} = 0, \qquad \frac{dQ}{dx} = -q(x). \tag{9.70}$$

Table 9.2 Comparison of the finite element solution with the exact solution of the beam problem considered in Example 9.2

		$w_0 \times 10^6$		$(-dw_0/dx) \times 10^6$		$-(d^2w_0/dx^2) \times 10^6$	
x	N[a]	FEM	Exact	FEM	Exact	FEM	Exact
2.0	3	−0.8570		1.1553		−1.0250	
	5	4.1405	5.3739	−3.3386	−4.1552	0.4663	0.3219
	6	5.2319		−4.1558		0.4638	
6.0	3	−18.4510		8.8348		−2.8147	
	5	8.3936	9.6048	4.4199	5.2328	−4.3456	−4.4727
	6	9.4751		5.2323		−4.3431	
12.0	3	−138.2300		33.7760		−5.4992	
	5	−122.5900	−120.8100	39.6630	39.6720	−5.4794	−5.9238
	6	−122.5900		39.6630		−5.4794	
21.0	3	−674.3600		71.5620		3.5507	
	5	−643.4900	−639.6300	62.3030	62.3020	3.5507	2.9335
	6	−643.4900		62.3030		3.5507	
31.0	3	−812.9300		−83.7970		27.5210	
	5	−782.0700	−778.1900	−74.5370	−74.5390	27.5210	26.9040
	6	−782.0700		−74.5370		27.5210	
42.0	3	2174.9000		−451.3600		22.2220	
	5	2174.9000	2174.8000	−451.3600	−451.3600	22.2220	22.2220
	6	2174.9000		−451.3600		22.2220	

[a] 3-Elements $h_1 = 16$, $h_2 = 20$, $h_3 = 12$. 5-Elements: $h_1 = 8$, $h_2 = 8$, $h_3 = 10$, $h_4 = 10$, $h_5 = 12$. 6-Elements: $h_1 = 4$, $h_2 = 4$, $h_3 = 8$, $h_4 = 10$, $h_5 = 10$, $h_6 = 12$.

Here, $q(x)$ is the distributed transverse load, E Young's modulus, G the shear modulus, A the area of cross section, I the moment of inertia, and K_s the shear correction factor. Using (9.69) in (9.70), we obtain the following equilibrium equations:

$$-\frac{d}{dx}\left(EI\frac{d\phi}{dx}\right) + GAK_s\left(\phi + \frac{dw_0}{dx}\right) = 0, \tag{9.71}$$

$$-\frac{d}{dx}\left[GAK_s\left(\phi + \frac{dw_0}{dx}\right)\right] = q(x). \tag{9.72}$$

In the following, a number of displacement finite element models of these equations are presented for element-wise constant properties, $E = E_e$, $G = G_e$, $A = A_e$, and $I = I_e$.

9.4.2 Displacement Finite Element Models

The displacement finite element model of the Timoshenko beam theory is constructed using the principle of minimum total potential energy, or equivalently, using the weak form,

$$0 = \int_{x_a}^{x_b} \left[E_e I_e \frac{d\delta\phi}{dx}\frac{d\phi}{dx} + GAK_s\left(\delta\phi + \frac{d\delta w_0}{dx}\right)\left(\phi + \frac{dw_0}{dx}\right) - q(x)\delta w_0 \right] dx$$
$$- V_a^e \delta w_0(x_a) - V_b^e \delta w_0(x_b) - M_a^e \delta\phi(x_a) - M_b^e \delta\phi(x_b) \tag{9.73}$$

and

$$\begin{aligned} V_a^e &\equiv -Q(x_a) = -\left[G_e A_e K_s \left(\frac{dw_0}{dx} + \phi\right)\right]_{x=x_a}, \\ M_a^e &\equiv -M(x_a) = -\left[E_e I_e \frac{d\phi}{dx}\right]_{x=x_a}, \\ V_b^e &\equiv Q(x_b) = \left[G_e A_e K_s \left(\frac{dw_0}{dx} + \phi\right)\right]_{x=x_b}, \\ M_b^e &\equiv M(x_b) = \left[E_e I_e \frac{d\phi}{dx}\right]_{x=x_b}. \end{aligned} \tag{9.74}$$

Suppose that w_0 and ϕ are approximated as

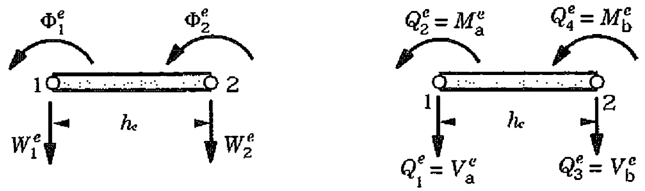
$$w_0(x) \approx \sum_{j=1}^{m} \psi_j^e W_j^e, \qquad \phi(x) \approx \sum_{j=1}^{n} \varphi_j^e \Phi_j^e, \tag{9.75}$$

where (W_j^e, Φ_j^e) are the nodal values of (w_0, ϕ) and (ψ_j^e, φ_j^e) are the associated interpolation functions. Substitution of (9.75) for w_0 and ϕ, and $\delta w_0 = \psi_i^e$ and $\delta\phi = \varphi_i^e$ into Eq. (9.73), yields the finite element model

$$\begin{bmatrix} [K^{11}] & [K^{12}] \\ [K^{12}]^{\mathrm{T}} & [K^{22}] \end{bmatrix} \begin{Bmatrix} \{W\} \\ \{\Phi\} \end{Bmatrix} = \begin{Bmatrix} \{F^1\} \\ \{F^2\} \end{Bmatrix}, \tag{9.76}$$

where

$$\begin{aligned} K_{ij}^{11} &= \int_{x_a}^{x_b} K_s G_e A_e \frac{d\psi_i^e}{dx}\frac{d\psi_j^e}{dx}\, dx, \\ K_{ij}^{12} &= \int_{x_a}^{x_b} K_s G_e A_e \frac{d\psi_i^e}{dx}\varphi_j^e\, dx, \\ K_{ij}^{22} &= \int_{x_a}^{x_b} \left(E_e I_e \frac{d\varphi_i^e}{dx}\frac{d\varphi_j^e}{dx} + K_s G_e A_e \varphi_i^e \varphi_j^e\right) dx, \\ F_i^1 &= \int_{x_a}^{x_b} \psi_i^e q\, dx + V_a^e \psi_i^e(x_a) + V_b^e \psi_i^e(x_b), \\ F_i^2 &= M_a^e \varphi_i^e(x_a) + M_b^e \varphi_i^e(x_b). \end{aligned} \tag{9.77}$$



(a) Generalized displacements (b) Generalized forces

Figure 9.13 Displacement finite element for the Timoshenko beam theory.

9.4.3 Reduced Integration Element (RIE)

For a linear interpolation of w_0 and ϕ (i.e., $m = n = 2$; see Fig. 9.13) and exact evaluation of the integrals of Eq. (9.77), Eq. (9.76) takes the form

$$\frac{G_e A_e K_s}{6h_e}\begin{bmatrix} 6 & -6 & -3h_e & -3h_e \\ -6 & 6 & 3h_e & 3h_e \\ -3h_e & 3h_e & 2h_e^2\lambda & h_e^2\xi \\ -3h_e & 3h_e & h_e^2\xi & 2h_e^2\lambda \end{bmatrix}\begin{Bmatrix} W_1^e \\ W_2^e \\ \Phi_1^e \\ \Phi_2^e \end{Bmatrix} = \begin{Bmatrix} q_1^e \\ q_2^e \\ 0 \\ 0 \end{Bmatrix} + \begin{Bmatrix} V_a^e \\ V_b^e \\ M_a^e \\ M_b^e \end{Bmatrix}, \tag{9.78}$$

or, rearranging the equations, we obtain

$$\left(\frac{2E_e I_e}{\mu_0 h_e^3}\right)\begin{bmatrix} 6 & -3h_e & -6 & -3h_e \\ -3h_e & 2h_e^2\lambda & 3h_e & h_e^2\xi \\ -6 & 3h_e & 6 & 3h_e \\ -3h_e & h_e^2\xi & 3h_e & 2h_e^2\lambda \end{bmatrix}\begin{Bmatrix} W_1^e \\ \Phi_1^e \\ W_2^e \\ \Phi_2^e \end{Bmatrix} = \begin{Bmatrix} q_1^e \\ 0 \\ q_2^e \\ 0 \end{Bmatrix} + \begin{Bmatrix} Q_1^e \\ Q_2^e \\ Q_3^e \\ Q_4^e \end{Bmatrix}, \tag{9.79}$$

where $Q_1^e = V_a^e$, $Q_2^e = M_a^e$, $Q_3^e = V_b^e$, $Q_4^e = M_b^e$, and

$$q_i^e = \int_{x_a}^{x_b} \psi_i^e q \, dx \qquad (i = 1, 2), \tag{9.80}$$

$$\Lambda = \frac{E_e I_e}{G_e A_e K_s h_e^2}, \qquad \mu_0 = 12\Lambda, \qquad \xi = 1 - 6\Lambda, \qquad \lambda = 1 + 3\Lambda. \tag{9.81}$$

In the thin beam limit, the effect of shear deformation must vanish, i.e., $\Lambda \to 0$. Then the first two equations of (9.79) imply the following relation among $(W_1^e, W_2^e, \Phi_1^e, \Phi_2^e)$:

$$\left(\frac{\Phi_1^e + \Phi_2^e}{2}\right) + \left(\frac{W_2^e - W_1^e}{h_e}\right) = 0, \tag{9.82}$$

which is equivalent to the (Kirchhoff) constraint $\phi + (dw_0/dx) = 0$ (or shear strain $\gamma_{xz} = 0$). The last two equations of (9.79), in view of (9.82), yield the constraint

$$\frac{\Phi_1^e - \Phi_2^e}{h_e} = 0. \tag{9.83}$$

This is equivalent to $d\phi/dx = 0$, which is an incorrect condition to be satisfied as it forces the curvature and hence the bending energy to zero. Thus, the finite element equations in (9.79), in an effort to satisfy the constraints (9.82) and (9.83), will yield the trivial solution $W_1^e = W_2^e = \Phi_1^e = \Phi_2^e = 0$. This is known in the finite element literature as *shear locking.*

The Kirchhoff condition (9.82) suggests that w_0 and ϕ be interpolated such that dw_0/dx is a polynomial of the same order as ϕ. If w_0 is approximated using a linear polynomial (a minimum requirement), then ϕ should be a constant. Since the minimum continuity requirement on ϕ is also linear, it follows that w_0 be approximated using a quadratic polynomial. This is termed a consistent interpolation. Alternatively, both w_0 and ϕ appearing in the bending energy may be approximated with linear polynomials while ϕ appearing in the shear energy is treated as a constant. This can be achieved by using numerical integration with one Gauss point to evaluate the shear stiffness. This procedure is known in the literature as the *reduced integration* of the shear stiffness. It amounts to evaluating the second term of K_{ij}^{22} in Eq. (9.77) with one-point integration as opposed to two-point integration required to exactly evaluate the integral. This results in the element equations

$$\left(\frac{2EI}{\mu_0 h_e^3}\right)\begin{bmatrix} 6 & -3h_e & -6 & -3h_e \\ -3h_e & h^2(1.5+6\Lambda) & 3h_e & h_e^2(1.5-6\Lambda) \\ -6 & 3h_e & 6 & 3h_e \\ -3h_e & h_e^2(1.5-6\Lambda) & 3h_e & h_e^2(1.5+6\Lambda) \end{bmatrix}\begin{Bmatrix} W_1^e \\ \Phi_1^e \\ W_2^e \\ \Phi_2^e \end{Bmatrix} = \begin{Bmatrix} q_1^e \\ 0 \\ q_2^e \\ 0 \end{Bmatrix} + \begin{Bmatrix} Q_1^e \\ Q_2^e \\ Q_3^e \\ Q_4^e \end{Bmatrix}. \tag{9.84}$$

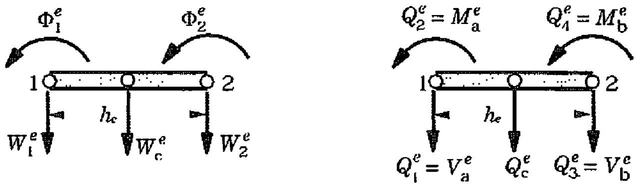
This element is designated as the reduced integration element (RIE). In the thin (or slender) beam limit, these element equations reduce to only one constraint, namely, the Kirchhoff condition in Eq. (9.82). While the element does not lock, it does not yield exact displacements at the nodes. However, with a sufficient number of elements in the mesh of a problem, one does get a very accurate solution.

9.4.4 Consistent Interpolation Element (CIE)

As suggested earlier, if we use a quadratic approximation of w_0 and linear approximation of Φ, Eq. (9.76) yields the following 5×5 system of equations:

$$\frac{GAK_s}{6h_e}\begin{bmatrix} 14 & -16 & 2 & -5h_e & -h_e \\ -16 & 32 & -16 & 4h_e & -4h_e \\ 2 & -16 & 14 & h_e & 5h_e \\ -5h_e & 4h_e & h_e & 2h_e^2\lambda & h_e^2\xi \\ -h_e & -4h_e & 5h_e & h_e^2\xi & 2h_e^2\lambda \end{bmatrix}\begin{Bmatrix} W_1^e \\ W_c^e \\ W_2^e \\ \Phi_1^e \\ \Phi_2^e \end{Bmatrix} = \begin{Bmatrix} q_1^e \\ q_c^e \\ q_2^e \\ 0 \\ 0 \end{Bmatrix} + \begin{Bmatrix} V_a^e \\ \hat{Q}_c^e \\ V_b^e \\ M_a^e \\ M_b^e \end{Bmatrix}, \tag{9.85}$$

where $(V_a^e, V_b^e, M_a^e, M_b^e)$ are the generalized forces defined in Eq. (9.74), W_c^e and $\hat{Q}_c^e$ are the deflection and applied external load, respectively, at the center node of the



(a) Generalized displacements (b) Generalized forces

Figure 9.14 Consistent finite element model of the Timoshenko beam theory.

quadratic element, and

$$q_i^e = \int_{x_a}^{x_b} \psi_i^e q \, dx \qquad (i = 1, 2, \mathrm{c}). \tag{9.86}$$

This element is designated as the consistent interpolation element (CIE) (see Fig. 9.14).

The center degree of freedom W_c^e may be condensed out to reduce the system (9.85) to a 4×4 system of equations. The second equation of (9.85) can be used to express W_c^e in terms of W_1^e, W_2^e, Φ_1^e, Φ_2^e, q_c^e, and $\hat{Q}_c^e$:

$$W_c^e = \frac{6h_e}{32G_eA_eK_s}(q_c^e + \hat{Q}_c^e) + \left(\frac{W_1^e + W_2^e}{2}\right) + h_e\left(\frac{\Phi_2^e - \Phi_1^e}{8}\right). \tag{9.87}$$

Substituting for W_c^e from Eq. (9.87) into the remaining equations of (9.85) [i.e., eliminating W_c^e from Eq. (9.85)], we obtain

$$\left(\frac{2E_eI_e}{\mu_0 h_e^3}\right)\begin{bmatrix} 6 & -3h_e & -6 & -3h_e \\ -3h_e & h_e^2(1.5+6\Lambda) & 3h_e & h_e^2(1.5-6\Lambda) \\ -6 & 3h_e & 6 & 3h_e \\ -3h_e & h_e^2(1.5-6\Lambda) & 3h_e & h_e^2(1.5+6\Lambda) \end{bmatrix}\begin{Bmatrix} W_1^e \\ \Phi_1^e \\ W_2^e \\ \Phi_2^e \end{Bmatrix}$$

$$= \begin{Bmatrix} q_1^e + \frac{1}{2}\hat{q}_c^e \\ -\frac{1}{8}\hat{q}_c^e h_e \\ q_2^e + \frac{1}{2}\hat{q}_c^e \\ \frac{1}{8}\hat{q}_c^e h_e \end{Bmatrix} + \begin{Bmatrix} Q_1^e \\ Q_2^e \\ Q_3^e \\ Q_4^e \end{Bmatrix}, \tag{9.88}$$

where $\hat{q}_c^e = q_c^e + \hat{Q}_c^e$. For simplicity, but without the loss of generality, we will assume that $\hat{Q}_c^e = 0$ so that $\hat{q}_c^e = q_c^e$. Note that the element has the same stiffness matrix as the reduced integration element but a different load vector. The load vector is equivalent to that of the Euler–Bernoulli beam element [see Eq. (9.61)]. In fact, for uniform load q, the load vector in Eq. (9.88) is identical to that of the Euler–Bernoulli beam element. Note that *the elimination of W_c is not possible for the dynamic case* because of the presence of $\ddot{W}_c$.

9.4.5 Superconvergent Element (SCE)

The next choice of consistent interpolation is to use the Lagrange cubic polynomials for w_0 and quadratic polynomials for ϕ. This will lead to a 7×7 system of equations. The displacement degrees of freedom associated with the interior nodes can again be condensed out, for the static case, to obtain a 4×4 system of equations. Here we will not consider this approach. Instead, we consider the Hermite cubic interpolation of w_0 and a related quadratic approximation of ϕ (see Reddy [5,7]). The 4×4 system of equations thus obtained yield exact values of the nodal displacements for the Timoshenko beam theory, as in the case of Euler–Bernoulli beam element. Such elements are called *superconvergent elements*.

The exact solution of Eqs. (9.71) and (9.72) for the homogeneous case (analogous to what was done in Example 5.3 in connection with the Euler–Bernoulli beam theory) is

$$w_0(x) = -\frac{1}{EI}\left(a_1\frac{x^3}{6} + a_2\frac{x^2}{2} + a_3 x + a_4\right) + \frac{1}{GAK_s}(a_1 x), \tag{9.89}$$

$$EI\phi(x) = a_1\frac{x^2}{2} + a_2 x + a_3, \tag{9.90}$$

where a_1 through a_4 are the constants of integration. Note that the constants a_1, a_2, and a_3 appearing in Eq. (9.90) are the same as those in Eq. (9.89). Thus, Eqs. (9.89) and (9.90) suggest that one may use cubic approximation of w_0 and an *interdependent* quadratic approximation of ϕ. The resulting finite element is termed the interdependent interpolation element (IIE).

The constants a_i appearing in Eqs. (9.89) and (9.90) can be expressed in terms of the nodal values of w_0 and ϕ (see Example 5.5). Then Eqs. (9.89) and (9.90) take the form

$$w_0(x) \approx \sum_{j=1}^{m} \psi_j^e \Delta_j^e, \qquad \phi(x) \approx \sum_{j=1}^{n} \varphi_j^e \Delta_j^e, \tag{9.91}$$

$$\Delta_1^e = W_1^e, \qquad \Delta_2^e = \Phi_1^e, \qquad \Delta_3^e = W_2^e, \qquad \Delta_4^e = \Phi_2^e, \tag{9.92}$$

where ψ_i^e and φ_i^e are the approximation functions

$$\begin{aligned}
\psi_1^e &= \frac{1}{\mu}\left[\mu - 12\Lambda\eta - (3 - 2\eta)\eta^2\right],\\
\psi_2^e &= -\frac{h_e}{\mu}\left[(1-\eta)^2\eta + 6\Lambda(1-\eta)\eta\right],\\
\psi_3^e &= \frac{1}{\mu}\left[(3-2\eta)\eta^2 + 12\Lambda\eta\right],\\
\psi_4^e &= \frac{h_e}{\mu}\left[(1-\eta)\eta^2 + 6\Lambda(1-\eta)\eta\right],
\end{aligned} \tag{9.93}$$

$$
\begin{aligned}
\varphi_1^e &= \frac{6}{h_e\mu}(1-\eta)\eta, \\
\varphi_2^e &= \frac{1}{\mu}(\mu - 4\eta + 3\eta^2 - 12\Lambda\eta), \\
\varphi_3^e &= -\frac{6}{h_e\mu}(1-\eta)\eta, \\
\varphi_4^e &= \frac{1}{\mu}(3\eta^2 - 2\eta + 12\Lambda\eta).
\end{aligned} \tag{9.94}
$$

Here, η is the nondimensional local coordinate

$$
\eta = \frac{x - x_a}{h_e}, \qquad \mu = 1 + 12\Lambda. \tag{9.95}
$$

Substitution of Eq. (9.91) into Eq. (9.73) yields the finite element model

$$
[K^e]\{\Delta^e\} = \{q^e\} + \{Q^e\}, \tag{9.96a}
$$

where

$$
\begin{aligned}
K_{ij}^e &= \int_{x_a}^{x_b} \left[E_e I_e \frac{d\varphi_i^e}{dx}\frac{d\varphi_j^e}{dx} + G_e A_e K_s \left(\varphi_i^e + \frac{d\psi_i^e}{dx}\right)\left(\varphi_j^e + \frac{d\psi_j^e}{dx}\right)\right] dx, \\
q_i^e &= \int_{x_a}^{x_b} \psi_i^e q(x)\, dx,
\end{aligned} \tag{9.96b}
$$

and $Q_1^e = V_1^e$, $Q_2^e = M_1^e$, $Q_3^e = V_2^e$, and $Q_4^e = M_2^e$. Equation (9.96a) has the explicit form [see Eq. (9.81) for the definitions of ξ and λ]:

$$
\left(\frac{2E_e I_e}{\mu h_e^3}\right)
\begin{bmatrix}
6 & -3h_e & -6 & -3h_e \\
-3h_e & 2h_e^2\lambda & 3h_e & h_e^2\xi \\
-6 & 3h_e & 6 & 3h_e \\
-3h_e & h_e^2\xi & 3h_e & 2h_e^2\lambda
\end{bmatrix}
\begin{Bmatrix} W_1^e \\ \Phi_1^e \\ W_2^e \\ \Phi_2^e \end{Bmatrix}
=
\begin{Bmatrix} q_1^e \\ q_2^e \\ q_3^e \\ q_4^e \end{Bmatrix}
+
\begin{Bmatrix} Q_1^e \\ Q_2^e \\ Q_3^e \\ Q_4^e \end{Bmatrix}. \tag{9.97}
$$

As stated earlier, this element leads to the exact nodal values for any distribution of the transverse load $q(x)$, provided that the bending stiffness EI and shear stiffness $K_s GA$ are element-wise constant. In the thin beam limit, Eq. (9.97) reduces to the Euler–Bernoulli beam equations (9.61), and it experiences no shear locking.

Example 9.3 Consider a simply supported beam. The following data are used:

$$
E = 10^6, \qquad \nu = 0.25, \qquad K = \tfrac{5}{6}, \qquad q_0 = 1.
$$

Two different beam length-to-height ratios, $L/H = 10$ and 100, are considered. Table 9.3 shows a comparison of the finite element solutions obtained with one, two, and four elements in half beam with the exact beam solutions for two different types of loads, namely, uniform load and sinusoidal load. Clearly, more than two elements

Table 9.3 Comparison of the finite element solutions with the exact maximum deflection and rotation of a simply supported isotropic beam (N = number of elements used in half beam)

	$w_0 \times 10^2$			$-\phi \times 10^3$		
Element	$N=1$	$N=2$	$N=4$	$N=1$	$N=2$	$N=4$
			Uniform load ($L/H = 10$)			
RIE	0.09750	0.14438	0.15609	0.37500	0.46875	0.49219
CIE	0.12875	0.15219	0.15805	0.50000	0.50000	0.50000
IIE[a]	0.16000	0.16000	0.16000	0.50000	0.50000	0.50000
EBE[a]	0.15265	0.15265	0.15265	0.50000	0.50000	0.50000
			Uniform load ($L/H = 100$)			
RIE	0.09379	0.14066	0.15238	0.37500	0.46875	0.49219
CIE	0.12504	0.14847	0.15433	0.50000	0.50000	0.50000
IIE[a]	0.15629	0.15629	0.15629	0.50000	0.50000	0.50000
EBE[a]	0.15265	0.15265	0.15265	0.50000	0.50000	0.50000
			Sinusoidal load ($L/H = 100$)			
RIE	0.07639	0.11079	0.12007	0.30543	0.36702	0.38204
CIE	0.09679	0.11682	0.12163	0.38705	0.38702	0.38702
IIE[a]	0.12322	0.12322	0.12322	0.38702	0.38702	0.38702
EBE[a]	0.12319	0.12319	0.12319	0.38702	0.38702	0.38702

[a]Exact values compared to the respective beam theories.

of CIE and RIE are required to obtain acceptable solutions. On the other hand, IIE yields exact nodal values with one element.

9.5 FINITE ELEMENT MODELS OF THE CLASSICAL PLATE THEORY

9.5.1 Introduction

Here we develop displacement finite element models of the equations governing the motion of plates according to the classical plate theory (CPT). While many of the basic ideas of the finite element method from one-dimensional beam problems carry over to two-dimensional plate problems, finite element models of plates are considerably more complicated due to the fact that two-dimensional problems are described by partial differential equations over geometrically complex regions. The boundary Γ of a two-dimensional domain Ω is, in general, a curve. Therefore, the two-dimensional finite elements must be simple geometric shapes that can be used to approximate the geometry of a given two-dimensional domain as well as the solution over it. Consequently, the finite element solution will have errors due to the approximation of the solution as well as the geometry of the domain.

In two dimensions, there is more than one geometric shape that can be used as a finite element (see Fig. 9.15). The interpolation functions depend not only on the

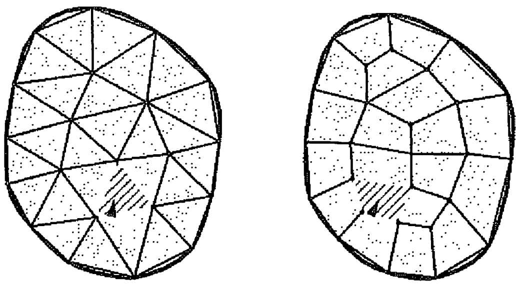


Figure 9.15 Finite element discretization of a two-dimensional domain using triangular and quadrilateral elements.

number of nodes in the element, but also on the shape of the element. The shape of the element must be such that its geometry is uniquely defined by a set of points, which serve as the element nodes in the development of the interpolation functions. A triangular element is the simplest geometric shape, followed by a rectangle.

9.5.2 General Formulation

Here we consider the general case of the dynamic response of plates subjected to in-plane compressive and shear forces. The virtual work statement of the classical plate theory over typical finite element Ω^e is given by [from Eq. (8.131a), except that the time derivative terms are integrated by parts in time]:

$$\begin{aligned}0 = \int_{t_1}^{t_2}\int_{\Omega^e}\Bigg[&-\frac{\partial^2\delta w_0}{\partial x^2}M_{xx} - 2\frac{\partial^2\delta w_0}{\partial x\partial y}M_{xy} - \frac{\partial^2\delta w_0}{\partial y^2}M_{yy}\\ &-\frac{\partial\delta w_0}{\partial x}\left(\hat{N}_{xx}\frac{\partial w_0}{\partial x} + \hat{N}_{xy}\frac{\partial w_0}{\partial y}\right) - \frac{\partial\delta w_0}{\partial y}\left(\hat{N}_{xy}\frac{\partial w_0}{\partial x} + \hat{N}_{yy}\frac{\partial w_0}{\partial y}\right)\\ &+ I_0\delta w_0\ddot{w}_0 + I_2\left(\frac{\partial\delta w_0}{\partial x}\frac{\partial\ddot{w}_0}{\partial x} + \frac{\partial\delta w_0}{\partial y}\frac{\partial\ddot{w}_0}{\partial y}\right) - \delta w_0 q\Bigg]\,dxdydt\\ &+\int_{t_1}^{t_2}\oint_{\Gamma^e}\left(-\delta w_0 V_n + \frac{\partial\delta w_0}{\partial x}T_x + \frac{\partial\delta w_0}{\partial y}T_y\right)ds dt,\end{aligned} \tag{9.98}$$

where

$$T_x \equiv M_{xx}n_x + M_{xy}n_y, \qquad T_y \equiv M_{xy}n_x + M_{yy}n_y, \tag{9.99a}$$

$$\begin{aligned}Q_n \equiv &\left(\frac{\partial M_{xx}}{\partial x} + \frac{\partial M_{xy}}{\partial y} - \hat{N}_{xx}\frac{\partial w_0}{\partial x} - \hat{N}_{xy}\frac{\partial w_0}{\partial y} - I_2\frac{\partial^3 w_0}{\partial x\partial t^2}\right)n_x\\ &+\left(\frac{\partial M_{xy}}{\partial x} + \frac{\partial M_{yy}}{\partial y} - \hat{N}_{xy}\frac{\partial w_0}{\partial x} - \hat{N}_{yy}\frac{\partial w_0}{\partial y} - I_2\frac{\partial^3 w_0}{\partial y\partial t^2}\right)n_y.\end{aligned} \tag{9.99b}$$

Here (n_x, n_y) denote the direction cosines of the unit normal on the element boundary Γ^e, (M_{xx}, M_{xy}, M_{yy}) are defined by Eq. (8.120), and $(\hat{N}_{xx}, \hat{N}_{xy}, \hat{N}_{yy})$ are the in-plane compressive and shear forces. It is clear from the variational statement (9.98) that the finite elements based on the classical plate theory require continuity of the transverse deflection and its normal derivative across element boundaries. Also, to satisfy the constant displacement (rigid-body mode) and constant strain requirements, the polynomial expansion for w_0 should be a complete quadratic.

Let us assume finite element approximation of the form

$$w_0(x, y, t) = \sum_{j=1}^{n} \Delta_j^e(t)\varphi_j^e(x, y), \tag{9.100}$$

where Δ_j^e are the values of w_0 and its derivatives at the nodes, and φ_j^e are the interpolation functions, the specific form of which will depend on the geometry of the element and the nodal degrees of freedom interpolated. Substituting approximations (9.100) for w_0 and φ_i^e for the virtual displacement δw_0 into Eq. (9.98), we obtain the ith equation of the finite element model:

$$\sum_{j=1}^{n}\left[\left(K_{ij}^e - G_{ij}^e\right)\Delta_j^e + M_{ij}^e\ddot{\Delta}_j^e\right] = F_i^e, \tag{9.101}$$

where $i, j = 1, 2, \ldots, n$. The coefficients of the stiffness matrix $K_{ij}^e = K_{ji}^e$, mass matrix $M_{ij}^e = M_{ji}^e$, geometric stiffness (or stability) matrix $G_{ij}^e = G_{ji}^e$, and force vectors F_i^e and F_i^{eT} are defined as follows:

$$K_{ij}^e = \int_{\Omega^e} D\left[T_{ij}^{xxxx} + \nu\left(T_{ij}^{xxyy} + T_{ij}^{yyxx}\right) + 2(1-\nu)T_{ij}^{xyxy} + T_{ij}^{yyyy}\right]dxdy,$$

$$G_{ij}^e = \int_{\Omega^e}\left[\hat{N}_{xx}S_{ij}^{xx} + \hat{N}_{xy}\left(S_{ij}^{xy} + S_{ij}^{yx}\right) + \hat{N}_{yy}S_{ij}^{yy}\right]dxdy,$$

$$M_{ij}^e = \int_{\Omega^e}\left[I_0S_{ij}^{00} + I_2\left(S_{ij}^{xx} + S_{ij}^{yy}\right)\right]dxdy,$$

$$F_i^e = \int_{\Omega^e} q\varphi_i^e\,dxdy + \oint_{\Gamma^e}\left(V_n\varphi_i^e - T_x\frac{\partial\varphi_i^e}{\partial x} - T_y\frac{\partial\varphi_i^e}{\partial y}\right)ds, \tag{9.102a}$$

where

$$T_{ij}^{\xi\eta\zeta\mu} = \frac{\partial^2\varphi_i^e}{\partial\xi\partial\eta}\frac{\partial^2\varphi_j^e}{\partial\zeta\partial\mu}, \qquad S_{ij}^{\xi\eta} = \frac{\partial\varphi_i^e}{\partial\xi}\frac{\partial\varphi_j^e}{\partial\eta}, \qquad S_{ij}^{00} = \varphi_i^e\varphi_j^e, \tag{9.102b}$$

and ξ, η, ζ, and μ take on the symbols x and y. In matrix notation, Eq. (9.101) can be expressed as

$$\left([K^e] - [G^e]\right)\{\Delta^e\} + [M^e]\{\ddot{\Delta}^e\} = \{F^e\}. \tag{9.103}$$

This completes the finite element model development of the classical laminate theory. The finite element model in Eq. (9.103) is called a *displacement finite element model* because it is based on equations of motion expressed in terms of the displacements, and the generalized displacements are the primary nodal degrees of freedom.

9.5.3 Conforming and Nonconforming Plate Elements

There exists a large body of literature on triangular and rectangular plate-bending finite elements of isotropic or orthotropic plates based on the classical plate theory (see, e.g., [8–16]). There are two kinds of plate-bending elements of the classical plate theory. A *conforming element* is one in which the interelement continuity of w_0, $\theta_x = \partial w_0/\partial x$, and $\theta_y = \partial w_0/\partial y$ (or $\partial w_0/\partial n$) are satisfied, and a *nonconforming element* is one in which the continuity of the normal slope, $\partial w_0/\partial n$, is not satisfied.

A nonconforming rectangular element has w_0, θ_x, and θ_y as the nodal variables (see Fig. 9.16a). The normal slope variation is cubic along an edge, whereas there are only two values of $\partial w_0/\partial n$ available on the edge. Therefore, the cubic polynomial for the normal derivative of w_0 is not the same on the edge common to two elements. The interpolation functions for this element can be expressed compactly as

$$\begin{aligned}
&\varphi_i^e = g_{i1}\ (i = 1, 4, 7, 10); \qquad \varphi_i^e = g_{i2}\ (i = 2, 5, 8, 11),\\
&\varphi_i^e = g_{i3}\ (i = 3, 6, 9, 12),
\end{aligned} \tag{9.104a}$$

$$\begin{aligned}
g_{i1} &= \tfrac{1}{8}(1 + \xi_0)(1 + \eta_0)(2 + \xi_0 + \eta_0 - \xi^2 - \eta^2),\\
g_{i2} &= \tfrac{1}{8}\xi_i(\xi_0 - 1)(1 + \eta_0)(1 + \xi_0)^2,\\
g_{i3} &= \tfrac{1}{8}\eta_i(\eta_0 - 1)(1 + \xi_0)(1 + \eta_0)^2,\\
\xi &= (x - x_c)/2, \qquad \eta = (y - y_c)/b, \qquad \xi_0 = \xi\xi_i, \qquad \eta_0 = \eta\eta_i,
\end{aligned} \tag{9.104b}$$

where $2a$ and $2b$ are the sides of the rectangle, and (x_c, y_c) are the global coordinates of the center of the rectangle.

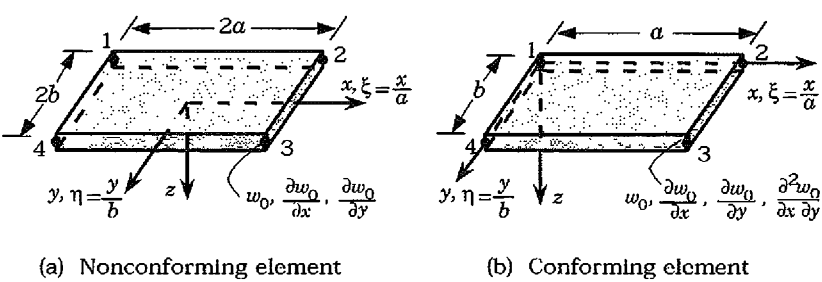


Figure 9.16 (a) Nonconforming and (b) conforming rectangular elements.

A conforming rectangular element has w_0, $\partial w_0/\partial x$, $\partial w_0/\partial y$, and $\partial^2 w_0/\partial x \partial y$ as the nodal variables. The interpolation functions for this element (see Fig. 9.16b) are

$$\varphi_i^e = g_{i1}\ (i = 1, 5, 9, 13); \qquad \varphi_i^e = g_{i2}\ (i = 2, 6, 10, 14),$$
$$\varphi_i^e = g_{i3}\ (i = 3, 7, 11, 15); \qquad \varphi_i^e = g_{i4}\ (i = 4, 8, 12, 16), \tag{9.105a}$$

$$\begin{aligned}
g_{i1} &= \tfrac{1}{16}(\xi + \xi_i)^2(\xi_0 - 2)(\eta + \eta_i)^2(\eta_0 - 2),\\
g_{i2} &= \tfrac{1}{16}\xi_i(\xi + \xi_i)^2(1 - \xi_0)(\eta + \eta_i)^2(\eta_0 - 2),\\
g_{i3} &= \tfrac{1}{16}\eta_i(\xi + \xi_i)^2(\xi_0 - 2)(\eta + \eta_i)^2(1 - \eta_0),\\
g_{i4} &= \tfrac{1}{16}\xi_i\eta_i(\xi + \xi_i)^2(1 - \xi_0)(\eta + \eta_i)^2(1 - \eta_0).
\end{aligned} \tag{9.105b}$$

The conforming element has four degrees of freedom per node, while the non-conforming element has three degrees of freedom per node. For the conforming rectangular element the total number of bending nodal degrees of freedom per element is 16, and for the nonconforming element the total number is 12.

9.5.4 Fully Discretized Finite Element Models

Static Bending In the case of static bending under applied mechanical loads, Eq. (9.103) reduces to

$$\left([K^e] - [G^e]\right)\{\Delta^e\} = \{F^e\}, \tag{9.106}$$

where it is understood that all time-derivative terms are zero.

Once the nodal values of generalized displacements (w_0, $\partial w_0/\partial x$, $\partial w_0/\partial y$) have been obtained by solving the assembled equations of a problem, the strains are evaluated in each element by differentiating the displacements. The strains and stresses are most accurate if they are computed at the center of the element [17,18].

Buckling In the case of buckling under applied in-plane compressive ($\hat{N}_{xx}$, $\hat{N}_{yy}$) and shear $\hat{N}_{xy}$ edge loads, Eq. (9.103) reduces to

$$\left([K^e] - \lambda[\hat{G}^e]\right)\{\Delta^e\} = \{\bar{F}\}, \tag{9.107}$$

where

$$\lambda = \hat{N}_{xx}/\hat{N}_{xx}^0 = \hat{N}_{yy}/\hat{N}_{yy}^0 = \hat{N}_{xy}/\hat{N}_{xy}^0, \tag{9.108a}$$

$$\hat{G}_{ij}^e = \int_{\Omega^e}\left[\hat{N}_{xx}^0 S_{ij}^{xx} + \hat{N}_{xy}^0\left(S_{ij}^{xy} + S_{ij}^{yx}\right) + \hat{N}_{yy}^0 S_{ij}^{yy}\right]dxdy,$$

$$\bar{F}_k = \oint_{\Gamma^e}\left(V_n\varphi_k^e - T_x\frac{\partial\varphi_k^e}{\partial x} - T_y\frac{\partial\varphi_k^e}{\partial y}\right)ds. \tag{9.108b}$$

Natural Vibration In the case of natural vibration, the response of the plate is assumed to be periodic. Equation (9.103) becomes

$$\left([K^e] - \omega^2[M^e]\right)\{\Delta^e\} = \{\bar{F}^e\}, \tag{9.109}$$

where ω denotes the frequency of natural vibration.

Transient Response In the case of transient response, Eq. (9.103) must be integrated with respect to time t to determine the nodal values $\Delta_j^e(t)$ as functions of time. Here we consider the Newmark family of time-integration schemes [1,3,19,20] that is used widely in structural dynamics.

In the Newmark method, the first and second time derivatives are approximated as

$$\begin{aligned} \{\dot{\Delta}^e\}_{s+1} &= \{\dot{\Delta}^e\}_s + a_1\{\ddot{\Delta}^e\}_s + a_2\{\ddot{\Delta}^e\}_{s+1}, \\ \{\ddot{\Delta}^e\}_{s+1} &= a_3\left(\{\Delta^e\}_{s+1} - \{\Delta^e\}_s\right) - a_4\{\dot{\Delta}^e\}_s - a_5\{\ddot{\Delta}^e\}_s, \end{aligned} \tag{9.110a}$$

where the superposed dot denotes differentiation with respect to time,

$$a_1 = (1-\alpha)dt_s, \quad a_2 = \alpha\, dt_s, \quad a_3 = \frac{2}{\gamma(dt_s)^2}, \quad a_4 = dt_s\, a_3, \quad a_5 = \frac{(1-\gamma)}{\gamma}, \tag{9.110b}$$

dt is the time increment, $dt_s = t_{s+1} - t_s$, and $\{\cdot\}_s$, for example, denotes the value of the enclosed vector at time t_s.

The parameters α and γ in the Newmark scheme are selected such that the scheme is either stable or conditionally stable; i.e., the error introduced through the time approximation (9.110a) does not grow unboundedly with time. All schemes for which $\gamma \geq \alpha \geq 1/2$ are unconditionally stable. Schemes for which $\gamma < \alpha$ and $\alpha \geq 0.5$ are conditionally stable, and the stability condition is

$$dt \leq dt_{cr} \equiv \left[\tfrac{1}{2}\omega_{max}(\alpha - \gamma)\right]^{-1/2}, \tag{9.111}$$

where ω_{max} denotes the maximum frequency associated with the finite element equations (9.109) for the same mesh.

The Newmark family contains the following widely used schemes:

$$\begin{aligned} &\alpha = \tfrac{1}{2}, \quad \gamma = \tfrac{1}{2}, \quad \text{the constant average acceleration method (stable).} \\ &\alpha = \tfrac{1}{2}, \quad \gamma = \tfrac{1}{3}, \quad \text{the linear acceleration method (conditionally stable).} \\ &\alpha = \tfrac{1}{2}, \quad \gamma = 0, \quad \text{the centered difference method (conditionally stable).} \\ &\alpha = \tfrac{3}{2}, \quad \gamma = 2, \quad \text{the backward difference method (stable).} \end{aligned} \tag{9.112}$$

Using Eq. (9.110a), Eq. (9.103) can be expressed as

$$[\hat{K}^e]\{w^e\}_{s+1} = \{\hat{F}^e\}, \tag{9.113a}$$

where

$$[\hat{K}^e] = ([K^e]_{s+1} - [G^e]_{s+1}) + a_3[M^e]_{s+1},$$
$$\{\hat{F}^e\} = \{F^e\}_{s+1} + [M^e]_{s+1}\left(a_3\{\Delta^e\}_s + a_4\{\dot{\Delta}^e\}_s + a_5\{\ddot{\Delta}^e\}_s\right). \tag{9.113b}$$

Note that for the centered difference scheme ($\gamma = 0$), it is necessary to use an alternative form of Eq. (9.113a).

Equation (9.113a) represents a system of algebraic equations among the (discrete) values of $\{\Delta^e(t)\}$ at time $t = t_{s+1}$ in terms of known values at time $t = t_s$. At the first time step (i.e., $s = 0$), the values $\{\Delta^e\}_0 = \{\Delta^e(0)\}$ and $\{\dot{\Delta}^e\}_0 = \{\dot{\Delta}^e(0)\}$ are known from the initial conditions of the problem, and Eq. (9.101) is used to determine $\{\ddot{\Delta}^e\}_0$ at $t = 0$:

$$\{\ddot{\Delta}^e\}_0 = [M^e]^{-1}\left\{\{F^e\} - \left([K^e] - [G^e]\right)\{\Delta^e\}_0\right\}. \tag{9.114}$$

The conforming (C) and nonconforming (NC) rectangular finite elements discussed herein are used to analyze plates for bending and natural vibration response, and the results are presented in the next two examples. We shall use the notation "$m \times n$ mesh" to indicate that m elements along the x-axis and n elements along the y-axis are used. In the case of the conforming element, it is necessary that the cross-derivative $\partial^2 w_0/\partial x\partial y$ also be set to zero at the center of the plate when a quarter-plate model is used, otherwise the results will be less accurate. The stresses in the finite element analysis were computed at the center of the elements.

Example 9.4 Consider a rectangular isotropic ($\nu = 0.25$) plate with simply supported edges, and subjected to uniformly distributed load of intensity q_0. Due to the symmetry, a quadrant of the plate may be used as the computational domain. Full integration (i.e., 4×4 Gauss rule) was used to evaluate the stiffness coefficients in both elements, although the nonconforming element gave the same results for 3×3 Gauss rule. The geometric boundary conditions of the computational domain (see the shaded quadrant in Fig. 9.17) are

$$\frac{\partial w_0}{\partial x} = 0 \text{ at } x = 0; \qquad \frac{\partial w_0}{\partial y} = 0 \text{ at } y = 0, \tag{9.115a}$$

$$w_0 = \frac{\partial w_0}{\partial x} = 0 \text{ at } x = \frac{a}{2}; \qquad w_0 = \frac{\partial w_0}{\partial y} = 0 \text{ at } y = \frac{b}{2}. \tag{9.115b}$$

In addition, $\partial^2 w_0/\partial x\partial y = 0$ was used at $x = y = 0$ for the conforming element. The boundary conditions (for CPT) along the symmetry lines are shown in Fig. 9.17.

Table 9.4 shows a comparison of finite element solutions with the analytical solutions developed in Chapter 8. The series solutions were evaluated using $m, n = 1, 3, \ldots, 19$. The exact maximum deflection occurs at $x = y = 0$, maximum stresses σ_{xx} and σ_{yy} occur at $(0, 0, h/2)$, and the maximum shear stress σ_{xy} occurs at $(a/2, b/2, -h/2)$. The locations of the maximum normal stresses are $(a/8, b/8)$,

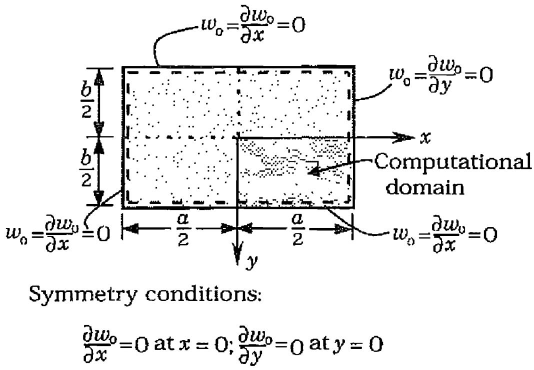


Figure 9.17 Symmetry boundary conditions for rectangular plates.

Table 9.4 A comparison of the maximum transverse deflections and stresses[a] of simply supported (SSSS) square plates under uniform load

	Nonconforming			Conforming			Analytical
Variable	2 × 2	4 × 4	8 × 8	2 × 2	4 × 4	8 × 8	Solution
$\bar{w}$	4.8571	4.6425	4.5883	4.7388	4.5952	4.5734	4.5701
$\bar{\sigma}_{xx}$	0.2405	0.2673	0.2740	0.2259	0.2637	0.2732	0.2762
$\bar{\sigma}_{yy}$	0.2405	0.2673	0.2740	0.2259	0.2637	0.2732	0.2762
$\bar{\sigma}_{xy}$	0.1713	0.1964	0.2050	0.1669	0.1935	0.2040	0.2085

[a] $\bar{w} = 10^2[w_0 E_2 h^3/(q_0 a^4)]$, $\bar{\sigma} = \sigma h^2/(q_0 a^2)$.

Table 9.5 Maximum transverse deflections and stresses[a] of clamped (CCCC), isotropic ($\nu = 0.25$) square plates under uniform load

	Nonconforming			Conforming		
Variable	2 × 2	4 × 4	8 × 8	2 × 2	4 × 4	8 × 8
$\bar{w}$[b]	1.5731	1.4653	1.4342	1.7245	1.5327	1.4539
$\bar{\sigma}_{xx}$	0.0987	0.1238	0.1301	0.1115	0.1247	0.1305
$\bar{\sigma}_{yy}$	0.0987	0.1238	0.1301	0.1115	0.1247	0.1305
$\bar{\sigma}_{xy}$	0.0497	0.0222	0.0067	0.0700	0.0297	0.0086

[a] $\bar{w} = 10^2[w_0 E_2 h^3/(q_0 a^4)]$, $\bar{\sigma} = \sigma h^2/(q_0 a^2)$.
[b] The analytical solution (see Example 8.12) is 1.4231.

$(a/16, b/16)$, and $(a/32, b/32)$ for uniform meshes 2×2, 4×4, and 8×8, respectively, while those of σ_{xy} are $(3a/8, 3b/8)$, $(7a/16, 7b/16)$, and $(15a/32, 15b/32)$ for the three meshes.

Similar results are presented in Table 9.5 for a clamped square plate. Reduced integration (i.e., 3×3 Gauss rule) was used to evaluate the stiffness coefficients in both elements. The locations of the normal and shear stresses reported for the three

Table 9.6 A comparison of the fundamental frequencies[a] of simply supported (SSSS) and clamped (CCCC) rectangular plates

	SSSS Plates			CCCC Plates		
a/b	2×2	4×4	Exact	2×2	4×4	Ritz[a]
0.5	4.8301	4.9752	4.9003	7.8213	9.1185	9.9891
1.0	1.9736	1.9959	1.9838	3.3103	3.5203	3.6476
1.5	1.4144	1.4395	1.4359	2.3435	2.5861	2.7404
2.0	1.2074	1.2439	1.2436	1.9551	2.2781	2.4973

[a] $\bar{\omega} = \omega(b^2/\pi^2)\sqrt{\rho h/D_{22}}$.
[b] Ritz = Ritz solution discussed in Chapter 8.

meshes are:

$$\sigma \to 2\times 2:\ (a/8, b/8);\qquad 4\times 4:\ (a/16, b/16);\qquad 8\times 8:\ (a/32, b/32);$$
$$\tau \to 2\times 2:\ (3a/8, 3b/8);\qquad 4\times 4:\ (7a/16, 7b/16);\qquad 8\times 8:\ (15a/32, 15b/32).$$

These stresses are not necessarily the maximum ones in the plate. For example, for an 8×8 mesh the maximum normal stress in the plate is found to be 0.2316 at $(0.46875a, 0.03125b, -h/2)$ and the maximum shear stress is 0.0225 at $(0.28125a, 0.09375b, -h/2)$ for the nonconforming element.

Both conforming and nonconforming elements show good convergence. Since the stresses in the finite element analysis are computed at locations different from the analytical solutions, they are expected to be different. A mesh refinement not only improves the accuracy of the solution, but the Gauss point locations also get closer to the true locations of the maximum values.

Example 9.5 Next consider natural vibrations of isotropic rectangular plates. Table 9.6 contains fundamental frequencies for isotropic ($\nu = 0.25$) simply supported and clamped plates. The results were obtained using the conforming plate-bending element. Only a quadrant was modeled and rotary inertia was included ($b/h = 0.1$). Because a quadrant is used to model the problem, only symmetric vibration modes can be calculated.

9.6 FINITE ELEMENT MODELS OF THE FIRST-ORDER SHEAR DEFORMATION PLATE THEORY

9.6.1 Governing Equations and Weak Forms

Here we develop the finite element models of the equations governing the first-order shear deformation plate theory (FDST). We consider the linear equations of motion of FSDT from Eqs. (8.282)–(8.284) that are expressed in terms of the stress resultants $(M_{xx}, M_{yy}, M_{xy}, Q_x, Q_y)$ defined in Eqs. (8.288) and (8.289). These equations are expressed in terms of the generalized displacements (w_0, ϕ_x, ϕ_y) in Eqs. (8.290)–(8.292). The weak forms of Eqs. (8.282)–(8.284) are given by

$$0 = \int_{\Omega^e} \left[\frac{\partial \delta w_0}{\partial x} Q_x + \frac{\partial \delta w_0}{\partial y} Q_y - \delta w_0 q + I_0 \delta w_0 \frac{\partial^2 w_0}{\partial t^2} \right.$$
$$\left. + \frac{\partial \delta w_0}{\partial x}\left(\hat{N}_{xx} \frac{\partial w_0}{\partial x} + \hat{N}_{xy} \frac{\partial w_0}{\partial y} \right) + \frac{\partial \delta w_0}{\partial y}\left(\hat{N}_{xy} \frac{\partial w_0}{\partial x} + \hat{N}_{yy} \frac{\partial w_0}{\partial y} \right) \right] dxdy$$
$$- \oint_{\Gamma^e} \left(\bar{Q}_x n_x + \bar{Q}_y n_y \right) \delta w_0 \, ds, \tag{9.116a}$$

$$0 = \int_{\Omega^e} \left(\frac{\partial \delta \phi_x}{\partial x} M_{xx} + \frac{\partial \delta \phi_x}{\partial y} M_{xy} + \delta \phi_x Q_x + I_2 \delta \phi_x \frac{\partial^2 \phi_x}{\partial t^2} \right) dxdy - \oint_{\Gamma^e} T_x \delta \phi_x \, ds, \tag{9.116b}$$

$$0 = \int_{\Omega^e} \left(\frac{\partial \delta \phi_y}{\partial x} M_{xy} + \frac{\partial \delta \phi_y}{\partial y} M_{yy} + \delta \phi_y Q_y + I_2 \delta \phi_y \frac{\partial^2 \phi_y}{\partial t^2} \right) dxdy - \oint_{\Gamma^e} T_y \delta \phi_y \, ds, \tag{9.116c}$$

where

$$\bar{Q}_x = Q_x - \left(\hat{N}_{xx} \frac{\partial w_0}{\partial x} + \hat{N}_{xy} \frac{\partial w_0}{\partial y} \right),$$
$$\bar{Q}_y = Q_y - \left(\hat{N}_{xy} \frac{\partial w_0}{\partial x} + \hat{N}_{yy} \frac{\partial w_0}{\partial y} \right), \tag{9.117a}$$
$$T_x \equiv M_{xx} n_x + M_{xy} n_y, \qquad T_y \equiv M_{xy} n_x + M_{yy} n_y.$$

We note from the boundary terms in Eqs. (9.116a–c) that (w_0, ϕ_x, ϕ_y) are the primary variables and $(\hat{Q}_n, T_x, T_y)$ are the secondary variables, where

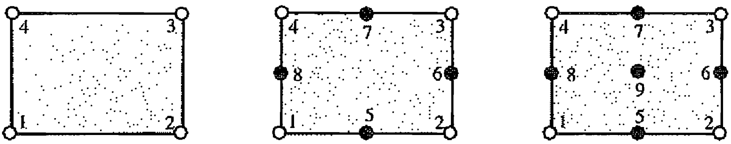
$$Q_n = \bar{Q}_x n_x + \bar{Q}_y n_y. \tag{9.117b}$$

Unlike in the classical plate theory, the rotations (ϕ_x, ϕ_y) are independent of w_0. Note also that no derivatives of w_0 are in the list of the primary variables and therefore the finite element to be developed is called a C^0 element.

9.6.2 Finite Element Model

The dependent variables (w_0, ϕ_x, ϕ_y) can all be approximated using the Lagrange interpolation functions, and, in principle, w_0 and (ϕ_x, ϕ_y) can be approximated with differing degrees of functions. Let

$$w_0(x, y, t) = \sum_{j=1}^{n} w_j(t) \psi_j^e(x, y), \tag{9.118a}$$



(a) Bilinear element (b) 8-node quadratic element (c) 9-node quadratic element

Figure 9.18 Linear and quadratic rectangular elements.

$$\phi_x(x, y, t) = \sum_{j=1}^{n} S_j^1(t)\varphi_j^e(x, y), \tag{9.118b}$$

$$\phi_y(x, y, t) = \sum_{j=1}^{n} S_j^2(t)\varphi_j^e(x, y), \tag{9.118c}$$

where (ψ_j^e, φ_j^e) are Lagrange interpolation functions, and (w_j, S_j^1, S_j^2) are the nodal values of (w_0, ϕ_x, ϕ_y), respectively. One can use linear, quadratic, or higher-order interpolation functions for each of the variables. Here we use rectangular elements.

The linear and quadratic Lagrange interpolation functions of rectangular elements (see Fig. 9.18) are given below in terms of the element coordinates (ξ, η), called the *natural coordinates*. The subscripts of these functions ψ_i correspond to the node numbers shown in Fig. 9.18. For a derivation of these functions, see Reddy [4,32].

Bilinear Interpolation Functions

$$\begin{Bmatrix} \psi_1^e \\ \psi_2^e \\ \psi_3^e \\ \psi_4^e \end{Bmatrix} = \frac{1}{4}\begin{Bmatrix} (1-\xi)(1-\eta) \\ (1+\xi)(1-\eta) \\ (1+\xi)(1+\eta) \\ (1-\xi)(1+\eta) \end{Bmatrix}. \tag{9.119}$$

Nine-Node Quadratic Interpolation Functions

$$\begin{Bmatrix} \psi_1^e \\ \psi_2^e \\ \psi_3^e \\ \psi_4^e \\ \psi_5^e \\ \psi_6^e \\ \psi_7^e \\ \psi_8^e \\ \psi_9^e \end{Bmatrix} = \frac{1}{4}\begin{Bmatrix} (1-\xi)(1-\eta)(-\xi-\eta-1) + (1-\xi^2)(1-\eta^2) \\ (1+\xi)(1-\eta)(\xi-\eta-1) + (1-\xi^2)(1-\eta^2) \\ (1+\xi)(1+\eta)(\xi+\eta-1) + (1-\xi^2)(1-\eta^2) \\ (1-\xi)(1+\eta)(-\xi+\eta-1) + (1-\xi^2)(1-\eta^2) \\ 2(1-\xi^2)(1-\eta) - (1-\xi^2)(1-\eta^2) \\ 2(1+\xi)(1-\eta^2) - (1-\xi^2)(1-\eta^2) \\ 2(1-\xi^2)(1+\eta) - (1-\xi^2)(1-\eta^2) \\ 2(1-\xi)(1-\eta^2) - (1-\xi^2)(1-\eta^2) \\ 4(1-\xi^2)(1-\eta^2) \end{Bmatrix}. \tag{9.120}$$

Eight-Node Quadratic Interpolation Functions

$$\begin{Bmatrix} \psi_1^e \\ \psi_2^e \\ \psi_3^e \\ \psi_4^e \\ \psi_5^e \\ \psi_6^e \\ \psi_7^e \\ \psi_8^e \end{Bmatrix} = \frac{1}{4} \begin{Bmatrix} (1-\xi)(1-\eta)(-\xi-\eta-1) \\ (1+\xi)(1-\eta)(\xi-\eta-1) \\ (1+\xi)(1+\eta)(\xi+\eta-1) \\ (1-\xi)(1+\eta)(-\xi+\eta-1) \\ 2(1-\xi^2)(1-\eta) \\ 2(1+\xi)(1-\eta^2) \\ 2(1-\xi^2)(1+\eta) \\ 2(1-\xi)(1-\eta^2) \end{Bmatrix}. \tag{9.121}$$

Substituting Eqs. (9.118a–c) for (w_0, ϕ_x, ϕ_y) into Eqs. (9.116a–c), we obtain the following semidiscrete finite element model of the FSDT:

$$\left(\begin{bmatrix} [K^{11}] & [K^{12}] & [K^{13}] \\ [K^{12}]^T & [K^{22}] & [K^{23}] \\ [K^{13}]^T & [K^{23}]^T & [K^{33}] \end{bmatrix} + \begin{bmatrix} [G] & [0] & [0] \\ [0] & [0] & [0] \\ [0] & [0] & [0] \end{bmatrix} \right) \begin{Bmatrix} \{w^e\} \\ \{S^1\} \\ \{S^2\} \end{Bmatrix}$$

$$+ \begin{bmatrix} [M^{11}] & [0] & [0] \\ [0] & [M^{22}] & [0] \\ [0] & [0] & [M^{33}] \end{bmatrix} \begin{Bmatrix} \{\ddot{w}^e\} \\ \{\ddot{S}^1\} \\ \{\ddot{S}^2\} \end{Bmatrix} = \begin{Bmatrix} \{F^1\} \\ \{F^2\} \\ \{F^3\} \\ \{0\} \end{Bmatrix} \tag{9.122a}$$

or

$$[K^e]\{\Delta^e\} + [M^e]\{\ddot{\Delta}^e\} = \{F^e\}. \tag{9.122b}$$

The coefficients of the submatrices $[K^{\alpha\beta}]$ and $[M^{\alpha\beta}]$ and vectors $\{F^\alpha\}$ are defined for $(\alpha, \beta = 1, 2, 3)$ by the expressions

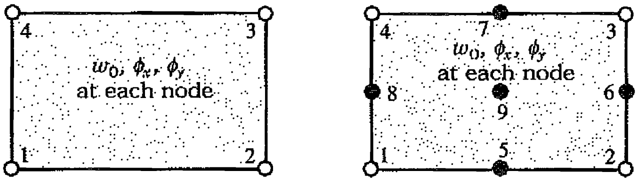
$$K_{ij}^{11} = \int_{\Omega^e} \left(K_s A_{55} \frac{\partial \psi_i^e}{\partial x} \frac{\partial \psi_j^e}{\partial x} + K_s A_{44} \frac{\partial \psi_i^e}{\partial y} \frac{\partial \psi_j^e}{\partial y} \right) dxdy,$$

$$K_{ij}^{12} = \int_{\Omega^e} K_s A_{55} \frac{\partial \psi_i^e}{\partial x} \varphi_j^e \, dxdy, \qquad K_{ij}^{13} = \int_{\Omega^e} K_s A_{44} \frac{\partial \psi_i^e}{\partial y} \varphi_j^e \, dxdy,$$

$$K_{ij}^{22} = \int_{\Omega^e} \left(D_{11} \frac{\partial \varphi_i^e}{\partial x} \frac{\partial \varphi_j^e}{\partial x} + D_{66} \frac{\partial \varphi_i^e}{\partial y} \frac{\partial \varphi_j^e}{\partial y} + K_s A_{55} \varphi_i^e \varphi_j^e \right) dxdy,$$

$$K_{ij}^{23} = \int_{\Omega^e} \left(D_{12} \frac{\partial \varphi_i^e}{\partial x} \frac{\partial \psi_j^e}{\partial y} + D_{66} \frac{\partial \varphi_i^e}{\partial y} \frac{\partial \varphi_j^e}{\partial x} \right) dxdy, \tag{9.123}$$

$$K_{ij}^{33} = \int_{\Omega^e} \left(D_{66} \frac{\partial \varphi_i^e}{\partial x} \frac{\partial \varphi_j^e}{\partial x} + D_{22} \frac{\partial \varphi_i^e}{\partial y} \frac{\partial \varphi_j^e}{\partial y} + K_s A_{44} \varphi_i^e \varphi_j^e \right) dxdy,$$



(a) Bilinear FSDT element (b) Quadratic FSDT element

Figure 9.19 C^0 rectangular plate-bending finite elements for the first-order shear deformation plate theory.

$$M_{ij}^{11} = \int_{\Omega^e} I_0 \psi_i^e \psi_j^e \, dxdy, \qquad M_{ij}^{22} = \int_{\Omega^e} I_2 \varphi_i^e \varphi_j^e \, dxdy = M_{ij}^{33},$$

$$F_i^1 = \int_{\Omega^e} q \psi_i^e \, dxdy + \oint_{\Gamma^e} \bar{Q}_n \psi_i^e \, ds,$$

$$F_i^2 = \oint_{\Gamma^e} T_x \varphi_i^e \, ds, \qquad F_i^3 = \oint_{\Gamma_e} T_y \varphi_i^e \, ds.$$

Here D_{ij} are the bending stiffnesses and A_{44} and A_{55} are shear stiffnesses of an orthotropic plate:

$$D_{11} = \frac{E_1 h^3}{12(1 - \nu_{12}\nu_{21})}, \qquad D_{22} = \frac{E_2 h^3}{12(1 - \nu_{12}\nu_{21})}, \qquad D_{12} = \nu_{12} D_{22},$$

$$D_{66} = \frac{G_{12} h^3}{12}, \qquad A_{44} = G_{23} h, \qquad A_{55} = G_{13} h. \tag{9.124}$$

When the bilinear interpolation functions are used with $\psi_i^e = \varphi_i^e$ for all generalized displacements (w_0, ϕ_x, ϕ_y), the element stiffness matrices are of the order 12×12; for the nine-node quadratic element they are 27×27 (see Fig. 9.19).

The C^0 plate-bending element of Eq. (9.122b) is among the simplest available in the literature. It is known, when lower-order (quadratic or less) equal interpolation of the transverse deflection and rotations is used, that the elements experience shear locking (similar to the Timoshenko beam element). A commonly used technique to alleviate the problem of shear locking is to under-integrate the transverse shear parts of stiffness coefficients (i.e., all coefficients in $K_{ij}^{\alpha\beta}$ that contain A_{44} and A_{55}). To better understand what is reduced integration in the present context, we visit briefly the concept of numerical integration using the Gauss–Legendre quadrature. For alternative finite element models of the FSDT, the reader may consult the papers [21–26] listed at the end of the chapter.

9.6.3 Numerical Integration

Numerical integration schemes, such as the Gauss–Legendre numerical integration scheme, require the integral to be evaluated on a specific domain or with respect to a

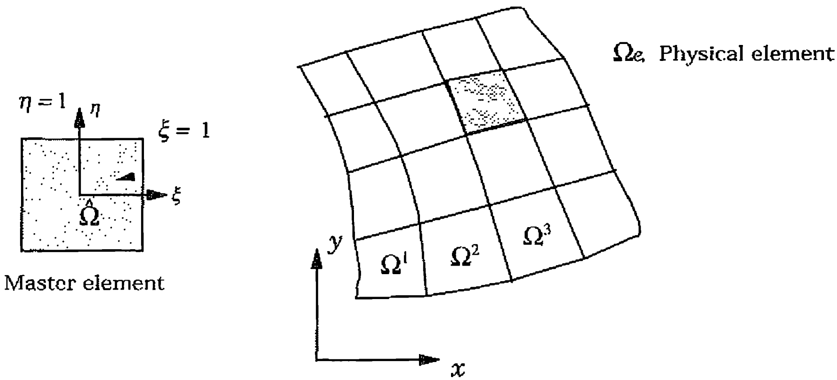


Figure 9.20 Transformation between the domain on which an integral is defined (actual element domain) and the domain on which the integral is evaluated.

specific coordinate system. Gauss quadrature, for example, requires the integral to be expressed over a square region $\hat{\Omega}$ of dimension 2×2 and for the coordinate system (ξ, η) be such that $-1 \leq (\xi, \eta) \leq 1$. The coordinates (ξ, η) are called *normalized* or *natural coordinates*. Thus, the transformation between (x, y) and (ξ, η) of a given integral expression defined over an element Ω^e to one on the domain $\hat{\Omega}$ facilitates the use of Gauss–Legendre quadrature to evaluate integrals. The element $\hat{\Omega}$ is called a *master element* (see Reddy [4], chap. 9).

The transformation between Ω^e and $\hat{\Omega}$ is accomplished by a coordinate transformation of the form (see Fig. 9.20)

$$x = \sum_{j=1}^{m} x_j^e \hat{\psi}_j^e(\xi, \eta), \qquad y = \sum_{j=1}^{m} y_j^e \hat{\psi}_j^e(\xi, \eta), \tag{9.125}$$

while a typical dependent variable $u(x, y)$ is approximated by

$$u(x, y) = \sum_{j=1}^{n} u_j^e \psi_j^e(x, y) = \sum_{j=1}^{n} u_j^e \psi_j^e(x(\xi, \eta), y(\xi, \eta)), \tag{9.126}$$

where $\hat{\psi}_j^e$ denote the interpolation functions of the master element $\hat{\Omega}$ and ψ_j^e are interpolation functions of a typical element Ω^e over which u is approximated. The transformation (9.125) maps a point (x, y) in a typical element Ω^e of the mesh to a point (ξ, η) in the master element $\hat{\Omega}$, and vice versa if the Jacobian of the transformation is positive-definite. The positive-definite requirement of the Jacobian dictates admissible geometries of elements in a mesh (see Reddy [4], pp. 421–448).

The interpolation functions ψ_j^e used for the approximation of the dependent variable are, in general, different from $\hat{\psi}_j^e$ used in the approximation of the geometry. Depending on the relative degree of approximations used for the geometry and the dependent variable(s), the finite element formulations are classified into three categories:

(a) *Superparametric formulation* ($m > n$). The polynomial degree of approximation used for the geometry is of higher order than that used for the dependent variable.

(b) *Isoparametric formulation* ($m = n$). Equal degree of approximation is used for both geometry and dependent variables.

(c) *Subparametric formulation* ($m < n$). Higher-order approximation of the dependent variable is used.

For example, the nonconforming and conforming finite elements are based on higher-order interpolation (e.g., cubic or higher order) of the deflection w_0 while the geometry is represented by linear or quadratic interpolation functions. Hence the formulation falls into the subparametric category. The first-order shear deformation plate-bending elements presented here are based on the isoparametric formulation.

Next, consider the evaluation of the integral

$$I^e = \int_{\Omega^e} F^e(x, y)\, dxdy. \tag{9.127}$$

The integrand $F^e(x, y)$ in the present case consists of the elastic coefficients D^e_{ij} and A^e_{ij} and the interpolation functions ψ^e_i and their derivatives. Hence, if D^e_{ij} and A^e_{ij} are constant within the element Ω^e, $F^e(x, y)$ is a polynomial of x and y of a certain degree, depending on the degree of ψ^e_i.

When a Gauss rule is used, the integral in (9.127) is expressed in terms of the natural coordinates (ξ, η) as

$$\begin{aligned} I^e &= \int_{\Omega^e} F^e(x, y)\, dxdy \\ &\approx \int_{\hat{\Omega}^e} F^e(x(\xi, \eta), y(\xi, \eta))\, |J^e|\, d\xi d\eta \\ &\equiv \int_{-1}^{1}\int_{-1}^{1} \hat{F}(\xi, \eta)\, d\xi d\eta, \end{aligned} \tag{9.128}$$

where $|J^e|$ is the determinant of the matrix of coordinate transformation, called the Jacobian:

$$[J^e] = \begin{bmatrix} \dfrac{\partial x}{\partial \xi} & \dfrac{\partial y}{\partial \xi} \\[2mm] \dfrac{\partial x}{\partial \eta} & \dfrac{\partial y}{\partial \eta} \end{bmatrix}^e . \tag{9.129}$$

For straight-sided elements $|J^e|$ is a constant, and the polynomial degree of $\hat{F}^e(\xi, \eta)$ is the same as that of $F^e(x, y)$.

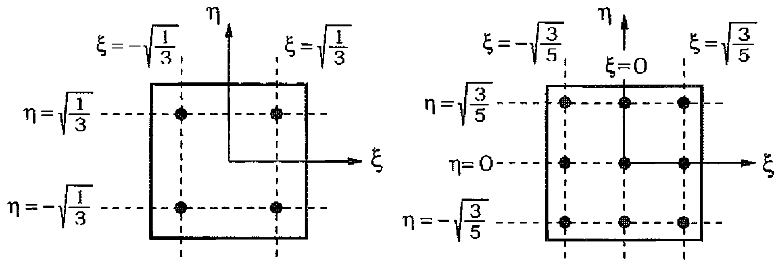


Figure 9.21 Gauss points in a master rectangular element.

The numerical evaluation of I^e by the Gauss quadrature amounts to evaluating the integrand at a number of points in $\hat{\Omega}^e$, called Gauss points, multiplying the function values with so-called Gauss weights, and adding them up:

$$I^e = \int_{-1}^{1} \int_{-1}^{1} \hat{F}(\xi, \eta)\, d\xi d\eta \approx \sum_{I=1}^{M} \sum_{J=1}^{N} \hat{F}(\xi_I, \eta_J) W_I W_J, \tag{9.130}$$

where (ξ_I, η_J) are the coordinates of the (I, J) Gauss point in $\hat{\Omega}^e$, W_I and W_J are the corresponding Gauss weights, and (M, N) are the number of Gauss points in each coordinate direction (see Fig. 9.21). If $\hat{F}(\xi, \eta)$ is a polynomial of degree p in ξ and degree q in η, then the choice

$$M = \left[\frac{p+1}{2}\right], \qquad N = \left[\frac{q+1}{2}\right], \tag{9.131}$$

gives the exact value of the integral I^e. Here $[\cdot]$ denotes the nearest integer equal to or greater than the enclosed value. For example, for the C^0 elements based on isoparametric formulation, the interpolation functions are of the same degree in both ξ and η. Hence, for the evaluation of the stiffness coefficients, we have $M = N$; and $p = q$ are at most equal to twice the degree r of the polynomials (of ξ or η) in $\psi_i^e(\xi, \eta)$. This can be seen by examining the coefficients $K_{ij}^{\alpha\alpha}$ $(\alpha = 1, 2, 3)$, which contain the products $\psi_i^e \psi_j^e$. Thus we have

$$M = N = \left[\frac{2r+1}{2}\right] = r + 1, \qquad r = \text{order of the element}. \tag{9.132}$$

Note that $r = 1$ for the linear element and $r = 2$ for the quadratic element. The Gauss rule $M \times N$ means that M Gauss points in the ξ-coordinate direction and N Gauss points in the η-coordinate direction. If $N \times N$ is the full Gauss rule, then $(N-1) \times (N-1)$ is the reduced Gauss rule.

Equation (9.122b) can be simplified for static bending, buckling, natural vibration, and transient analysis, as described for the classical plate element in Section 9.5.4. The simplifications are obvious and are therefore not repeated here. Numerical results for bending, buckling, and natural vibration will be discussed next.

Example 9.6 This example is designed to illustrate the effect of full and reduced integration Gauss rules used to evaluate the stiffness coefficients on the deflections and stresses. We consider a simply supported, isotropic square plate under the (sinusoidal) load

$$q(x, y) = q_0 \cos \frac{\pi x}{a} \cos \frac{\pi y}{a}, \tag{9.133}$$

where the origin of the coordinate system is taken at the center of the plate, $-a/2 \leq x \leq a/2, -a/2 \leq y \leq a/2$, and $-h/2 \leq z \leq h/2$. Because of the biaxial symmetry of the solution, the quadrant $0 \leq x \leq a/2$ and $0 \leq y \leq a/2$ is used as the computational domain.

We use the notation nL for $n \times n$ uniform mesh of linear rectangular elements, nQ8 for $n \times n$ uniform mesh of eight-node quadratic elements, and nQ9 for $n \times n$ uniform mesh of nine-node quadratic elements in a quarter plate. Three different Gauss rules are used to evaluate the stiffness coefficients:

1. *Full integration (F).* All stiffness coefficients $[K^{\alpha\beta}]$ of Eq. (9.123) are evaluated using the integration rule $N \times N$ that would yield their exact values.
2. *Selective (mixed) integration (S). All terms of the stiffness coefficients except those containing* A_{44} and A_{55} are evaluated using the full integration rule, and the terms involving A_{44} and A_{55} are evaluated using the reduced integration rule.
3. *Reduced integration (R).* All stiffness coefficients $[K^{\alpha\beta}]$ of Eq. (9.123) are evaluated using the reduced integration rule $(N-1) \times (N-1)$.

Various stresses in the finite element analysis are evaluated at the reduced Gauss points as indicated below:

$$\sigma_{xx}\left(A, A, \frac{h}{2}\right), \quad \sigma_{yy}\left(A, A, \frac{h}{2}\right), \quad \sigma_{xy}\left(B, B, -\frac{h}{2}\right),$$
$$\sigma_{xz}(B, A) \quad \text{at } z = \pm\frac{h}{2}, \qquad \sigma_{yz}(A, B) \quad \text{at } z = 0, \tag{9.134}$$

where the values of (A, B) are given in Table 9.7.

Table 9.7 The Gauss point locations at which the stresses are computed in the finite element analysis

Coord.	Exact	2L	4L	8L	2Q8/2Q9	4Q8/4Q9
A	0.0	$0.125a$	$0.0625a$	$0.03125a$	$0.05283a$	$0.02642a$
B	$0.5a$	$0.375a$	$0.4375a$	$0.46875a$	$0.44717a$	$0.47358a$

The maximum deflection and stresses obtained for the problem are presented in Table 9.8. The following nondimensionalizations are used:

$$\bar{w} = w_0(A, A) \times 10^2 \frac{D}{a^4 q_0}, \qquad \bar{\sigma}_{xx} = \sigma_{xx}\left(A, A, \frac{h}{2}\right)\frac{h^2}{a^2 q_0},$$

$$\bar{\sigma}_{yy} = \sigma_{yy}\left(A, A, \frac{h}{2}\right)\frac{h^2}{a^2 q_0}, \qquad \bar{\sigma}_{xy} = \sigma_{xy}\left(B, B, -\frac{h}{2}\right)\frac{h^2}{a^2 q_0},$$

$$\bar{\sigma}_{xz} = -\sigma_{xz}(B, A)\frac{h}{a q_0}, \qquad \bar{\sigma}_{yz} = -\sigma_{yz}(A, B)\frac{h}{a q_0}. \tag{9.135}$$

An examination of the numerical results presented in Table 9.8 shows that the FSDT finite element with equal interpolation of all generalized displacements does not experience shear locking for thick plates even when the full integration rule is used. Shear locking is evident when the element is used to model thin plates ($a/h \geq 100$) with the full integration rule (F). Also, higher-order elements show less sensitivity for locking but with slower convergence. The element behaves uniformly well for thin and thick plates when the reduced (R) or selectively reduced integration (S) rule is used. It is clear that the effect of shear deformation is to increase the deflections. It should be noted that the nondimensionalization in Eq. (9.210) is such that the CPT solution is independent of the side-to-thickness ratio.

Higher-order elements or refined meshes of lower-order elements experience relatively less locking, but sometimes at the expense of rate of convergence. With the suggested Gauss rule, highly distorted elements tend to have slower rates of convergence but they give reasonably accurate results.

Plates composed of multiple layers are called *laminated plates* (see Reddy [27] and Fig. 9.22). A laminated plate composed of multiple orthotropic layers, with the material axes (x_1, x_2) of each layer oriented at either 0° or 90° to the plate axes (x, y), has the following bending moment–deflection relationships:

$$\begin{Bmatrix} M_{xx} \\ M_{yy} \\ M_{xy} \end{Bmatrix} = \begin{bmatrix} D_{11} & D_{12} & 0 \\ D_{12} & D_{22} & 0 \\ 0 & 0 & D_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon^1_{xx} \\ \varepsilon^1_{yy} \\ \gamma^1_{xy} \end{Bmatrix}, \tag{9.136}$$

where ε^1_{ij} are the bending strains, defined by Eq. (8.117) for the CPT and Eq. (8.276b) for the FSDT. The laminate bending stiffnesses D_{ij} are defined by

$$D_{ij} = \frac{1}{3}\sum_{k=1}^{N} \bar{Q}^{(k)}_{ij}\left(z^3_{k+1} - z^3_k\right). \tag{9.137a}$$

Here N denotes the total number of layers in the laminate, z_k is the z-coordinate of the bottom of the kth layer (measured in the positive z-direction), and $\bar{Q}^{(k)}_{ij}$ are the

Table 9.8 Effect of reduced integration on the nondimensionalized maximum deflections $\bar{w}$ and stresses $\bar{\sigma}$ of simply supported laminate (0/90/90/0) square plates subjected to sinusoidal loading

a/h	Source	$\bar{w}$	$\bar{\sigma}_{xx}$	$\bar{\sigma}_{yy}$	$\bar{\sigma}_{xy}$	$\bar{\sigma}_{xz}$	$\bar{\sigma}_{yz}$
			Finite Element Solutions				
10	2L–F	0.1382	0.0744	0.0744	0.0446	0.1545	0.1545
	2L–R	0.2571	0.1458	0.1458	0.0874	0.1545	0.1545
	2L–S	0.2621	0.1483	0.1483	0.0889	0.1634	0.1634
	1Q9-F	0.2461	0.1402	0.1402	0.0965	0.1626	0.1626
	1Q9-R	0.2720	0.1645	0.1645	0.0985	0.1658	0.1658
	1Q9-S	0.2715	0.1638	0.1638	0.0987	0.405	0.1665
	4L–F	0.2179	0.1436	0.1436	0.0862	0.1537	0.1537
	4L–R	0.2670	0.1781	0.1781	0.1068	0.1814	0.1814
	4L–S	0.2682	0.1788	0.1788	0.1073	0.1837	0.1837
	2Q9-F	0.2682	0.1812	0.1812	0.1100	0.1879	0.1879
	2Q9-R	0.2703	0.1844	0.1844	0.1106	0.1854	0.1854
	2Q9-S	0.2703	0.1844	0.1844	0.1106	0.1855	0.1855
	8L–S	0.2697	0.1871	0.1871	0.1123	0.1892	0.1892
	4Q9-S	0.2702	0.1886	0.1886	0.1132	0.1897	0.1897
			Analytical Solutions				
	FSDT	0.2702	0.1900	0.1900	0.1140	0.1910	0.1910
	CPT	0.2566	0.1900	0.1900	0.1140	0.2387	0.2387
			Finite Element Solutions				
100	2L–F	0.0026	0.0015	0.0015	0.0009	0.1545	0.1545
	2L–R	0.2431	0.1458	0.1458	0.0875	0.1546	0.1546
	2L–S	0.2472	0.1483	0.1483	0.0890	0.1634	0.1634
	1Q9-F	0.2133	0.1223	0.1223	0.0953	0.1567	0.1567
	1Q9-R	0.2583	0.1645	0.1645	0.0985	0.1658	0.1658
	1Q9-S	0.2581	0.1638	0.1638	0.0990	0.1659	0.1659
	4L–F	0.0101	0.0070	0.0070	0.0042	0.1813	0.1813
	4L–R	0.2535	0.1781	0.1781	0.1068	0.1814	0.1814
	4L–S	0.2545	0.1788	0.1788	0.1073	0.1837	0.1837
	2Q9-F	0.2494	0.1731	0.1731	0.1083	0.1846	0.1846
	2Q9-R	0.2569	0.1844	0.1844	0.1106	0.1854	0.1854
	2Q9-S	0.2569	0.1844	0.1844	0.1106	0.1850	0.1850
	8L–S	0.2562	0.1871	0.1871	0.1123	0.1819	0.1819
	4Q9-S	0.2568	0.1886	0.1886	0.1132	0.1896	0.1896
			Analytical Solutions				
	FSDT	0.2568	0.1900	0.1900	0.1140	0.1910	0.1910
	CPT	0.2566	0.1900	0.1900	0.1140	0.2387	0.2387

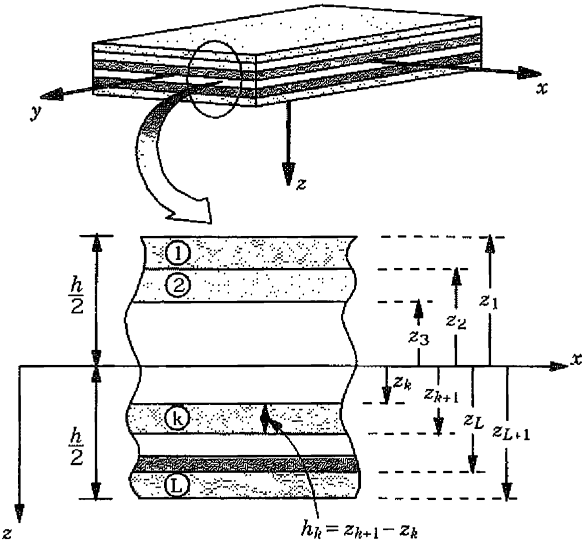


Figure 9.22 The layer numbering and coordinate system used for a typical laminated plate.

elastic stiffnesses of the kth layer referred to the plate coordinates (x, y):

$$\bar{Q}_{11}^{(k)} = \frac{E_1^k}{1 - \nu_{12}^k \nu_{21}^k}, \qquad \bar{Q}_{12}^{(k)} = \frac{\nu_{21}^k E_1^k}{1 - \nu_{12}^k \nu_{21}^k}, \qquad \bar{Q}_{22}^{(k)} = \frac{E_2^k}{1 - \nu_{12}^k \nu_{21}^k},$$

$$\bar{Q}_{16}^{(k)} = 0, \qquad \bar{Q}_{26}^{(k)} = 0, \qquad \bar{Q}_{66}^{(k)} = G_{12}^k, \qquad \bar{Q}_{44}^{(k)} = G_{23}^k, \qquad \bar{Q}_{55}^{(k)} = G_{13}^k. \tag{9.137b}$$

The engineering constants E_1^k, E_2^k, G_{12}^k, G_{23}^k, and ν_{12}^k referred to the principal material axes of the kth layer. Note that in laminated plates the principal material axis x_3 is always taken along the plate axis z that is transverse to the plane of the laminae.

Example 9.7 Consider a square sandwich plate subjected to sinusoidally distributed transverse loading. The sandwich plate is treated as a three-layer plate with different thickness of the face sheets (layers 1 and 3) and the core (layer 2). The face sheets are assumed to be orthotropic, with the principal material coordinates (x_1, x_2, x_3) coinciding with the plate axes (x, y, z), and the material properties are given by

$$E_1 = 25E_2, \qquad E_2 = 10^6 \text{ psi}, \qquad G_{12} = G_{13} = 0.5E_2,$$
$$G_{23} = 0.2E_2, \qquad \nu_{12} = 0.25. \tag{9.138a}$$

The core material is assumed to be transversely isotropic and is characterized by the following material properties:

$$E_1 = E_2 = 10^6 \text{ psi}, \qquad G_{13} = G_{23} = 0.06 \times 10^6 \text{ psi}, \qquad \nu_{12} = 0.25,$$

$$G_{12} = \frac{E_1}{2(1+\nu_{12})} = 0.016 \times 10^6 \text{ psi}, \qquad K_s = \tfrac{5}{6}. \tag{9.138b}$$

Each face sheet is assumed to be one-tenth of the total thickness of the sandwich plate. The finite element results obtained with a uniform 4×4 mesh of eight-node quadratic elements with reduced integration (4Q8–R) are compared with the exact solutions based on FSDT and elasticity solution (ELS) of Pagano [28] in Table 9.9. The deflection is nondimensionalized as

$$\bar{w} = w_0(0,0) \times 10^2 \left(\frac{E_2 h^3}{a^4 q_0}\right). \tag{9.139a}$$

The stresses are nondimensionalized as in Eq. (9.135), and their locations with respect to a coordinate system whose origin is at the center of the plate are as follows:

$$\sigma_{xx}\left(0,0,\frac{h}{2}\right), \quad \sigma_{yy}\left(0,0,\frac{h}{2}\right), \quad \sigma_{xy}\left(\frac{a}{2},\frac{b}{2},-\frac{h}{2}\right),$$
$$\sigma_{xz}\left(0,\frac{b}{2},0\right), \quad \sigma_{yz}\left(\frac{a}{2},0,0\right). \tag{9.139b}$$

The results of Table 9.9 indicate that the effect of shear deformation on deflections is significant in sandwich plates even at large values of a/h (i.e., the thin plate range). The equilibrium-derived transverse shear stresses [see, for example, Eqs. (8.144a–c) for the CLPT] are very close to those predicted by the elasticity theory for $a/h \geq 10$, while those computed from constitutive equations are considerably

Table 9.9 Comparison of nondimensionalized maximum deflections and stresses in simply supported sandwich plates subjected to sinusoidal load ($h_1 = h_3 = 0.1h$, $h_2 = 0.8h$, $K_s = 5/6$)

a/h	Source	$\bar{w}$	$\bar{\sigma}_{xx}$	$\bar{\sigma}_{yy}$	$\bar{\sigma}_{xy}$	$\bar{\sigma}_{xz}$	$\bar{\sigma}_{yz}$
10	ELS[a]	—	1.153	0.1104	0.0717	0.300	0.0527
	Exact[b]	1.5604	1.0457	0.0798	0.0552	0.1374	0.0293
						(0.3134)	(0.0408)[c]
	FEM	1.5603	1.0384	0.0792	0.0548	0.1365	0.0278
20	ELS	—	1.110	0.0700	0.0511	0.317	0.0361
	Exact	1.0524	1.0831	0.0612	0.0466	0.1409	0.0234
						(0.3213)	(0.0325)
	FEM	1.0523	1.0755	0.0608	0.0462	0.1399	0.0233
100	ELS	—	1.098	0.0550	0.0437	0.324	0.0297
	Exact	0.8852	1.0964	0.0546	0.0435	0.1422	0.0213
						(0.3242)	(0.0296)
	FEM	0.8851	1.0887	0.0542	0.0432	0.1412	0.0161
	CLPT	0.8782	1.0970	0.0543	0.0433	(0.3243)	(0.0295)

[a] 3D elasticity solution from [40].
[b] Based on the first-order shear deformation plate theory (FSDT).
[c] Values computed from equilibrium equations.

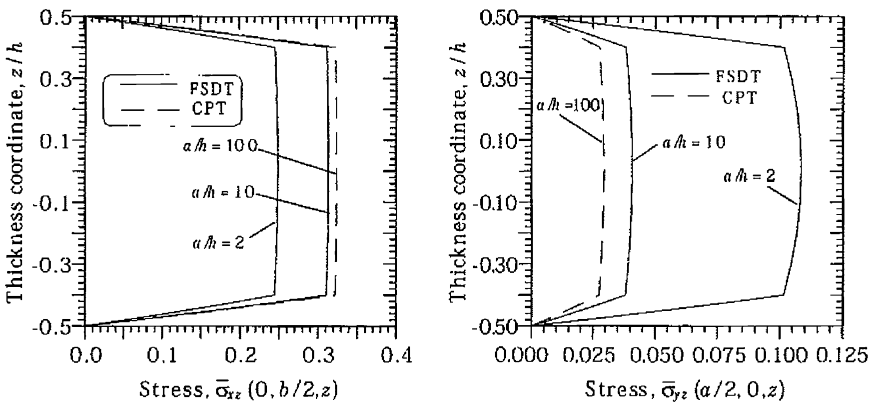


Figure 9.23 Distribution of shear stresses through the thickness of a simply supported sandwich plate under sinusoidal load. (a) σ_{xz}. (b) σ_{yz}.

underestimated for small side-to-thickness ratios. The transverse shear stress component σ_{yz} is significantly overestimated by CLPT. Figures 9.23a and 9.23b show the variation of the transverse shear stresses through the thickness of the sandwich plates for side-to-thickness ratios $a/h = 2$, 10, and 100.

The same sandwich plate discussed above is analyzed for simply supported and clamped boundary conditions when a uniformly distributed load is used. Once again a quarter-plate model is used with 4×4 mesh of quadratic FSDT elements and 8×8 mesh of CPT conforming cubic elements. The results are presented in Table 9.10. The effect of shear deformation on the deflections is even more significant in clamped plates than in simply supported plates.

Example 9.8 Here we consider natural vibration of a simply supported laminated plate. The plate is made of four orthotropic layers. The principal material axes (x_1, x_2) of the first and fourth layers coincide with the plate axes while those of the middle two layers are oriented at 90° to the plate axes (x, y). Such a laminate is designated as (0/90/90/0) or simply $(0/90)_s$, where the latter notation implies that the laminate is a symmetric (about the midplane) laminate. The material properties of each layer (or lamina) are taken to be

$$E_1 = 40E_2, \qquad G_{12} = G_{13} = 0.6E_2, \qquad G_{23} = 0.5E_2, \qquad \nu_{12} = 0.25. \tag{9.140}$$

A 2×2 mesh in a quarter plate is used to obtain the results (rotary inertia included).

Effects of side-to-thickness ratio, integration, and type of element on the nondimensionalized fundamental frequency $\bar{\omega} = \omega(a^2/h)\sqrt{\rho/E_2}$ can be seen from the results presented in Table 9.11. From the results obtained, it is clear that both the reduced integration (R) and selective integration (S) rules give good results for a wide range of side-to-thickness ratios.

Table 9.10 Nondimensionalized maximum deflections and stresses in square sandwich plates with (a) simply supported and (b) clamped boundary conditions ($h_1 = h_3 = 0.1h$, $h_2 = 0.8h$, $K_s = 5/6$)

a/h	Source	$\bar{w} \times 10^2$	$\bar{\sigma}_{xx}$	$\bar{\sigma}_{yy}$	$\bar{\sigma}_{xy}$	$\bar{\sigma}_{xz}$	$\bar{\sigma}_{yz}$
		(a) Simply supported plate under uniformly distributed load					
10	4Q8-R	2.3370	1.5430	0.0883	0.1136	0.2396	0.0991
50	4Q8-R	1.3671	1.5964	0.0526	0.0916	0.2433	0.0881
100	4Q8-R	1.3359	1.5978	0.0514	0.0906	0.2394	0.0880
CLPT	8CC-F[a]	1.3296	1.5830	0.0509	0.0906	—	—
		(b) Clamped plate under uniformly distributed load					
10	4Q9-R[b]	1.2654	0.5018	0.0550	0.0120	0.2318	0.1445
50	4Q9-R	0.3111	0.5356	0.0108	0.0039	0.2406	0.1160
100	4Q9-R	0.2785	0.5347	0.0094	0.0030	0.2400	0.1148
CLPT	8CC-F[a]	0.2951	0.5401	0.0145	0.0605	—	—

[a] 8×8 mesh of conforming cubic elements with full integration for stiffness coefficient evaluation and one-point Gauss rule for stresses.
[b] The 4Q9-S element gives the same results as 4Q9-R.

Table 9.11 Effects of side-to-thickness ratio, integration, and type of element on the nondimensionalized fundamental frequency $\bar{\omega}$ of simply supported square laminates (0/90/90/0)

	Serendipity Element			Lagrange Element			
a/h	F	R	S	F	R	S	Exact
2	5.502	5.503	5.501	5.502	5.503	5.501	5.500
10	15.174	15.179	15.159	15.182	15.193	15.172	15.143
100	19.171	18.841	18.808	19.225	18.883	18.933	18.836

EXERCISES

9.1 Derive the finite element model of the equation

$$-\frac{d}{dx}\left[(1+x)\frac{du}{dx}\right] = f, \qquad 0 < x < 1,$$

for the boundary conditions of the type

$$u(0) = \hat{u}, \qquad \left[(1+x)\frac{du}{dx} + u\right]_{x=1} = \hat{q},$$

and solve the problem for the data $f = 0$, $\hat{u} = 1$, and $\hat{q} = 0$. Use two linear finite elements.

9.2 Use the following procedure to derive the cubic interpolation functions for a line element with equally spaced (four) nodes. Set up the local coordinate $\bar{x}$ at the left-end node (i.e., node 1), and then use the interpolation property of ψ_i,

$$\psi_i(\bar{x}_j) = \delta_{ij}, \qquad i, j = 1, 2, 3, 4,$$

to write ψ_i as a polynomial that vanishes at all $\bar{x}_j, \bar{x}_j \neq \bar{x}_i$,

$$\psi_1(\bar{x}) = c_1(\bar{x} - \bar{x}_2)(\bar{x} - \bar{x}_3)(\bar{x} - \bar{x}_4),$$
$$\psi_2(\bar{x}) = c_2(\bar{x} - \bar{x}_1)(\bar{x} - \bar{x}_3)(\bar{x} - \bar{x}_4),$$

and determine c_1, c_2, etc., such that $\psi_1(\bar{x}_1) = 1, \quad \psi_2(\bar{x}_2) = 1$, etc.

Use the minimum possible number of linear finite elements to analyze the axially loaded structures of Exercises 9.3–9.9.

9.3 Find the stresses and compressions in each section of the composite member shown in Fig. E9.3. Use $E_s = 30 \times 10^6$ psi, $E_a = 10^7$ psi, $E_b = 15 \times 10^6$ psi, and the minimum number of linear elements.

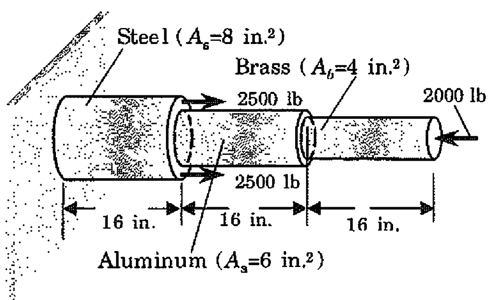


Figure E9.3

9.4 A solid circular brass cylinder ($E_b = 15 \times 10^6$ psi, $d_s = 0.25$ in.) is encased in a hollow circular steel shell ($E_s = 30 \times 10^6$ psi, $d_s = 0.21$ in.). A load of $P = 1{,}330$ lb compresses the assembly as shown in Fig. E9.4. Determine (a) the compression and (b) the compressive forces and stresses in the steel shell and brass cylinder. Use the minimum number of linear finite elements. Assume that the Poisson effect is negligible.

9.5 A rectangular steel bar ($E_s = 30 \times 10^6$ psi) of length 24 in. has a slot in the middle half of its length, as shown in Fig. E9.5. Determine the displacement of the ends due to the axial loads $P = 2{,}000$ lb. Use the minimum number of linear elements.

9.6 The two members in Fig. E9.6 are fastened together and to rigid walls. If the members are stress-free before they are loaded, what will be the stresses and deformations in each after the two 50,000 lb. loads are applied? Use $E_s = 30 \times 10^6$ psi and $E_a = 10^7$ psi; the aluminum rod is 2 in. in diameter and the steel rod is 1.5 in. in diameter.

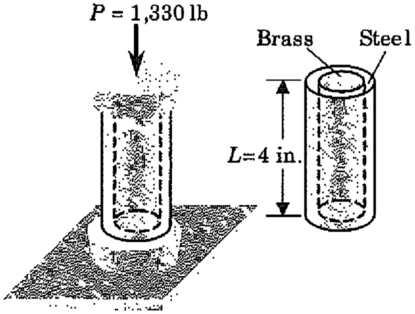


Figure E9.4

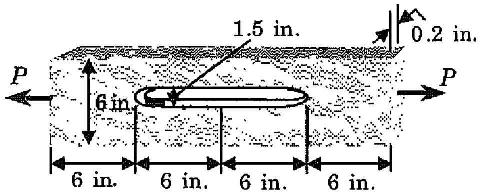


Figure E9.5

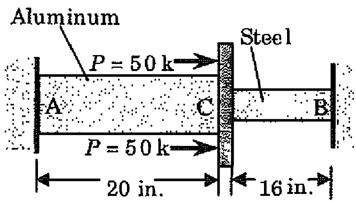


Figure E9.6

9.7 The aluminum and steel pipes shown in Fig. E9.7 are fastened to rigid supports at ends A and B and to a rigid plate C at their junction. Determine the displacement of point C and stresses in the aluminum and steel pipes. Use the minimum number of linear finite elements.

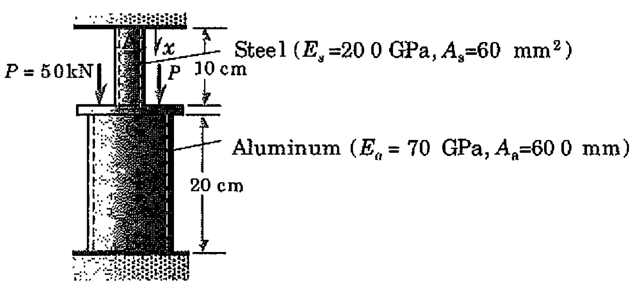


Figure E9.7

9.8 A steel bar ABC is pin-supported at its upper end A to an immovable wall and loaded by a force F_1 at its lower end C, as shown in Fig. E9.8. A rigid horizontal beam BDE is pinned to the vertical bar at B, supported at point D, and carries a load F_2 at end E. Determine the displacements u_B and u_C at points B and C.

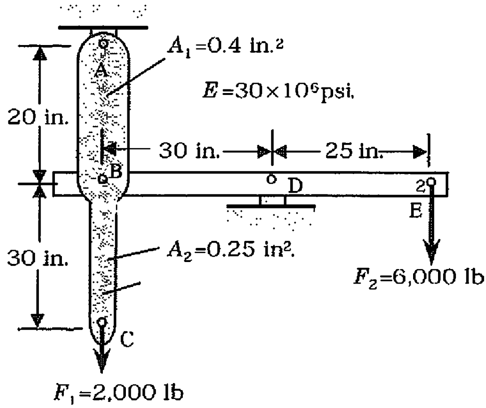


Figure E9.8

9.9 Repeat Exercise 9.8 when point C is spring-supported (see Fig. E9.9).

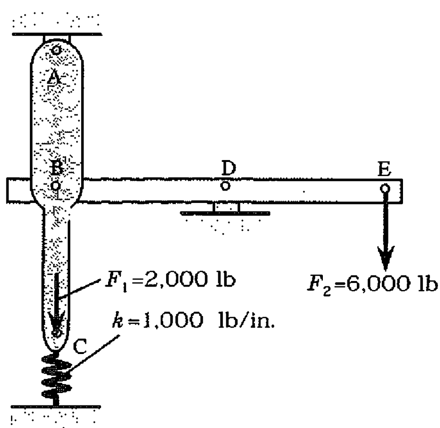


Figure E9.9

9.10–9.17 Use the minimum possible number of Euler–Bernoulli beam elements to analyze (i.e., determine deflections and slopes at the nodes of) the beam structures shown in Figs. E9.10–E9.17.

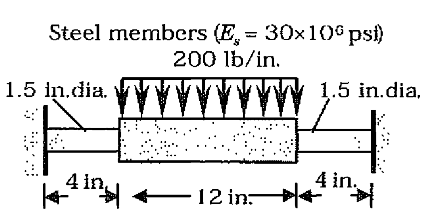


Figure E9.10

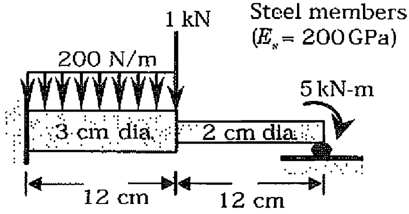


Figure E9.11

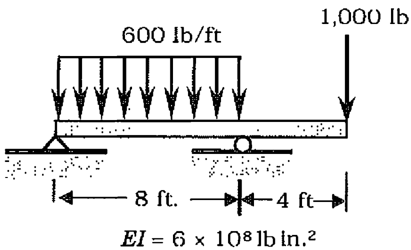


Figure E9.12

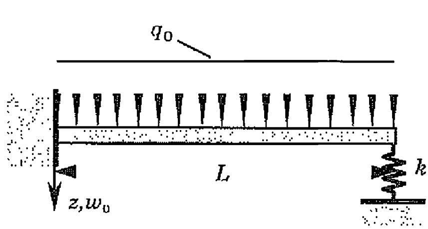


Figure E9.13

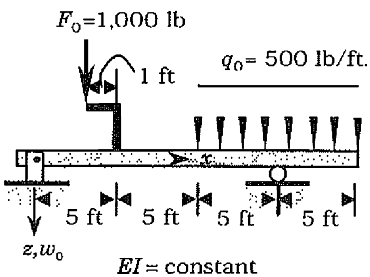


Figure E9.14

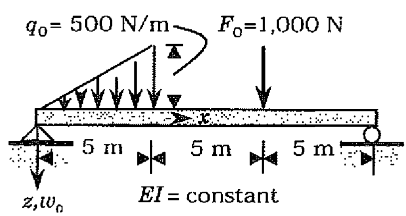


Figure E9.15

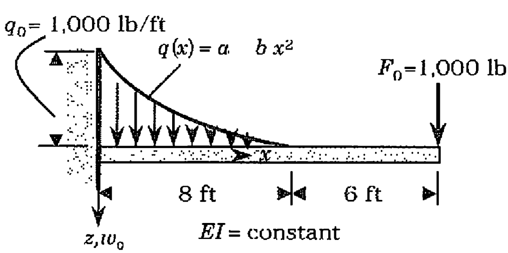


Figure E9.16

9.18 Construct the finite element model, based on the weak form, of the following differential equation, which arises in connection with the buckling of columns:

$$\frac{d^2}{dx^2}\left(EI\frac{d^2w}{dx^2}\right) + \hat{N}\frac{d^2w}{dx^2} = 0, \qquad 0 < x < L,$$

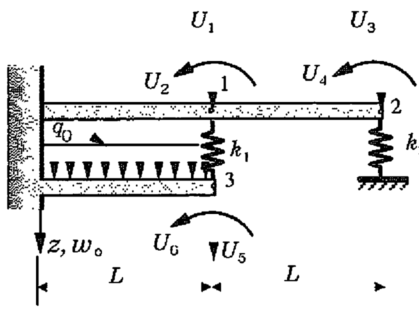


Figure E9.17

where $\hat{N}$ is the buckling load to be determined and EI is the flexural rigidity.

9.19 Evaluate the finite element matrices of Exercise 9.18 using the Hermite cubic polynomials. Take EI constant and make use of the results already available in the text.

9.20 Find the critical buckling load of a simply supported beam using the least number of Euler–Bernoulli beam elements. This problem requires the solution of an eigenvalue problem.

9.21 The axisymmetric bending of a thin, uniform thickness, circular plate can be described by a one-dimensional differential equation. For isotropic material the differential equation is given by [see Eqs. (8.36) and (8.80)]

$$D\frac{1}{r}\frac{d}{dr}\left[-\left(r\frac{d^3w}{dr^3}+\frac{d^2w}{dr^2}\right)+\frac{1}{r}\frac{dw}{dr}\right]+f(r)=0,$$

where r is the radial coordinate and $D = Eh^3/[12(1-\nu^2)]$ is the bending stiffness. Derive the variational formulation and associated finite element model of the equation over a typical element.

9.22 The differential equations governing axisymmetric bending of circular plates according to the first-order shear deformation plate theory are

$$-\frac{1}{r}\frac{d}{dr}\left[rA_{55}\left(\phi+\frac{dw_0}{dr}\right)\right]=q,$$

$$-\frac{1}{r}\frac{d}{dr}\left[\left(rD_{11}\frac{d\phi}{dr}+D_{12}\phi\right)\right]$$
$$+\frac{1}{r}\left(D_{12}\frac{d\phi}{dr}+\frac{D_{22}}{r}\phi\right)+A_{55}\left(\phi+\frac{dw_0}{dr}\right)=0,$$

where A_{55}, D_{11}, D_{12}, and D_{22} are plate material stiffnesses, ϕ is the rotation, w_0 is the transverse deflection, and q is the transverse load. Develop (a) the weak form of the equations over an element and (b) the finite element model of the equations.

9.23 Consider the fourth-order equation governing the Euler–Bernoulli beam theory. Suppose that a two-node element is employed, with *three* primary variables at each node (w_0, θ, and κ), where $\theta = dw_0/dx$ and $\kappa = d^2w_0/dx^2$. Show that the associated Hermite interpolation functions are given by

$$\phi_1 = 2 - 10\frac{x^3}{h^3} + 15\frac{x^4}{h^4} - 6\frac{x^5}{h^5}, \qquad \phi_2 = x\left(1 - 6\frac{x^2}{h^2} + 8\frac{x^3}{h^3} - 3\frac{x^4}{h^4}\right),$$

$$\phi_3 = \frac{x^2}{2}\left(1 - 3\frac{x}{h} + 3\frac{x^2}{h^2} - \frac{x^3}{h^3}\right), \qquad \phi_4 = -10\frac{x^3}{h^3} + 15\frac{x^4}{h^4} - 6\frac{x^5}{h^5},$$

$$\phi_5 = x\left(4\frac{x^2}{h^2} - 7\frac{x^3}{h^3} + 3\frac{x^4}{h^4}\right), \qquad \phi_6 = \frac{x^2}{2}\left(\frac{x}{h} - 2\frac{x^2}{h^2} + \frac{x^3}{h^3}\right),$$

where x is the element coordinate with the origin at node 1.

9.24 Consider the weak form (9.54a) of the Euler–Bernoulli beam element. Use a three-node element with *two* degrees of freedom (w_0, θ), where $\theta \equiv -dw_0/dx$. Derive the Hermite interpolation functions for the element. Compute the element stiffness matrix and force vector.

9.25 Give the variational formulation and associated finite element model of the equation

$$-\frac{\partial}{\partial x}\left(c_{11}\frac{\partial u}{\partial x} + c_{12}\frac{\partial u}{\partial y}\right) - \frac{\partial}{\partial y}\left(c_{21}\frac{\partial u}{\partial x} + c_{22}\frac{\partial u}{\partial y}\right) + c_0 u = f, \quad \text{in } \Omega. \qquad \text{(a)}$$

The coefficients $c_{ij}(i, j = 1, 2)$ and c_0 and the source term f are given functions of position (x, y) in the domain Ω. Equation (a) arises in the study of a number of engineering problems, including torsion of constant cross-section members, conduction heat transfer, irrotational flow in a viscid fluid, transverse deflection of a membrane, and so on.

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10

MIXED VARIATIONAL FORMULATIONS

10.1 INTRODUCTION

10.1.1 General Comments

In this chapter we will cover mixed variational principles and their applications. We define a "mixed" (or sometimes called "hybrid") variational formulation as one where secondary variables of the conventional formulation are also treated as dependent variables along with the primary variables. Most often these formulations are developed with the objective of determining the secondary variables, which are often quantities of practical interest, directly rather than from postcomputations.

In this chapter we will present a brief exposure of mixed variational principles of elasticity, including Reissner's variational principle and the Hellinger–Reissner variational principles [1,2]. Use of variational methods as well as the finite element method to determine solutions based on the mixed variational formulations will also be discussed.

10.1.2 Mixed Variational Principles

To illustrate the basic ideas in the development of mixed variational principles, we shall consider the simple case of axial deformation of a bar. From equilibrium considerations (see Example 4.3) we have

$$-\frac{dN}{dx} = f(x),\ 0 < x < L, \qquad N = \int_A \sigma_{xx}\, dA, \tag{10.1}$$

where L is the length of the bar, f is the body force along the axis x, and σ_{xx} is the axial stress in the bar. The axial force N is related to the axial displacement u through the kinematic and constitutive relations

$$\varepsilon_{xx} = \frac{du}{dx}, \qquad \sigma_{xx} = E\varepsilon_{xx}, \tag{10.2}$$

by [substitute Eqs. (10.2) into the definition of N in Eq. (10.1)]

$$N = EA\frac{du}{dx} \qquad \text{or} \qquad \frac{du}{dx} - \frac{N}{EA} = 0. \tag{10.3}$$

Substituting Eq. (10.3) for N in Eq. (10.1), we arrive at

$$-\frac{d}{dx}\left(EA\frac{du}{dx}\right) = f(x), \qquad 0 < x < L. \tag{10.4}$$

The conventional (or displacement) variational formulation is based on Eq. (10.4). Consequently, the displacement u is the primary dependent variable of the formulation, and N is postcomputed from the known displacement using Eq. (10.3). The variational (or weak) formulation of Eq. (10.4) is obtained by either following the steps presented in Section 7.4.1 (Model Equation 1) or using the principle of minimum total potential energy. The total potential energy functional is given by [see Eq. (4.109)]

$$\Pi(u) = \int_0^L \left[\frac{EA}{2}\left(\frac{du}{dx}\right)^2 - fu\right] dx - \text{WPL}, \tag{10.5}$$

where WPL stands for work done by applied point loads. Equation (10.4) is the Euler equation of this functional.

In a mixed formulation, we treat Eqs. (10.1) and (10.3) as a pair of coupled equations in two independent variables u and N. The mixed variational formulation of this pair can be developed using the same steps as discussed in Section 7.4.1 (or see Examples 7.10 and 7.11). We have

$$0 = \int_0^L w_1\left(-\frac{dN}{dx} - f\right) dx, \tag{10.6a}$$

$$0 = \int_0^L w_2\left(\frac{du}{dx} - \frac{N}{EA}\right) dx, \tag{10.6b}$$

where w_1 and w_2 are the weight functions, which may be interpreted as follows. If $w_1 \cdot f\, dx$ were to represent the work done, w_1 must be an axial displacement. Similarly, $w_2 \cdot (du/dx)dx$ represents work done if w_2 is an axial force. Thus, $w_1 \sim \delta u$ and $w_2 \sim \delta N$. Since N appears without differential in Eq. (10.6b), in the interest of lowering the order of approximation used for N, we carry out the integration by parts of the first term in Eq. (10.6a). One reason is that we do not wish to make u a secondary variable by carrying out integration by parts of the first term in Eq. (10.6b). There are other reasons that will be apparent below. We obtain

$$0 = \int_0^L w_1 \left(-\frac{dN}{dx} - f \right) dx$$

$$= \int_0^L \left(\frac{dw_1}{dx} N - w_1 f \right) dx - [w_1 N]_0^L, \tag{10.7a}$$

$$0 = \int_0^L w_2 \left(\frac{du}{dx} - \frac{N}{EA} \right) dx, \tag{10.7b}$$

Now it is clear that, since $w_1 \sim u$, u is the primary variable and N is the secondary variable. Note that N is also an independent variable of the formulation but it is *not* a primary variable.

Equations (10.7a,b) can be written, since $w_1 = \delta u$ and $w_2 = \delta N$ are linearly independent, as

$$0 = \int_0^L \left(\frac{d\delta u}{dx} N - \delta u f + \delta N \frac{du}{dx} - \frac{N}{EA} \delta N \right) dx - [\delta u N]_0^L$$

$$= \delta \int_0^L \left(\frac{du}{dx} N - \frac{N^2}{2EA} - f u \right) dx - [\delta u N]_0^L \, . \tag{10.8}$$

The boundary term can be simplified for a specific problem. Whenever u is specified at $x = 0$ or $x = L$, δu is zero there. If u is not specified at a boundary point, then N is known (either as zero or some specified value) at that point. Suppose that u is zero at $x = 0$ (essential boundary condition) and u is not specified at $x = L$, but N is specified to be P at $x = L$. Then we have

$$0 = \delta \left[\int_0^L \left(\frac{du}{dx} N - \frac{N^2}{2EA} - f u \right) dx - u(L) P \right]. \tag{10.9}$$

This is the mixed formulation associated with Eqs. (10.1) and (10.3). Clearly, the functional

$$\Pi_m(u, N) = \int_0^L \left(\frac{du}{dx} N - \frac{N^2}{2EA} - f u \right) dx - u(L) P \tag{10.10}$$

yields Eqs. (10.1) and (10.3) as its Euler equations along with the natural boundary condition

$$N - P = 0 \qquad \text{at } x = L. \tag{10.11}$$

In other words, $\delta \Pi_m = 0$ is equivalent to Eqs. (10.1), (10.3), and (10.11). Note that Π_m does not attain a minimum or maximum with respect to u and N. It may be minimum with respect to u but maximum with respect to N. Thus the mixed variational principle can be stated as: find (u, N) such that

$$0 = \delta\Pi_m = \delta_u \Pi_m + \delta_N \Pi_m, \qquad \text{or} \qquad \delta_u \Pi_m = 0 \qquad \text{and} \qquad \delta_N \Pi_m = 0. \tag{10.12}$$

10.1.3 Extremum and Stationary Behavior of Functionals

It can be shown that $\Pi(u)$ of Eq. (10.5) attains a minimum when u is the solution of Eq. (10.4) and satisfies the boundary conditions

$$u(0) = u_0 = 0, \qquad \left(EA\frac{du}{dx}\right)_{x=L} = P. \tag{10.13}$$

To establish this, suppose that $\bar{u}(x) = u(x) + \alpha v(x)$ is an arbitrary element from the set of admissible functions (i.e., $\bar{u}$ satisfies the specified geometric boundary conditions and is differentiable as required by the functional Π), where u is the solution of Eq. (10.4), α is a real number, and v is an element from the space of admissible variations (i.e., v satisfies the homogeneous form of the specified geometric boundary conditions and is differentiable as required by Π). Then we have

$$\begin{aligned}
\Pi(\bar{u}) &= \int_0^L \left[\frac{EA}{2}\left(\frac{d\bar{u}}{dx}\right)^2 - f\bar{u}\right] dx - \bar{u}(L)P \\
&= \int_0^L \left[\frac{EA}{2}\left(\frac{du}{dx}\right)^2 - fu\right] dx - u(L)P \\
&\quad + \alpha^2 \int_0^L \frac{EA}{2}\left(\frac{dv}{dx}\right)^2 dx \\
&\quad + \alpha\left[\int_0^L \left(EA\frac{du}{dx}\frac{dv}{dx} - fv\right) dx - v(L)P\right] \\
&= \Pi(u) + \alpha^2 \int_0^L \frac{EA}{2}\left(\frac{dv}{dx}\right)^2 dx \\
&\quad + \alpha \int_0^L \left[-\frac{d}{dx}\left(EA\frac{du}{dx}\right) - f\right] v\, dx + \alpha\left[EA\frac{du}{dx} - P\right] v(L) \\
&= \Pi(u) + \alpha^2 \int_0^L \frac{EA}{2}\left(\frac{dv}{dx}\right)^2 dx \geq \Pi(u),
\end{aligned} \tag{10.14}$$

where in arriving at the last step we have used the fact that u satisfies Eqs. (10.4) and (10.13). It is clear from Eq. (10.13) that $\Pi(\bar{u}) > \Pi(u)$ for any $\alpha \neq 0$, and $\Pi(\bar{u}) = \Pi(u)$ if and only if $\alpha = 0$ (i.e., $\bar{u} = u$). Hence $\Pi(u)$ attains its minimum only at $\bar{u} = u$, and for any other admissible function $\bar{u}$ we have $\Pi(\bar{u}) > \Pi(u)$. This is essentially the statement of the principle of minimum total potential energy [see Eqs. (4.87)–(4.90) and (5.30)].

Next, we establish the stationarity of $\Pi_m(u, N)$. Let $\bar{u} = u + \alpha v$ and $\bar{N} = N + \beta Q$, where α and β are real numbers, and (u, N) satisfy Eqs. (10.1) and (10.3) inside the domain, Eq. (10.11) at $x = L$, and $u = 0$ at $x = 0$. Then

$$\begin{aligned}
\Pi_m(\bar{u}, \bar{N}) &= \int_0^L \left(\frac{d\bar{u}}{dx}\bar{N} - \frac{\bar{N}^2}{2EA} - f\bar{u} \right) dx - \bar{u}(L)P \\
&= \int_0^L \left(\frac{du}{dx}N - \frac{N^2}{2EA} - fu \right) dx - u(L)P \\
&\quad + \alpha \left[\int_0^L \left(\frac{dv}{dx}N - fv \right) dx - v(L)P \right] \\
&\quad + \beta \int_0^L \left(\frac{du}{dx} - \frac{N}{EA} \right) Q\, dx \\
&\quad - \frac{\beta^2}{2} \int_0^L \frac{Q^2}{2EA}\, dx + \alpha\beta \int_0^L \frac{dv}{dx} Q\, dx \qquad (10.15a) \\
&= \Pi_m(u, N) - \frac{\beta^2}{2} \int_0^L \frac{Q^2}{2EA}\, dx + \alpha\beta \int_0^L \frac{dv}{dx} Q\, dx. \qquad (10.15b)
\end{aligned}$$

The third and fourth lines of Eq. (10.15a) vanish by virtue of Eqs. (10.1), (10.3), and (10.11). It is clear that $\Pi_m(u, \bar{N})$ decreases from its value at equilibrium when u is kept at its actual value u (i.e., $\alpha = 0$) and N is changed to $\bar{N} = N + \beta Q$; that is,

$$\Pi_m(u, \bar{N}) < \Pi_m(u, N). \qquad (10.16a)$$

The value of the functional is unchanged when N is kept at its actual value (i.e., $\beta = 0$) and u is changed to $\bar{u} = u + \alpha v$. In fact, if the end $x = L$ is connected to an elastic spring, one can show that

$$\Pi_m(\bar{u}, N) \geq \Pi_m(u, N). \qquad (10.16b)$$

Thus, Π_m exhibits a stationary behavior and not an extremum character.

10.2 STATIONARY VARIATIONAL PRINCIPLES

10.2.1 The Minimum Total Potential Energy Principle

The principles of minimum total potential energy and maximum total complementary energy are both extremum principles. Recall from Eq. (4.87) that a functional $\Pi(\mathbf{u})$ is a minimum at the displacement vector $\mathbf{u}$ if and only if

$$\Pi(\bar{\mathbf{u}}) \geq \Pi(\mathbf{u}) \qquad \text{for all} \quad \bar{\mathbf{u}} \qquad (10.17)$$

and the equality holds only if $\bar{\mathbf{u}} = \mathbf{u}$.

To show that the total potential energy of a linear elasticity body is the minimum at its equilibrium configuration, consider the total potential energy functional [of an anisotropic, linear elastic body under the assumption of infinitesimal strains; cf. Eq. (4.114)]:

$$\Pi(\mathbf{u}) = \int_V \left(\frac{1}{2}C_{ijkl}e_{kl}e_{ij} - f_i u_i\right) dV - \int_{S_2} \hat{t}_i u_i \, ds, \tag{10.18a}$$

where $\mathbf{u}$ is the displacement vector, $\mathbf{f}$ is the body force vector, $\hat{\mathbf{t}}$ is the specified traction vector on boundary S_2 of the volume V occupied by the body, C_{ijkl} are the components of the fourth-order elasticity tensor, and e_{ij} are the components of the infinitesimal strain tensor:

$$e_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}). \tag{10.18b}$$

In Eqs. (10.18a,b), summation on repeated indices is implied, and a comma followed by a subscript i denotes differentiation with respect to x_i. The geometric boundary condition on $\mathbf{u}$ is

$$\mathbf{u} = \hat{\mathbf{u}} \qquad \text{on } S_1, \tag{10.19}$$

where $S_1 + S_2 = S$, the total boundary of V.

Let $\mathbf{u}$ be the true displacement field and $\bar{\mathbf{u}}$ be an arbitrary but admissible displacement field. Then $\bar{\mathbf{u}}$ is of the form

$$\bar{\mathbf{u}} = \mathbf{u} + \alpha\mathbf{v},$$

where $\mathbf{v}$ is a sufficiently differentiable function that satisfies the homogeneous form of the essential boundary condition $\mathbf{v} = \mathbf{0}$ on S_1. From Eq. (10.18a), we have

$$\Pi(\mathbf{u} + \alpha\mathbf{v}) = \int_V \left[\frac{1}{2}C_{ijkl}\left(e_{kl} + \alpha g_{kl}\right)\left(e_{ij} + \alpha g_{ij}\right) - f_i\left(u_i + \alpha v_i\right)\right] dV - \int_{S_2} \hat{t}_i (u_i + \alpha v_i)\, ds,$$

where

$$g_{ij} = \frac{1}{2}(v_{i,j} + v_{j,i}).$$

Collecting the terms, we obtain (because $C_{ijkl} = C_{klij}$)

$$\Pi(\bar{\mathbf{u}}) = \Pi(\mathbf{u}) + \alpha\left[\int_V \left(-f_i v_i + C_{ijkl}e_{kl}g_{ij} + \frac{\alpha}{2}C_{ijkl}g_{ij}g_{kl}\right) dV - \int_{S_2} \hat{t}_i v_i \, ds\right]. \tag{10.20}$$

Using the equilibrium equations, we can write

$$
\begin{aligned}
-\int_V f_i v_i \, dV &= \int_V \sigma_{ij,j} v_i \, dV = \int_V C_{ijkl} e_{kl,j} v_i \, dV \\
&= -\int_V C_{ijkl} e_{kl} v_{i,j} \, dV + \int_{S_2} C_{ijkl} e_{kl} v_i n_j \, ds \\
&= -\int_V C_{ijkl} e_{kl} g_{ij} \, dV + \int_{S_2} \hat{t}_i v_i \, dS, \qquad (10.21)
\end{aligned}
$$

where the condition $v_i = 0$ on S_1 is used in arriving at the last step. Substituting Eq. (10.21) into Eq. (10.20), we obtain

$$
\Pi(\bar{\mathbf{u}}) = \Pi(\mathbf{u}) + \frac{\alpha^2}{2} \int_V C_{ijkl} g_{ij} g_{kl} \, dV. \qquad (10.22)
$$

In view of the nonnegative nature of the second term on the right-hand side of Eq. (10.22), it follows that

$$
\Pi(\bar{\mathbf{u}}) \geq \Pi(\mathbf{u}), \qquad (10.23)
$$

and $\Pi(\bar{\mathbf{u}}) = \Pi(\mathbf{u})$ only if the quadratic expression $\frac{1}{2} C_{ijkl} g_{ij} g_{kl}$ is zero. Due to the positive-definiteness of the strain energy density, the quadratic expression is zero only if $v_i = 0$, which in turn implies $\bar{u}_i = u_i$. Thus, Eq. (10.23) implies that of all admissible displacement fields the body can assume, the true one is that which makes the total potential energy a minimum. Therefore, the total potential energy principle is a minimum principle. Similar arguments can be made for the total complementary energy. It should be noted that the consideration of whether a functional is minimum or maximum does not enter the calculation of the Euler equations. Such a consideration may be used in determining approximate solutions.

10.2.2 The Hellinger–Reissner Variational Principle

A stationary principle is one in which the functional attains neither a minimum nor a maximum in its arguments. In fact, a functional can attain a maximum with respect to one set of variables (while the others are fixed) and a minimum with respect to another set of variables involved in the functional, as discussed earlier. An example of such functionals is provided by the functional based on the Lagrange multiplier method (see Section 4.4.8). For example, the functional in Eq. (4.132) attains a minimum with respect to (w_0, ϕ) and a maximum with respect to λ. Stationary principles, also called *mixed* variational principles, are of special importance in the analysis of structures. Here we consider the Hellinger–Reissner variational principle [3–6] for an elastic body. The stationary functional is constructed from the equations of an elastic continuum by treating all the dependent variables as independent of each other. The stationary conditions of the principle are the strain–displacement equations, stress–strain equations, stress–equilibrium equations, and both natural and essential boundary conditions—in short, all of the governing equations of elasticity (see Oden and Reddy [1]). Here we consider the principle for an elastic body undergoing large displacements (static case).

Recall the total potential energy functional Π from Eq. (4.114):

$$\Pi(\mathbf{u}) = \int_V \left[\frac{\mu}{4}(u_{i,j} + u_{j,i})(u_{i,j} + u_{j,i}) + \frac{\lambda}{2}u_{i,i}u_{k,k} - f_i u_i\right] dV - \int_{S_2} \hat{t}_i u_i \, ds, \tag{10.24}$$

where μ and λ denote the Lamé constants. In writing the total potential energy functional Π, it was assumed that the displacements and strains are related by a kinematic relationship (10.18b) and that the displacement field satisfies the specified boundary conditions on S_1 [see Eq. (10.19)]. The principle of minimum total potential energy gives (see Example 4.11) the equilibrium equations (4.117a) and traction boundary conditions (4.117b) as the Euler equations:

$$\mu(u_{i,jj} + u_{j,ij}) + \lambda u_{k,ki} + f_i = 0 \qquad \text{in } V, \tag{10.25a}$$

$$t_i \equiv \left[\mu\left(u_{i,j} + u_{j,i}\right) + \lambda u_{k,k}\delta_{ij}\right] n_j = \hat{t}_i \qquad \text{on } S_2. \tag{10.25b}$$

Here n_i denote the components of the unit normal vector on the boundary S of the volume V occupied by the elastic body.

Now suppose that we wish to include the strain–displacement relations (3.20)

$$\varepsilon_{ij} = \tfrac{1}{2}(u_{i,j} + u_{j,i} + u_{m,i}u_{m,j}) \tag{10.26}$$

and displacement boundary conditions (10.19)

$$u_i = \hat{u}_i \quad (i = 1, 2, 3) \quad \text{on } S_1 \tag{10.27}$$

into the variational statement by treating them as constraints. Toward this end, we first rewrite the total potential energy functional Π of Eq. (10.24) in terms of the nonlinear strains [i.e., revert to Eq. (10.18a)]:

$$\Pi(u_i, \varepsilon_{ij}) = \int_V (U_0(\varepsilon_{ij}) - f_i u_i) dV - \int_{S_2} \hat{t}_i u_i \, ds, \tag{10.28}$$

where U_0 is the strain energy density function. Next, let $\lambda_{ij} = \lambda_{ji}$ $(i, j = 1, 2, 3)$ and $\mu_i (i = 1, 2, 3)$ be the Lagrange multipliers associated with the strain–displacement relations (10.26) and the displacement boundary conditions (10.27), respectively.

Using the Lagrange multiplier method, we introduce the new functional:

$$\begin{aligned}
\Pi_{HW}(u_i, \varepsilon_{ij}, \lambda_{ij}, \mu_i) &= \Pi(u_i, \varepsilon_{ij}) - \int_{S_1} \mu_i \left(u_i - \hat{u}_i\right) ds \\
&\quad + \int_V \lambda_{ij}\left[\tfrac{1}{2}\left(u_{i,j} + u_{j,i} + u_{m,i}u_{m,j}\right) - \varepsilon_{ij}\right] dV \\
&= \int_V \left\{\left[\tfrac{1}{2}\left(u_{i,j} + u_{j,i} + u_{m,i}u_{m,j}\right) - \varepsilon_{ij}\right]\lambda_{ij} + U_0(\varepsilon_{ij})\right. \\
&\quad \left. - f_i u_i\right\} dV - \int_{S_1} \mu_i\left(u_i - \hat{u}_i\right) ds - \int_{S_2} \hat{t}_i u_i \, ds \qquad (10.29)
\end{aligned}$$

where $S_1 + S_2 = S$ is the total boundary, and $\hat{u}_i$ and $\hat{t}_i$ are the specified displacement and traction components, respectively, on S_1 and S_2.

We now wish to check if the Euler equations of the new functional Π_{HW} are indeed the governing equations of elasticity. Setting the first variation of Π_{HW} to zero,

$$0 = \delta\Pi_{HW} \equiv \delta_{\mathbf{u}}\Pi_{HW} + \delta_\varepsilon\Pi_{HW} + \delta_\lambda\Pi_{HW} + \delta_\mu\Pi_{HW}, \qquad (10.30)$$

and noting that the variations in $\mathbf{u}$, ε, λ and μ are arbitrary, it follows that

$$\delta_{\mathbf{u}}\Pi_{HW} = 0, \qquad \delta_\varepsilon\Pi_{HW} = 0, \qquad \delta_\lambda\Pi_{HW} = 0, \qquad \delta_\mu\Pi_{HW} = 0. \qquad (10.31)$$

Carrying out the variations with respect to the dependent variables, we obtain

$$\begin{aligned}
\delta\Pi_{HW} &= \int_V \left[\tfrac{1}{2}\left(\delta u_{i,j} + \delta u_{j,i} + \delta u_{m,i}u_{m,j} + u_{m,i}\delta u_{m,j}\right)\lambda_{ij} - f_i\delta u_i\right] dV \\
&\quad - \int_{S_1} \mu_i \delta u_i \, ds - \int_{S_2} \hat{t}_i \delta u_i \, ds + \int_V \left(-\delta\varepsilon_{ij}\lambda_{ij} + \frac{\partial U_0}{\partial \varepsilon_{ij}}\delta\varepsilon_{ij}\right) dV \\
&\quad + \int_V \left[\tfrac{1}{2}\left(u_{i,j} + u_{j,i} + u_{m,i}u_{m,j}\right) - \varepsilon_{ij}\right]\delta\lambda_{ij} \, dV - \int_{S_1} \delta\mu_i (u_i - \hat{u}_i) \, ds \\
&= \int_V \left[\delta u_{i,j}(\delta_{im} + u_{i,m})\lambda_{mj} + \left(\frac{\partial U_0}{\partial \varepsilon_{ij}} - \lambda_{ij}\right)\delta\varepsilon_{ij} - f_i \delta u_i\right] dV \\
&\quad + \int_V \left[\tfrac{1}{2}(u_{i,j} + u_{j,i} + u_{m,i}u_{m,j}) - \varepsilon_{ij}\right]\delta\lambda_{ij} \, dV \\
&\quad - \int_{S_1} \mu_i \delta u_i \, ds - \int_{S_2} \hat{t}_i \delta u_i \, ds - \int_{S_1} \delta\mu_i (u_i - \hat{u}_i) \, ds. \qquad (10.32)
\end{aligned}$$

Using integration by parts, the first term in Eq. (10.32) can be expressed as

$$\begin{aligned}
\int_V \delta u_{i,j}(\delta_{im} + u_{i,m})\lambda_{mj} \, dV &= -\int_V \lambda_{mj}\left(\delta_{im} + u_{i,m}\right)_{,j} \delta u_i \, dV \\
&\quad + \int_S n_j \lambda_{mj}(\delta_{im} + u_{i,m})\delta u_i \, ds. \qquad (10.33)
\end{aligned}$$

Substituting Eq. (10.33) into Eq. (10.32), and setting the coefficients of δu_i, $\delta\varepsilon_{ij}$, and $\delta\lambda_{ij}$ in V, δu_i on S_1 and S_2, and $\delta\mu_i$ on S_1 to zero, we obtain

$$\delta u_i\colon \quad [\lambda_{mj}(\delta_{im} + u_{i,m})]_{,j} + f_i = 0 \quad \text{in } V, \tag{10.34}$$

$$\delta\varepsilon_{ij}\colon \quad \frac{\partial U_0}{\partial \varepsilon_{ij}} - \lambda_{ij} = 0 \quad \text{in } V, \tag{10.35}$$

$$\delta\lambda_{ij}\colon \quad \tfrac{1}{2}(u_{i,j} + u_{j,i} + u_{m,i}u_{m,j}) - \varepsilon_{ij} = 0 \quad \text{in } V, \tag{10.36}$$

$$\delta u_i\colon \quad n_j\lambda_{mj}(\delta_{im} + u_{i,m}) - \mu_i = 0 \quad \text{on } S_1, \tag{10.37}$$

$$\delta u_i\colon \quad n_j\lambda_{mj}(\delta_{im} + u_{i,m}) - \hat{t}_i = 0 \quad \text{on } S_2, \tag{10.38}$$

$$\delta\mu_i\colon \quad u_i - \hat{u}_i = 0 \quad \text{on } S_1. \tag{10.39}$$

Equations (10.35) and (10.37) can be interpreted as the definitions of the Lagrange multipliers λ_{ij} and μ_i, respectively. Clearly, they have the meaning of stress and traction components, respectively. Thus the functional Π_{HW} gives the equilibrium equations (10.34), constitutive equations (10.35), strain–displacement relations (10.36), traction boundary conditions (10.38), and displacement boundary conditions (10.39) as the Euler equations. Consequently, finding the stationary values of Π_{HW} is stated as a variational principle, and it is termed the Hellinger–Reissner variational principle for an elastic body.

Example 10.1 Consider the following equations of the Euler–Bernoulli theory of beams:

$$\text{Kinematics} \qquad \kappa = -\frac{d^2w}{dx^2}, \qquad w(0) = w_0 \tag{10.40}$$

$$\text{Constitutive equation} \qquad M = EI\kappa, \qquad -\frac{dw}{dx}(0) = \theta_0 \tag{10.41}$$

$$\text{Kinetics (equilibrium)} \qquad -\frac{d^2M}{dx^2} = q, \qquad M(L) = M_L, \qquad \frac{dM}{dx}(L) = V_L, \tag{10.42}$$

where w_0 and θ_0 denote specified displacement and rotation, respectively. The Hellinger–Reissner type variational principle for the Euler–Bernoulli beams is given by

$$\Pi_{HW}(w, \kappa, \lambda, \mu_1, \mu_2) = \Pi(\kappa) + \int_0^L \left(\frac{d^2w}{dx^2} + \kappa\right)\lambda\, dx + \mu_1\,[w(0) - w_0] + \mu_2\left[\left(\frac{dw}{dx}\right)\Big|_0 + \theta_0\right], \tag{10.43}$$

where $\Pi(\kappa)$ is the total potential energy functional expressed in terms of the curvature κ. The functional Π accounts for the moment and shear force boundary conditions because they are the natural boundary conditions for Π. We have

$$\Pi_{HW}(w, \kappa, \lambda, \mu_1, \mu_2) = \int_0^L \left[\frac{EI}{2}\kappa^2 + \left(\frac{d^2w}{dx^2} + \kappa\right)\lambda - qw\right] dx$$
$$- M_L\left(-\frac{dw}{dx}\right)\bigg|_L + V_L w(L)$$
$$+ \mu_1[w(0) - w_0] + \mu_2\left[\left(\frac{dw}{dx}\right)\bigg|_0 + \theta_0\right]. \qquad (10.44)$$

The Euler equations of functional Π_{HW} are

$$\delta\kappa:\quad EI\kappa + \lambda = 0, \qquad \text{in } 0 < x < L, \qquad (10.45)$$

$$\delta w:\quad \frac{d^2\lambda}{dx^2} - q = 0, \qquad \text{in } 0 < x < L, \qquad (10.46)$$

$$\delta\lambda:\quad \frac{d^2w}{dx^2} + \kappa = 0, \qquad \text{in } 0 < x < L, \qquad (10.47)$$

$$\frac{d\delta w}{dx}:\quad -\lambda(0) + \mu_2 = 0, \qquad \lambda(L) + M_L = 0, \qquad (10.48)$$

$$\delta w:\quad -\left(\frac{d\lambda}{dx}\right)\bigg|_0 + \mu_1 = 0, \qquad \left(\frac{d\lambda}{dx}\right)\bigg|_L + V_L = 0, \qquad (10.49)$$

$$\delta\mu_1:\quad w(0) - w_0 = 0, \qquad (10.50)$$

$$\delta\mu_2:\quad \frac{dw}{dx}(0) + \theta_0 = 0. \qquad (10.51)$$

From a comparison of Eqs. (10.40)–(10.42) with (10.45)–(10.49), we note that the Lagrange multipliers λ, μ_1, and μ_2 are given by

$$\lambda(x) = -M(x), \qquad \mu_1 = \left(\frac{dM}{dx}\right)\bigg|_0, \qquad \mu_2 = -M(0). \qquad (10.52)$$

Example 10.2 Here we develop a mixed formulation of the Timoshenko beam theory. In this formulation the displacements (w_0, ϕ) and strains $(\kappa_{xx}, \gamma_{xz})$ are treated as the dependent variables. The variational statement associated with this mixed formulation is given by the stationarity of the following functional (see Oden and Reddy [1], p. 116):

$$\Pi_m = \int_0^L \left[EI\left(\frac{d\phi}{dx} - \frac{1}{2}\kappa_{xx}\right)\kappa_{xx} + GAK_s\left(\frac{dw_0}{dx} + \phi - \frac{1}{2}\gamma_{xz}\right)\gamma_{xz} - qw_0\right] dx$$
$$- V_1 w_0(0) - V_2 w_0(L) - M_1\phi(0) - M_2\phi(L), \qquad (10.53)$$

where

$$V_1 = [-GAK_s\gamma_{xz}]_0, \qquad V_2 = [GAK_s\gamma_{xz}]_L,$$
$$M_1 = [-EI\kappa_{xx}]_0, \qquad M_2 = [EI\kappa_{xx}]_L. \tag{10.54}$$

The Euler–Lagrange equations associated with the functional in Eq. (10.53) are

$$w_0: \quad -\frac{d}{dx}[GAK_s\gamma_{xz}] - q = 0, \tag{10.55}$$

$$\delta\phi: \quad -\frac{d}{dx}[EI\kappa_{xx}] + GAK_s\gamma_{xz} = 0, \tag{10.56}$$

$$\delta\kappa_{xx}: \quad \frac{d\phi}{dx} - \kappa_{xx} = 0, \tag{10.57}$$

$$\delta\gamma_{xz}: \quad \left(\frac{dw_0}{dx} + \phi\right) - \gamma_{xz} = 0. \tag{10.58}$$

Note that the first two equations represent the equilibrium equations (9.72) and (9.71) expressed in terms of the curvature κ_{xx} and shear strain γ_{xz}, while the last two equations are the kinematic relations [see Eq. (9.68)].

10.2.3 The Reissner Variational Principle

A special case of the Hellinger–Reissner variational principle is the Reissner principle, which is obtained by eliminating strains ε_{ij} by assuming that the strains are related to the stresses $\sigma_{ij} = \lambda_{ij}$ by the constitutive relations

$$\varepsilon_{ij} = \frac{\partial U_0^*}{\partial \sigma_{ij}}, \qquad U_0^* = \sigma_{ij}\varepsilon_{ij} - U_0. \tag{10.59}$$

where U_0^* is the complementary strain energy density. Then we obtain from Eq. (10.29)

$$\Pi_R(u_i, \sigma_{ij}) = \int_V \left[\tfrac{1}{2}(u_{i,j} + u_{j,i} + u_{m,i}u_{m,j})\sigma_{ij} - U_0^*(\sigma_{ij}) - f_i u_i\right] dV$$
$$- \int_{S_1} t_i(u_i - \hat{u}_i)\, ds - \int_{S_2} \hat{t}_i u_i\, ds. \tag{10.60}$$

The Euler equations are given by Eqs. (10.34), (10.37)–(10.39), and

$$\delta\sigma_{ij}: \quad \tfrac{1}{2}(u_{i,j} + u_{j,i} + u_{m,i}u_{m,j}) - \frac{\partial U_0^*}{\partial \sigma_{ij}} = 0 \qquad \text{in } V. \tag{10.61}$$

10.3 VARIATIONAL SOLUTIONS BASED ON MIXED FORMULATIONS

The advantage of stationary principles lies in incorporating all of the governing equations into a single functional. Such functionals form the basis of a variety of new finite element models that often yield better accuracies for stresses than the displacement finite element models (which are based on the total potential energy functional).

The variational methods presented in Chapter 7 can also be used to determine solutions of problems based on mixed formulations. The variational methods require a variational statement to determine the undetermined parameters of the approximation, and the variational statement can be anything that is equivalent to the governing equations. Thus the Ritz solution, for example, of an Euler–Bernoulli beam can be determined using the principle of minimum potential energy or a mixed variational principle/statement. The conditions on the approximation functions are dictated by the variational principle. If all governing equations as well as the boundary conditions are included in the variational statement, then the approximation functions are required to satisfy only the continuity requirements. Here we illustrate these ideas through an example.

Example 10.3 Consider the bending of a beam using the Euler–Bernoulli beam theory. The governing equations are [see Eqs. (4.30b,c) and (4.34)]

$$-\frac{d^2 w_0}{dx^2} - \frac{M}{EI} = 0, \tag{10.62a}$$

$$-\frac{d^2 M}{dx^2} = q. \tag{10.62b}$$

In Examples 7.7 and 7.8 we have solved the Euler–Bernoulli beam problem using the principle of minimum total potential energy, whose Euler equation is the fourth-order equation governing the deflection:

$$\frac{d^2}{dx^2}\left(EI\frac{d^2 w_0}{dx^2}\right) = q, \tag{10.63}$$

which is obtained by combining Eqs. (10.62a,b) above. Here we wish to consider a mixed formulation in which both $w_0(x)$ and $M(x)$ are treated as dependent variables with independent approximations. That is, we solve Eqs. (10.62a,b) simultaneously (without eliminating M). For comparison purposes, we shall also solve the same problem using the Ritz method based on the principle of minimum total potential energy.

As a specific example problem, we consider a cantilever beam under uniformly distributed load, q_0. The boundary conditions are

$$w_0(0) = 0, \qquad \left(\frac{dw_0}{dx}\right)\bigg|_{x=0} = 0,$$

$$M(L) = 0, \qquad V(L) \equiv \left(\frac{dM}{dx}\right)\bigg|_{x=L} = 0. \tag{10.64}$$

In the Ritz approximation based on the minimum total potential energy principle

$$0 = \delta\Pi(w_0) \equiv \int_0^L \left(EI \frac{d^2\delta w_0}{dx^2}\frac{d^2 w_0}{dx^2} - q_0\delta w_0 \right) dx,$$

we use the two-parameter approximation $W_2(x) = c_1x^2 + c_2x^3$ (which satisfies only the geometric boundary conditions) and obtain the solution

$$W_2(x) = c_1x^2 + c_2x^3 = \frac{q_0L^4}{24EI}\left(5\frac{x^2}{L^2} - 2\frac{x^3}{L^3}\right), \qquad W_2(L) = \frac{q_0L^4}{8EI}. \quad (10.65)$$

The bending moment obtained from this Ritz solution is

$$M_2(x) = -EI\frac{d^2W_2}{dx^2} = -\frac{q_0L^2}{12}\left(5 - 6\frac{x}{L}\right), \qquad M_2(0) = -\frac{5q_0L^2}{12}. \quad (10.66)$$

The exact solutions for $w_0(x)$ and $M(x)$ are given by

$$w_0(x) = \frac{q_0L^4}{24EI}\left(6\frac{x^2}{L^2} - 4\frac{x^3}{L^3} + \frac{x^4}{L^4}\right), \qquad w_0(L) = \frac{q_0L^4}{8EI},$$

$$M(x) = -\frac{q_0L^2}{2}\left(1 - \frac{x}{L}\right)^2, \qquad M(0) = -\frac{q_0L^2}{2}. \quad (10.67)$$

Thus the two-parameter Ritz solution based on the total potential energy principle (or displacement model) yields exact maximum deflection, but maximum bending moment is in 16.67% error.

Let us consider the pair in Eqs. (10.62a,b) and construct their weak forms. Using the three-step procedure, we obtain after two steps

$$0 = \int_0^L \left(\frac{dv_1}{dx}\frac{dM}{dx} - v_1q_0\right) dx - \left[v_1 \cdot \frac{dM}{dx}\right]_0^L, \quad (10.68)$$

$$0 = \int_0^L \left(\frac{dv_2}{dx}\frac{dw_0}{dx} - v_2\frac{M}{EI}\right) dx - \left[v_2 \cdot \frac{dw_0}{dx}\right]_0^L, \quad (10.69)$$

where (v_1, v_2) are the weight functions.

To physically interpret the weight functions, we examine the expressions in the integrands. The most obvious one to interpret is the product $v_1q_0\,dx$. In order for it to have the units of "work done," v_1 should have the meaning of displacement. Thus, $v_1 \sim w_0$. Since $(M/EI)\,dx$ has the units of rotation, v_2 must have the units of a moment, giving $v_2 \sim M$. This information is needed in developing the weak forms (and later in applying the Ritz method, where we must replace v_1 and v_2 with the proper approximation functions). It is clear that v_1 must be zero at points where w_0 is specified, and v_2 should be zero where M is specified. It is also clear from the boundary expressions in Eqs. (10.68) and (10.69) that w_0 and M are the primary variables, while dw_0/dx (rotation) and dM/dx (shear force) are the secondary variables. Thus, the nature of the boundary conditions (i.e., classification into essential and natural) depends on the set of equations used.

Returning to the weak form development, we use the specified boundary conditions in Eq. (10.64) to simplify the boundary terms [$(dw_0/dx)(0) = 0$, $(dM/dx)(L) = 0$, $v_1(0) = 0$, and $v_2(L) = 0$ make all boundary terms vanish] and obtain the final weak forms:

$$0 = \int_0^L \left(\frac{dv_1}{dx}\frac{dM}{dx} - v_1 q_0 \right) dx, \tag{10.70}$$

$$0 = \int_0^L \left(\frac{dv_2}{dx}\frac{dw_0}{dx} - v_2 \frac{M}{EI} \right) dx. \tag{10.71}$$

Let us consider the two-parameter Ritz approximations of w_0 and M:

$$\begin{aligned} W_2(x) &= a_1\phi_1(x) + a_2\phi_2(x) = a_1 x + a_2 x^2, \\ M_2(x) &= b_1\psi_1(x) + b_2\psi_2(x) = b_1(L-x) + b_2(L-x)^2, \end{aligned} \tag{10.72}$$

which satisfy the specified essential boundary conditions [$w(0) = 0$ and $M(L) = 0$]. In general, there is a relationship between the number of parameters chosen for $w_0(x)$ and $M(x)$ to insure the invertibility of the resulting algebraic equations for the unknown parameters (a_i, b_i).

Substituting Eq. (10.72) for w_0 and M, and $v_1 = \phi_i$ and $v_2 = \psi_i$ into Eqs. (10.70) and (10.71), we obtain

$$0 = \int_0^L \left[\frac{d\phi_i}{dx} \left(\sum_{j=1}^{2} b_j \frac{d\psi_j}{dx} \right) - \phi_i q_0 \right] dx = 0, \qquad i = 1, 2, \tag{10.73}$$

$$0 = \int_0^L \left[\frac{d\psi_i}{dx} \left(\sum_{j=1}^{2} a_j \frac{d\phi_j}{dx} \right) - \frac{1}{EI}\psi_i \left(\sum_{j=1}^{2} b_j \psi_j \right) \right] dx = 0, \qquad i = 1, 2. \tag{10.74}$$

In matrix form, we have

$$\sum_{j=1}^{2} K_{ij} b_j = f_i, \qquad \sum_{j=1}^{2} K_{ji} a_j - \sum_{j=1}^{2} G_{ij} b_j = 0, \tag{10.75a}$$

where

$$K_{ij} = \int_0^L \frac{d\phi_i}{dx}\frac{d\psi_j}{dx}\, dx, \qquad G_{ij} = \frac{1}{EI}\int_0^L \psi_i \psi_j \, dx, \qquad f_i = \int_0^L q_0 \phi_i \, dx. \tag{10.75b}$$

Evaluating the integrals for the choice of ϕ_i and ψ_i from Eq. (10.72), we obtain

$$[K] = -\begin{bmatrix} L & L^2 \\ L^2 & \frac{2}{3}L^3 \end{bmatrix}, \qquad [G] = \frac{1}{EI}\begin{bmatrix} \frac{L^3}{3} & \frac{L^4}{4} \\ \frac{L^4}{4} & \frac{L^5}{5} \end{bmatrix}, \qquad \{f\} = \frac{q_0 L^2}{6}\begin{Bmatrix} 3 \\ 2L \end{Bmatrix}.$$

Solving the first equation in (10.75a) for b's yields

$$b_1 = 0, \qquad b_2 = -\frac{q_0}{2}.$$

Substituting for b's in the second equation of (10.75a), and solving for a's, we obtain

$$a_1 = \frac{1}{20}\frac{q_0 L^3}{EI}, \qquad a_2 = \frac{3}{40}\frac{q_0 L^2}{EI}.$$

The two-parameter Ritz solutions based on the mixed formulation are

$$\begin{aligned} W_2(x) &= \frac{q_0 L^4}{40EI}\left(2\frac{x}{L} + 3\frac{x^2}{L^2}\right), \qquad & W_2(L) &= \frac{q_0 L^4}{8EI}, \\ M_2(x) &= -\frac{q_0 L^2}{2}\left(1 - \frac{x}{L}\right)^2, \qquad & M_2(0) &= -\frac{q_0 L^2}{2}. \end{aligned} \tag{10.76}$$

Note that maximum values of both deflection and moment coincide with the exact values, and that the expression for the bending moment coincides with the exact solution. Although the maximum deflection predicted by the mixed Ritz approximation coincides with the exact solution, the deflection obtained violates the slope boundary condition at $x = 0$ (which is a natural boundary condition in the present case). Thus, the mixed variational formulation gives more accurate solutions for the force variable because they are approximated independently; however, the accuracy of the displacements is somewhat degraded. It can be improved by increasing the number of parameters (note that the polynomial degree used for w_0 is only 2 in the mixed formulation, while it is 3 in the displacement formulation). In cases where the analysis is carried out to support design based on stresses, it is desirable to have accurate values of the force variables.

Table 10.1 contains a comparison of pointwise deflections, slope, bending moment, and shear force obtained by the two approaches with the exact solution.

10.4 MIXED FINITE ELEMENT MODELS OF BEAMS

10.4.1 The Euler–Bernoulli Beam Theory

Governing Equations and Weak Forms Here, we develop a mixed finite element model of the equations governing the Euler–Bernoulli beam theory. Consider the following equations of the Euler–Bernoulli beam theory [see Eqs. (10.62a,b)

Table 10.1 Comparison of the scaled[a] generalized displacements and forces computed by the two-parameter Ritz approximations based on the total potential energy formulation (PEF) and mixed variational formulation (MVF)

	Deflection, $\bar{w}_0(x)$			Bending Moment, $\bar{M}(x)$		
x	Exact	PEF	MVF	Exact	PEF	MVF
0.0	0.0000	0.0000	0.0000	−0.500	−0.4167	−0.500
0.1	0.0023	0.0020	0.0057	−0.405	−0.3667	−0.405
0.2	0.0087	0.0077	0.0130	−0.320	−0.3167	−0.320
0.3	0.0183	0.0165	0.0218	−0.245	−0.2667	−0.245
0.4	0.0304	0.0280	0.0320	−0.180	−0.2167	−0.180
0.5	0.0443	0.0417	0.0438	−0.125	−0.1667	−0.125
0.6	0.0594	0.0570	0.0570	−0.080	−0.1167	−0.080
0.7	0.0753	0.0735	0.0717	−0.045	−0.0667	−0.045
0.8	0.0917	0.0907	0.0880	−0.020	−0.0167	−0.020
0.9	0.1083	0.1080	0.1057	−0.005	−0.0333	−0.005
1.0	0.1250	0.1250	0.1250	−0.000	−0.0833	−0.000
	Slope, $d\bar{w}_0/dx$			Shear Force, $\bar{V}(x)$		
x	Exact	PEF	MVF	Exact	PEF	MVF
0.0	0.0000	0.0000	−0.050	1.0	0.5	1.0
0.1	0.0452	0.0392	−0.065	0.9	0.5	0.9
0.2	0.0813	0.0733	−0.080	0.8	0.5	0.8
0.3	0.1095	0.1025	−0.095	0.7	0.5	0.7
0.4	0.1307	0.1267	−0.110	0.6	0.5	0.6
0.5	0.1458	0.1458	−0.125	0.5	0.5	0.5
0.6	0.1560	0.1600	−0.140	0.4	0.5	0.4
0.7	0.1622	0.1692	−0.155	0.3	0.5	0.3
0.8	0.1653	0.1733	−0.170	0.2	0.5	0.2
0.9	0.1665	0.1725	−0.185	0.1	0.5	0.1
1.0	0.1667	0.1667	−0.200	0.0	0.5	0.0

[a] $\bar{w} = w\frac{EI}{q_0L^4}$; $\frac{d\bar{w}_0}{dx} = \left(\frac{dw_0}{dx}\right)\frac{EI}{q_0L^3}$; $\bar{M} = M\frac{1}{q_0L^2}$; $\bar{V} = \left(\frac{dM}{dx}\right)\frac{1}{q_0L}$.

and (4.30b,c), and Example 10.3]:

$$-\frac{d^2w_0}{dx^2} - \frac{M}{EI} = 0, \qquad -\frac{d^2M}{dx^2} = q. \tag{10.77a,b}$$

We can construct a mixed finite element model based on either the weak forms of the equations (10.77a,b) or least-squares statement of the equations over an element $\Omega^e = (x_a, x_b)$. The least-squares finite element model necessarily requires higher-order interpolation of the variables. First, we develop the weak forms for the Ritz finite element model.

We have [see Eqs. (10.68) and (10.69)]

$$0 = \int_{x_a}^{x_b} \left(\frac{dv_1}{dx}\frac{dM}{dx} - v_1 q \right) dx - v_1(x_a)\bar{Q}_1 - v_1(x_b)\bar{Q}_2, \tag{10.78a}$$

$$0 = \int_{x_a}^{x_b} \left(\frac{dv_2}{dx}\frac{dw_0}{dx} - v_2\frac{M}{EI} \right) dx - v_2(x_a)\Theta_1 + v_2(x_b)\Theta_2, \tag{10.78b}$$

where (v_1, v_2) are the weight functions that have the interpretation of (see Example 10.3) virtual deflection δw_0 and virtual moment δM, respectively, and

$$\bar{Q}_1 = -\left(\frac{dM}{dx}\right)_{x=x_a}, \qquad \bar{Q}_2 = \left(\frac{dM}{dx}\right)_{x=x_b},$$
$$\Theta_1 = \left(-\frac{dw_0}{dx}\right)_{x=x_a}, \qquad \Theta_2 = \left(-\frac{dw_0}{dx}\right)_{x=x_b}. \tag{10.79}$$

Weak Form Finite Element Model The weak forms (10.78) suggest that both w_0 and M may be approximated using the Lagrange interpolation. Suppose that w_0 and M are approximated as

$$w_0(x) \approx \sum_{i=1}^{m} w_i\phi_i(x), \qquad M(x) \approx \sum_{i=1}^{n} M_i\psi_i(x), \tag{10.80}$$

where (ϕ_i, ψ_i) are the Lagrange interpolation functions of different degree used for w_0 and M, respectively. Substituting Eq. (10.80) for w_0 and M, and $v_1 = \phi_i$ and $v_2 = \psi_i$ into Eqs. (10.78a,b), we obtain

$$0 = \int_{x_a}^{x_b} \left[\frac{d\phi_i}{dx}\left(\sum_{j=1}^{n} M_j\frac{d\psi_j}{dx}\right) - \phi_i q \right] dx - \bar{Q}_i \qquad (i = 1, 2, \ldots, m), \tag{10.81a}$$

$$0 = \int_{x_a}^{x_b} \left[\frac{d\psi_i}{dx}\left(\sum_{j=1}^{m} w_j\frac{d\phi_j}{dx}\right) - \frac{1}{EI}\psi_i\left(\sum_{j=1}^{n} M_j\psi_j\right) \right] dx - (-1)^{i+1}\Theta_i$$
$$(i = 1, 2, \ldots, n), \tag{10.81b}$$

or

$$\sum_{j=1}^{m} K_{ij}^e M_j^e = f_i^e + \bar{Q}_i^e, \qquad \sum_{j=1}^{m} K_{ji}^e w_j^e - \sum_{j=1}^{n} G_{ij}^e M_j^e = (-1)^{i+1}\Theta_i^e, \tag{10.82a}$$

where

$$K_{ij}^e = \int_{x_a}^{x_b} \frac{d\phi_i}{dx}\frac{d\psi_j}{dx}\,dx, \qquad G_{ij}^e = \frac{1}{EI}\int_{x_a}^{x_b} \psi_i\psi_j\,dx, \qquad f_i^e = \int_{x_a}^{x_b} q\phi_i\,dx. \tag{10.82b}$$

In matrix notation, we have

$$[K^e]\{M^e\} = \{f^e\} + \{\bar{Q}^e\}, \quad [K^e]^{\mathrm{T}}\{w^e\} - [G^e]\{M^e\} = \{\bar{\Theta}^e\}, \tag{10.83a}$$

or

$$\begin{bmatrix} [0] & [K^e] \\ [K^e]^{\mathrm{T}} & -[G^e] \end{bmatrix} \begin{Bmatrix} \{w^e\} \\ \{M^e\} \end{Bmatrix} = \begin{Bmatrix} \{f^e\} + \{\bar{Q}^e\} \\ \{\bar{\Theta}^e\} \end{Bmatrix}, \tag{10.83b}$$

where

$$\bar{\Theta}_i^e = (-1)^{i+1}\Theta_i^e. \tag{10.83c}$$

The pair of matrix equations (10.83a) in nodal values of the displacements and moments may be reduced to a single matrix equation in $\{w^e\}$ and $\{\Theta^e\}$. Solving the second equation in (10.83a) for $\{M^e\}$, we obtain

$$\{M^e\} = [G^e]^{-1}\big([K^e]^{\mathrm{T}}\{w^e\} - \{\bar{\Theta}^e\}\big). \tag{10.84a}$$

Substituting the result into the first equation in (10.83a), we obtain

$$[K^e][G^e]^{-1}[K^e]^{\mathrm{T}}\{w^e\} - [K^e][G^e]^{-1}\{\bar{\Theta}^e\} = \{f^e\} + \{\bar{Q}^e\}. \tag{10.84b}$$

We can now write Eqs. (10.84a,b) in a single matrix equation as

$$\begin{bmatrix} [K^e][G^e]^{-1}[K^e]^{\mathrm{T}} & -[K^e][G^e]^{-1} \\ -[G^e]^{-1}[K^e]^{\mathrm{T}} & [G^e]^{-1} \end{bmatrix} \begin{Bmatrix} \{w^e\} \\ \{\Theta^e\} \end{Bmatrix} = \begin{Bmatrix} \{f^e\} + \{Q^e\} \\ -\{M^e\} \end{Bmatrix}. \tag{10.85}$$

For the choice of linear interpolation of both w_0 and M, we obtain

$$[K^e] = \frac{1}{h_e}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}, \quad [G^e] = \frac{h_e}{6E_eI_e}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}, \quad \{f^e\} = \begin{Bmatrix} f_1^e \\ f_2^e \end{Bmatrix},$$

$$[G^e]^{-1} = \frac{2E_eI_e}{h_e}\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}, \quad [G^e]^{-1}[K^e]^{\mathrm{T}} = \frac{6E_eI_e}{h_e^2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix},$$

$$[K^e][G^e]^{-1}[K^e]^{\mathrm{T}} = \frac{12E_eI_e}{h_e^3}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}, \quad [G^e]^{-1}[K^e]^{\mathrm{T}} = \big([K^e][G^e]^{-1}\big)^{\mathrm{T}}. \tag{10.86}$$

Hence, Eq. (10.85) becomes

$$\frac{2E_eI_e}{h_e^3}\begin{bmatrix} 6 & -3h_e & -6 & -3h_e \\ -3h_e & 2h_e^2 & 3h_e & h_e^2 \\ -6 & 3h_e & 6 & 3h_e \\ -3h_e & h_e^2 & 3h_e & 2h_e^2 \end{bmatrix} \begin{Bmatrix} u_1^e \\ u_2^e \\ u_3^e \\ u_4^e \end{Bmatrix} = \begin{Bmatrix} f_1^e \\ 0 \\ f_2^e \\ 0 \end{Bmatrix} + \begin{Bmatrix} Q_1^e \\ Q_2^e \\ Q_3^e \\ Q_4^e \end{Bmatrix}. \tag{10.87}$$

Interestingly, the stiffness matrix of the mixed finite element model with linear interpolation of both w_0 and M is the same as that of the displacement finite element model derived in Section 9.3 using the C^1 (Hermite cubic) interpolation. However, the load vector differs in the sense that the mixed model does not contain contributions of distributed load $q(x)$ to the nodal moments.

Weighted-Residual Finite Element Models It is possible to construct a mixed finite element model of Eqs. (10.77a,b) using a weighted-residual formulation. As will be seen shortly, this model requires higher-order approximations of both w_0 and M because they must satisfy both essential and natural boundary conditions. This leads to a complicated set of finite element equations that are not practical. Here we present the main ideas behind the development of the model.

The weighted-residual statement of Eqs. (10.77a,b) is

$$0 = \int_{x_a}^{x_b} v_1 \left(-\frac{d^2 w_0}{dx^2} - \frac{M}{EI} \right) dx, \tag{10.88}$$

$$0 = \int_{x_a}^{x_b} v_2 \left(-\frac{d^2 M}{dx^2} - q \right) dx, \tag{10.89}$$

where (v_1, v_2) are the weight functions. A close examination of the above statements indicate that $v_1 \sim M$ and $v_2 \sim w_0$. Using approximations of the form

$$w_0(x) \approx \sum_{i=1}^{4} \Delta_i \varphi_i^{(1)}(x), \qquad M(x) \approx \sum_{i=1}^{4} \Lambda_i \varphi_i^{(2)}(x), \tag{10.90}$$

we obtain the following Galerkin (i.e., $v_1 \sim \varphi_i^{(2)}$ and $v_2 \sim \varphi_i^{(1)}$) finite element model:

$$\begin{bmatrix} [0] & [A^e] \\ [B^e] & [D^e] \end{bmatrix} \begin{Bmatrix} \{\Delta^e\} \\ \{\Lambda^e\} \end{Bmatrix} = \begin{Bmatrix} \{f^e\} \\ \{0\} \end{Bmatrix}, \tag{10.91}$$

where

$$A_{ij}^e = \int_{x_a}^{x_b} \varphi_i^{(1)} \frac{d^2 \varphi_j^{(2)}}{dx^2}\, dx, \qquad f_i^e = -\int_{x_a}^{x_b} q \varphi_i^{(1)}\, dx,$$

$$B_{ij}^e = \int_{x_a}^{x_b} \varphi_i^{(2)} \frac{d^2 \varphi_j^{(1)}}{dx^2}\, dx, \qquad D_{ij}^e = \int_{x_a}^{x_b} \varphi_i^{(2)} \frac{d^2 \varphi^{(2)}}{dx^2}\, dx. \tag{10.92}$$

For Hermite cubic interpolation of both w_0 and M, we have $\varphi_i^{(1)} = \varphi_i^{(2)}$. The coefficient matrix in Eq. (10.91) is *not* symmetric.

The least-squares finite element model of the pair (10.77a,b) is based on the variational statement

$$0=\delta\int_{x_a}^{x_b}\left[p_1\left(\frac{d^2w_0}{dx^2}+\frac{M}{EI}\right)^2+p_2\left(\frac{d^2M}{dx^2}+q\right)^2\right]dx$$

$$=2\int_{x_a}^{x_b}\left[p_1\left(\frac{d^2w_0}{dx^2}+\frac{M}{EI}\right)\left(\frac{d^2\delta w_0}{dx^2}+\frac{\delta M}{EI}\right)+p_2\left(\frac{d^2M}{dx^2}+q\right)\frac{d^2\delta M}{dx^2}\right]dx,$$

or

$$0=\int_{x_a}^{x_b}\frac{d^2\delta w_0}{dx^2}\left(\frac{d^2w_0}{dx^2}+\frac{M}{EI}\right)dx, \tag{10.93}$$

$$0=\int_{x_a}^{x_b}\left[\delta M\left(\frac{d^2w_0}{dx^2}+\frac{M}{EI}\right)+\frac{EIp_2}{p_1}\frac{d^2\delta M}{dx^2}\left(\frac{d^2M}{dx^2}+q\right)\right]dx, \tag{10.94}$$

where (p_1, p_2) are weights used to make the entire statement have the same units. Substituting the approximation (10.90) into Eqs. (10.93) and (10.94), we obtain

$$\begin{bmatrix}[A^e] & [B^e]\\ [B^e]^{\mathrm{T}} & [D^e]\end{bmatrix}\begin{Bmatrix}\{\Delta^e\}\\ \{\Lambda^e\}\end{Bmatrix}=\begin{Bmatrix}\{q^e\}\\ \{0\}\end{Bmatrix}, \tag{10.95}$$

where

$$A_{ij}^e=\int_{x_a}^{x_b}\left(\frac{EIp_2}{p_1}\frac{d^2\varphi_i^{(2)}}{dx^2}\frac{d^2\varphi_j^{(2)}}{dx^2}+\varphi_i^{(2)}\varphi_j^{(2)}\right)dx,\qquad q_i^e=-\int_{x_a}^{x_b}\frac{d^2\varphi_i^{(2)}}{dx^2}q\,dx,$$

$$B_{ij}^e=\int_{x_a}^{x_b}\varphi_i^{(2)}\frac{d^2\varphi_j^{(1)}}{dx^2}dx,\qquad D_{ij}^e=\int_{x_a}^{x_b}\frac{d^2\varphi_i^{(1)}}{dx^2}\frac{d^2\varphi_j^{(1)}}{dx^2}dx. \tag{10.96}$$

Obviously, p_1 and p_2 should be selected such that $EI(p_2/p_1)$ has the dimension of $1/L^2$.

We close this section with the comment that finite element models other than the displacement finite element model are not commonly used in practice. In fact, finite element models based on the Timoshenko beam theory that accounts for transverse shear strain γ_{xz} may be used to analyze beams irrespective of whether shear deformation is significant. The next section is devoted to the discussion of Timoshenko beam finite elements.

10.4.2 The Timoshenko Beam Theory

Governing Equations The governing equations of the Timoshenko beam theory for pure bending are given by (see Eqs. (9.71) and (9.72); also see Reddy [17–19])

$$-\frac{d}{dx}\left(EI\frac{d\phi}{dx}\right) + GAK_s\left(\phi + \frac{dw_0}{dx}\right) = 0, \tag{10.97}$$

$$-\frac{d}{dx}\left[GAK_s\left(\phi + \frac{dw_0}{dx}\right)\right] = q(x). \tag{10.98}$$

Here $q(x)$ denotes the distributed transverse load, E Young's modulus, G the shear modulus, A the area of cross section, I the moment of inertia, and K_s the shear correction factor.

Displacement finite element models of Eqs. (10.97) and (10.98) were presented in Section 9.4.2. Here we shall discuss a number of mixed finite element models of these equations or their equivalent.

General Finite Element Model The mixed finite element model to be discussed here is based on the stationary functional (10.53) in which the displacements (w_0, ϕ) and strains $(\kappa_{xx}, \gamma_{xz})$ are treated as the dependent variables (see Example 10.2). The Euler equations of this functional are given in Eqs. (10.55)–(10.58).

The weak forms associated with Eqs. (10.55)–(10.58) over a typical element $\Omega^e = (x_a, x_b)$ are

$$\int_{x_a}^{x_b}\left(GAK_s\frac{d\delta w_0}{dx}\gamma_{xz} - \delta w_0 q\right)dx - V_a\delta w_0(x_a) - V_b\delta w_0(x_b) = 0, \tag{10.99}$$

$$\int_{x_a}^{x_b}\left(EI\frac{d\delta\phi}{dx}\kappa_{xx} + GAK_s\delta\phi\gamma_{xz}\right)dx - M_a\delta\phi(x_a) - M_b\delta\phi(x_b) = 0, \tag{10.100}$$

$$\int_{x_a}^{x_b} EI\delta\kappa_{xx}\left(\frac{d\phi}{dx} - \kappa_{xx}\right)dx = 0, \tag{10.101}$$

$$\int_{x_a}^{x_b} GAK_s\delta\gamma_{xz}\left(\frac{dw_0}{dx} + \phi - \gamma_{xz}\right)dx = 0, \tag{10.102}$$

where

$$\begin{aligned} V_1 &= [-GAK_s\gamma_{xz}]_{x_a}, \qquad & V_2 &= [GAK_s\gamma_{xz}]_{x_b},\\ M_1 &= [-EI\kappa_{xx}]_{x_a}, \qquad & M_2 &= [EI\kappa_{xx}]_{x_b}. \end{aligned} \tag{10.103}$$

Let the variables $(w_0, \phi, \kappa_{xx}, \gamma_{xz})$ be approximated as

$$w_0 \approx \sum_{j=1}^{m}\psi_j^{(1)}W_j^e, \quad \phi \approx \sum_{j=1}^{n}\psi_j^{(2)}\Phi_j^e, \quad \kappa_{xx} \approx \sum_{j=1}^{p}\psi_j^{(3)}K_j^e, \quad \gamma_{xz} \approx \sum_{j=1}^{q}\psi_j^{(4)}\Gamma_j^e, \tag{10.104}$$

where $(W_j^e, \Phi_j^e, \mathcal{K}_j^e, \Gamma_j^e)$ are the nodal values of $(w_0, \phi, \kappa_{xx}, \gamma_{xz})$ and $\psi_j^{(\alpha)}(x)$ ($\alpha = 1, 2, 3, 4$) are the associated interpolation functions whose choice is yet to be made. Substituting (10.104) into (10.99)–(9.102), we obtain the following finite element model:

$$\begin{bmatrix} [0] & [0] & [0] & [A^e] \\ [0] & [0] & [B^e] & [C^e] \\ [0] & [B^e]^T & -[D^e] & [0] \\ [A^e]^T & [C^e]^T & [0] & -[G^e] \end{bmatrix} \begin{Bmatrix} \{W^e\} \\ \{\Phi^e\} \\ \{\mathcal{K}^e\} \\ \{\Gamma^e\} \end{Bmatrix} = \begin{Bmatrix} \{F^e\} \\ \{0\} \\ \{0\} \\ \{0\} \end{Bmatrix} + \begin{Bmatrix} \{V^e\} \\ \{M^e\} \\ \{0\} \\ \{0\} \end{Bmatrix}, \tag{10.105a}$$

where

$$\begin{aligned} A_{ij}^e &= \int_{x_a}^{x_b} GAK_s \frac{d\psi_i^{(1)}}{dx} \psi_j^{(4)}\, dx, & B_{ij}^e &= \int_{x_a}^{x_b} EI \frac{d\psi_i^{(2)}}{dx} \psi_j^{(3)}\, dx, \\ C_{ij}^e &= \int_{x_a}^{x_b} GAK_s \psi_i^{(2)} \psi_j^{(4)}\, dx, & D_{ij}^e &= \int_{x_a}^{x_b} EI \psi_i^{(3)} \psi_j^{(3)}\, dx, \\ G_{ij}^e &= \int_{x_a}^{x_b} GAK_s \psi_i^{(4)} \psi_j^{(4)}\, dx, & F_i^e &= \int_{x_a}^{x_b} q \psi_i^{(1)}\, dx, \\ V_1^e = V_a^e, \quad & V_m^e = V_b^e, & M_1^e = M_a^e, \quad & M_n^e = M_b^e. \end{aligned} \tag{10.105b}$$

A couple of observations are in order concerning the finite element model in Eq. (10.105a). We note that $[A^e]$ is a vector $\{A^e\}$ when γ_{xz} is approximated as a constant, Γ_0. In addition, the first equation of (10.105a) has the form

$$G_e A_e K_s \begin{Bmatrix} -1 \\ 0 \\ \vdots \\ 0 \\ 1 \end{Bmatrix} \Gamma_0 = \begin{Bmatrix} F_1^e \\ F_2^e \\ F_3^e \\ \vdots \\ F_m^e \end{Bmatrix} + \begin{Bmatrix} V_1^e \\ 0 \\ \vdots \\ 0 \\ V_m^e \end{Bmatrix}, \tag{10.106}$$

when w_0 is interpolated using quadratic or higher-order polynomials. The nonzero entries correspond to the deflection degrees of freedom at node 1 and node m. For linear interpolation of w_0, we have $m = 2$ and Eq. (10.106) is correct. However, when $m > 2$, Eq. (10.106) implies that $F_i = 0$ for $i = 2, \ldots, m-1$, which, in general, is not true. Thus, either the distributed load is zero or it is converted to generalized point forces at the end nodes through the Hermite cubic polynomials φ_i of Eq. (9.58). In the latter case, the force components can be added to V_1^e and V_2^e and the moment components to M_1^e and M_2^e at nodes 1 and m, respectively.

ASD-LLCC Element For linear (L) interpolation of (w_0, ϕ) and constant (C) representation of $(\kappa_{xx}, \gamma_{xz})$, and for constant values of EI and GAK_s, the element equations become ($m = n = 2$ and $p = q = 1$)

$$G_e A_e K_s \begin{Bmatrix} -1 \\ 1 \end{Bmatrix} \Gamma_0^e = \begin{Bmatrix} q_1^e \\ q_2^e \end{Bmatrix} + \begin{Bmatrix} V_1^e \\ V_2^e \end{Bmatrix}, \tag{10.107}$$

$$E_e I_e \begin{Bmatrix} -1 \\ 1 \end{Bmatrix} \mathcal{K}_0^e + G_e A_e K_s \frac{h_e}{2} \begin{Bmatrix} 1 \\ 1 \end{Bmatrix} \Gamma_0^e = \begin{Bmatrix} M_1^e \\ M_2^e \end{Bmatrix}, \tag{10.108}$$

$$\{-1 \;\; 1\} \begin{Bmatrix} W_1^e \\ W_2^e \end{Bmatrix} + \frac{h_e}{2} \{1 \;\; 1\} \begin{Bmatrix} \Phi_1^e \\ \Phi_2^e \end{Bmatrix} - h_e \Gamma_0^e = 0, \tag{10.109}$$

$$\{-1 \;\; 1\} \begin{Bmatrix} \Phi_1^e \\ \Phi_2^e \end{Bmatrix} - h_e \mathcal{K}_0^e = 0. \tag{10.110}$$

Solving Eqs. (10.110) and (10.109) for $\mathcal{K}_0$ and Γ_0 and substituting into Eqs. (10.107) and (10.108) (i.e., condensing out $\mathcal{K}_0$ and Γ_0), we arrive at the following 4×4 system of equations:

$$\frac{GAK_s}{4h_e} \begin{bmatrix} 4 & -2h_e & -4 & -2h \\ -2h_e & h_e^2(1+4\Lambda) & 2h_e & h_e^2(1-4\Lambda) \\ -4 & 2h_e & 4 & 2h_e \\ -2h_e & h_e^2(1-4\Lambda) & 2h_e & h_e^2(1+4\Lambda) \end{bmatrix} \begin{Bmatrix} W_1^e \\ \Phi_1^e \\ W_2^e \\ \Phi_2^e \end{Bmatrix} = \begin{Bmatrix} q_1^e \\ 0 \\ q_2^e \\ 0 \end{Bmatrix} + \begin{Bmatrix} Q_1^e \\ Q_2^e \\ Q_3^e \\ Q_4^e \end{Bmatrix}, \tag{10.111}$$

where $\Lambda = E_e I_e / G_e A_e K_s h_e^2$. These are exactly the same equations obtained in the displacement formulation with the linear interpolation of w_0 and ϕ and using one-point Gauss quadrature to evaluate the shear stiffnesses. The element was referred to as the reduced integration element (RIE). Thus the assumed strain–displacement formulation eliminates the need for reduced integration concepts.

ASD-QLCC Element A quadratic interpolation of w_0, linear interpolation of ϕ, and constant representation of κ_{xx} and γ_{xz} will also yield the same stiffness matrix as in Eq. (10.111). However, as noted earlier, the distributed load q must be calculated using Hermite cubic polynomials φ_i ($i = 1, 2, 3, 4$):

$$q_i^{(h)} = \int_{x_a}^{x_b} q(x) \varphi_i(x)\, dx, \tag{10.112}$$

and the force components of $q_1^{(h)}$ and $q_3^{(h)}$ must be added to V_a^e and V_b^e, and the moment components $q_2^{(h)}$ and $q_4^{(h)}$ to M_1^e and M_2^e, respectively:

$$\begin{Bmatrix} \bar{V}_1^e \\ \bar{M}_1^e \\ \bar{V}_2^e \\ \bar{M}_2^e \end{Bmatrix} = \begin{Bmatrix} V_1 + q_1^{(h)} \\ M_1^e + q_2^{(h)} \\ V_2^e + q_3^{(h)} \\ M_2^e + q_4^{(h)} \end{Bmatrix}. \tag{10.113}$$

Thus the load vector is the same as that of the Euler–Bernoulli beam equations. This element was referred to as the consistent interpolation element (CIE).

ASD-HQLC Element Suppose that the distributed load is represented using Eq. (10.112). A Lagrange cubic interpolation of w_0, quadratic interpolation of ϕ, linear interpolation of κ_{xx}, and constant representation of γ_{xz} yields the equations

$$G_e A_e K_s \begin{Bmatrix} -1 \\ 1 \end{Bmatrix} \Gamma_0 = \begin{Bmatrix} \bar{V}_a^e \\ \bar{V}_b^e \end{Bmatrix}, \tag{10.114}$$

$$\frac{E_e I_e}{6}\begin{bmatrix} -5 & -1 \\ 4 & -4 \\ 1 & 5 \end{bmatrix}\begin{Bmatrix} \mathcal{K}_1^e \\ \mathcal{K}_2^e \end{Bmatrix} + \frac{G_e A_e K_s h_e}{6}\begin{Bmatrix} 1 \\ 4 \\ 1 \end{Bmatrix}\Gamma_0 = \begin{Bmatrix} \bar{M}_a^e \\ 0 \\ \bar{M}_b^e \end{Bmatrix}, \tag{10.115}$$

$$\frac{E_e I_e}{6}\begin{bmatrix} -5 & 4 & 1 \\ -1 & -4 & 5 \end{bmatrix}\begin{Bmatrix} \Phi_1^e \\ \Phi_c^e \\ \Phi_2^e \end{Bmatrix} - \frac{E_e I_e h_e}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\begin{Bmatrix} \mathcal{K}_1^e \\ \mathcal{K}_2^e \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}, \tag{10.116}$$

$$\{-1 \;\; 1\}\begin{Bmatrix} W_1^e \\ W_2^e \end{Bmatrix} + \frac{h_e}{6}\{1 \;\; 4 \;\; 1\}\begin{Bmatrix} \Phi_1^e \\ \Phi_c^e \\ \Phi_2^e \end{Bmatrix} - h_e \Gamma_0^e = 0, \tag{10.117}$$

where the end nodes of the element are designated as "1" and "2," and the middle node as "c," and the interior nodal degrees of freedom associated with w are omitted as they do not contribute to the equations. Solving Eq. (10.116) for $\{\mathcal{K}\}$ and Eq. (10.117) for Γ_0, and substituting the result into Eqs. (10.114) and (10.115), we obtain

$$S_e\left(\frac{1}{h_e}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} W_1^e \\ W_2^e \end{Bmatrix} + \frac{1}{6}\begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix}\begin{Bmatrix} \Phi_1^e \\ \Phi_2^e \end{Bmatrix} + \frac{4}{6}\begin{Bmatrix} -1 \\ 1 \end{Bmatrix}\Phi_c^e\right) = \begin{Bmatrix} \bar{V}_a^e \\ \bar{V}_b^e \end{Bmatrix}, \tag{10.118}$$

$$S_e\left\{\frac{1}{6}\begin{bmatrix} -1 & 1 \\ -4 & 4 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} W_1^e \\ W_2^e \end{Bmatrix} + \frac{\bar{\Lambda}}{3h_e}\begin{bmatrix} 7 & 1 \\ -8 & -8 \\ 1 & 7 \end{bmatrix}\begin{Bmatrix} \Phi_1^e \\ \Phi_2^e \end{Bmatrix} + \frac{1}{36}\begin{bmatrix} 1 & 1 \\ 4 & 4 \\ 1 & 1 \end{bmatrix}\begin{Bmatrix} \Phi_1^e \\ \Phi_2^e \end{Bmatrix} + \left(\frac{8\bar{\Lambda}}{3h_e}\begin{Bmatrix} -1 \\ 2 \\ -1 \end{Bmatrix} + \frac{h_e}{9}\begin{Bmatrix} 1 \\ 4 \\ 1 \end{Bmatrix}\right)\Phi_c^e\right\} = \begin{Bmatrix} \bar{M}_a^e \\ 0 \\ \bar{M}_b^e \end{Bmatrix}, \tag{10.119}$$

where $S_e = G_e A_e K_s$ and $\bar{\Lambda} = (E_e I_e / S_e)$. The second equation of (10.119) can be used to eliminate Φ_c^e from Eqs. (10.118) and (10.119). Thus, we obtain

$$\frac{2E_e I_e}{\mu h_e^3}\left(\begin{bmatrix} 6 & -6 \\ -6 & 6 \end{bmatrix}\begin{Bmatrix} W_1^e \\ W_2^e \end{Bmatrix} + \begin{bmatrix} -3h_e & -3h_e \\ 3h_e & 3h_e \end{bmatrix}\begin{Bmatrix} \Phi_1^e \\ \Phi_2^e \end{Bmatrix}\right) = \begin{Bmatrix} \bar{V}_a^e \\ \bar{V}_b^e \end{Bmatrix}, \tag{10.120}$$

$$\frac{2E_e I_e}{\mu h_e^3}\left(\begin{bmatrix} -3h_e & 3h_e \\ -3h_e & 3h_e \end{bmatrix}\begin{Bmatrix} W_1^e \\ W_2^e \end{Bmatrix} + \begin{bmatrix} 2h_e^2\lambda & h_e^2\xi \\ h_e^2\xi & 2h_e^2\lambda \end{bmatrix}\begin{Bmatrix} \Phi_1^e \\ \Phi_2^e \end{Bmatrix}\right) = \begin{Bmatrix} \bar{M}_a^e \\ \bar{M}_b^e \end{Bmatrix}. \tag{10.121}$$

Superposing Eqs. (10.120) and (10.121), we obtain

$$\frac{2E_eI_e}{\mu h_e^3}\begin{bmatrix} 6 & -3h_e & -6 & -3h_e \\ -3h_e & 2h_e^2\lambda & 3h_e & h_e^2\xi \\ -6 & 3h_e & 6 & 3h_e \\ -3h_e & h_e^2\xi & 3h_e & 2h_e^2\lambda \end{bmatrix} \begin{Bmatrix} W_1^e \\ \Phi_1^e \\ W_2^e \\ \Phi_2^e \end{Bmatrix} = \begin{Bmatrix} \bar{V}_a^e \\ \bar{M}_a^e \\ \bar{V}_b^e \\ \bar{M}_b^e \end{Bmatrix}. \tag{10.122}$$

The stiffness matrix is the same as that of the superconvergent element derived in (9.97); however, the load vector is different. It is the same when either the applied load q is element-wise uniform or the load vector is computed using Eq. (9.96b).

It should be noted that the degree of the polynomial interpolation used for w_0 does not enter the equations presented in all models discussed in this section. However, the load representation implies that w_0 be interpolated with Hermite cubic polynomials φ_i^e of Eq. (9.58) or φ_i^e of Eq. (9.94).

10.5 MIXED FINITE ELEMENT MODELS OF THE CLASSICAL PLATE THEORY

10.5.1 Preliminary Comments

As mentioned in the previous section, a conforming plate finite element based on the displacement formulation of the classical plate theory is algebraically complex and computationally demanding. A quintic polynomial would satisfy the continuity requirements at element interfaces, and results in 21 degrees of freedom per element. The continuity requirements can be relaxed by reformulating the fourth-order equation (8.143a) as a set of second-order equations in terms of the deflection and bending moments. Such a formulation is called a *mixed formulation*. Here we discuss two such models for the static bending of orthotropic plates (see Reddy and Tsay [20,21]). There exist many other papers dealing with mixed and hybrid formulations based on modified total potential energy functional, modified complementary energy functional, and variants of the Hellinger–Reissner and Hu–Washizu principles.

10.5.2 Mixed Model I

Governing Equations Consider the following set of equations [cf. Eqs. (8.123a) and 8.132)]:

$$-\left(\frac{\partial^2 M_x}{\partial x^2} + 2\frac{\partial^2 M_{xy}}{\partial x \partial y} + \frac{\partial^2 M_y}{\partial y^2}\right) = q, \tag{10.123}$$

$$M_{xx} = -\left(D_{11}\frac{\partial^2 w_0}{\partial x^2} + D_{12}\frac{\partial^2 w_0}{\partial y^2}\right),$$

$$M_{yy} = -\left(D_{12}\frac{\partial^2 w_0}{\partial x^2} + D_{22}\frac{\partial^2 w_0}{\partial y^2}\right), \tag{10.124}$$

$$M_{xy} = -2D_{66}\frac{\partial^2 w_0}{\partial x \partial y},$$

where D_{ij} are the bending stiffnesses of an orthotropic plate [see Eq. (8.123b)]:

$$D_{ij} = Q_{ij}\frac{h^3}{12}, \qquad (i, j = 1, 2, 6), \tag{10.125a}$$

and the elastic stiffnesses Q_{ij} are defined in terms of the principal moduli (E_1, E_2), shear modulus G_{12}, and Poisson's ratio ν_{12} as

$$Q_{11} = \frac{E_1}{1-\nu_{12}\nu_{21}}, \qquad Q_{12} = \frac{\nu_{12}E_2}{1-\nu_{12}\nu_{21}}, \qquad Q_{22} = \frac{E_2}{1-\nu_{12}\nu_{21}},$$
$$Q_{66} = G_{12}, \qquad \nu_{21} = \nu_{12}\frac{E_2}{E_1}. \tag{10.125b}$$

Substitution of Eqs. (10.124) into Eq. (10.123) leads to the fourth-order equation in Eq. (8.143a), whose finite element models were discussed in Section 9.5. Here we formulate a mixed finite element model of Eqs. (10.123) and (10.124), which involve second-order equations among the dependent variables $(w_0, M_{xx}, M_{yy}, M_{xy})$.

First we express the three equations in (10.124) in the alternative form

$$\frac{\partial^2 w_0}{\partial x^2} = -\left(\bar{D}_{22}M_{xx} - \bar{D}_{12}M_{yy}\right),$$
$$\frac{\partial^2 w_0}{\partial y^2} = -\left(\bar{D}_{12}M_{xx} - \bar{D}_{11}M_{yy}\right), \tag{10.126}$$
$$2\frac{\partial^2 w_0}{\partial x \partial y} = -(D_{66})^{-1}M_{xy},$$

where

$$\bar{D}_{ij} = \frac{D_{ij}}{D_0}, \qquad D_0 = D_{11}D_{22} - D_{12}^2. \tag{10.127}$$

Weak Forms The weak forms of Eqs. (10.123) and (10.126) over a typical element Ω_e can be derived using the inverse procedure described in Section 7.4.1. We have

$$0 = \int_{\Omega_e}\left[\frac{\partial \delta w_0}{\partial x}\left(\frac{\partial M_{xx}}{\partial x} + \frac{\partial M_{xy}}{\partial y}\right) + \frac{\partial \delta w_0}{\partial y}\left(\frac{\partial M_{yy}}{\partial y} + \frac{\partial M_{xy}}{\partial x}\right) - q\delta w_0\right]dx\,dy$$
$$- \oint_{\Gamma_e}\delta w_0\left[\left(\frac{\partial M_{xx}}{\partial x} + \frac{\partial M_{xy}}{\partial y}\right)n_x + \left(\frac{\partial M_{xy}}{\partial x} + \frac{\partial M_{yy}}{\partial y}\right)n_y\right]ds, \tag{10.128}$$

$$0 = \int_{\Omega_e} \left[\frac{\partial w_0}{\partial x} \frac{\partial \delta M_{xx}}{\partial x} - \delta M_{xx} \left(\bar{D}_{22} M_{xx} - \bar{D}_{12} M_{yy} \right) \right] dxdy$$
$$- \oint_{\Gamma_e} \delta M_{xx} \frac{\partial w_0}{\partial x} n_x \, ds, \tag{10.129}$$

$$0 = \int_{\Omega_e} \left[\frac{\partial w_0}{\partial y} \frac{\partial \delta M_{yy}}{\partial y} - \delta M_{yy} \left(\bar{D}_{11} M_{yy} - \bar{D}_{12} M_{xx} \right) \right] dxdy$$
$$- \oint_{\Gamma_e} \delta M_{yy} \frac{\partial w_0}{\partial y} n_y \, ds, \tag{10.130}$$

$$0 = \int_{\Omega_e} \left(\frac{\partial w_0}{\partial x} \frac{\partial \delta M_{xy}}{\partial y} + \frac{\partial w_0}{\partial y} \frac{\partial \delta M_{xy}}{\partial x} - (D_{66})^{-1} \delta M_{xy} M_{xy} \right) dxdy$$
$$- \oint_{\Gamma_e} \delta M_{xy} \left(\frac{\partial w_0}{\partial y} n_x + \frac{\partial w_0}{\partial x} n_y \right) ds. \tag{10.131}$$

It is clear that the primary variables of the formulation are

$$w_0, \; M_{xx}, \; M_{yy}, \; M_{xy}, \tag{10.132a}$$

and the secondary variables are

$$Q_n \equiv \left(\frac{\partial M_{xx}}{\partial x} + \frac{\partial M_{xy}}{\partial y} \right) n_x + \left(\frac{\partial M_{xy}}{\partial x} + \frac{\partial M_{yy}}{\partial y} \right) n_y,$$
$$\frac{\partial w_0}{\partial x} n_x, \qquad \frac{\partial w_0}{\partial y} n_y, \qquad \frac{\partial w_0}{\partial x} n_y + \frac{\partial w_0}{\partial y} n_x. \tag{10.132b}$$

Note that the natural boundary conditions involving the bending moments in the displacement formulations become the essential boundary conditions in the mixed formulation.

The functional associated with the weak forms (10.128)–(10.131) is given by [see Eq. (7.53)]

$$J_1^e(\Lambda) = \int_{\Omega_e} \left[\frac{1}{2} \left(-\bar{D}_{22} M_{xx}^2 + 2\bar{D}_{12} M_{xx} M_{yy} - \bar{D}_{11} M_{yy}^2 - \frac{1}{D_{66}} M_{xy}^2 \right) \right.$$
$$\left. + \frac{\partial w_0}{\partial x} \left(\frac{\partial M_{xx}}{\partial x} + \frac{\partial M_{xy}}{\partial y} \right) + \frac{\partial w_0}{\partial y} \left(\frac{\partial M_{xy}}{\partial x} + \frac{\partial M_{yy}}{\partial y} \right) - q w_0 \right] dxdy$$
$$- \oint_{\Gamma_e} [M_{xx} \theta_x n_x + M_{yy} \theta_y n_y + M_{xy} (\theta_x n_y + \theta_y n_x) + Q_n w_0] \, ds, \tag{10.133a}$$

where

$$\theta_x = \frac{\partial w_0}{\partial x}, \qquad \theta_y = \frac{\partial w_0}{\partial y}, \qquad \Lambda = (w_0, M_{xx}, M_{yy}, M_{xy}). \tag{10.133b}$$

The condition $\delta J_1 = 0$ gives Eqs. (10.123) and (10.126) as the Euler equations.

Finite Element Model Let $(w_0, M_{xx}, M_{yy}, M_{xy})$ be interpolated by expressions of the form

$$w_0 = \sum_{i=1}^{r} w_i \psi_i^{(1)}, \qquad M_{xx} = \sum_{i=1}^{s} M_{xi} \psi_i^{(2)}, \qquad M_{yy} = \sum_{i=1}^{p} M_{yi} \psi_i^{(3)},$$
$$M_{xy} = \sum_{i=1}^{q} M_{xyi} \psi_i^{(4)}, \tag{10.134}$$

where $\psi_i^{(\alpha)}$, $(\alpha = 1, 2, 3, 4)$ are appropriate interpolation functions. Substituting Eq. (10.134) into Eqs. (10.128)–(10.131) [or substituting Eq. (10.134) into J_1 and setting the partial variations of J_1 with respect to w_j, M_{xj}, M_{yj}, and M_{xyj} to zero separately], we obtain

$$\begin{bmatrix} [K^{11}] & [K^{12}] & [K^{13}] & [K^{14}] \\ & [K^{22}] & [K^{23}] & [K^{24}] \\ \text{symm.} & & [K^{33}] & [K^{34}] \\ & & & [K^{44}] \end{bmatrix} \begin{Bmatrix} \{w\} \\ \{M_x\} \\ \{M_y\} \\ \{M_{xy}\} \end{Bmatrix} = \begin{Bmatrix} \{F^1\} \\ \{F^2\} \\ \{F^3\} \\ \{F^4\} \end{Bmatrix}, \tag{10.135}$$

where

$$\begin{aligned}
K_{ij}^{11} &= 0, \qquad i, j = 1, 2, \ldots, r, \\
K_{ij}^{12} &= \int_{\Omega_e} \frac{\partial \psi_i^1}{\partial x} \frac{\partial \psi_j^2}{\partial x}\, dxdy, \qquad i = 1, 2, \ldots, r; \quad j = 1, 2, \ldots, s, \\
K_{ij}^{13} &= \int_{\Omega_e} \frac{\partial \psi_i^1}{\partial y} \frac{\partial \psi_j^3}{\partial y}\, dxdy, \qquad i = 1, 2, \ldots, r; \quad j = 1, 2, \ldots, p, \\
K_{ij}^{14} &= \int_{\Omega_e} \left(\frac{\partial \psi_i^1}{\partial x} \frac{\partial \psi_j^4}{\partial y} + \frac{\partial \psi_i^1}{\partial y} \frac{\partial \psi_j^4}{\partial x} \right) dxdy, \qquad i = 1, 2, \ldots, r; \quad j = 1, 2, \ldots, q, \\
K_{ij}^{22} &= \int_{\Omega_e} (-\bar{D}_{22}) \psi_i^2 \psi_j^2\, dxdy, \qquad i, j = 1, 2, \ldots, s, \\
K_{ij}^{23} &= \int_{\Omega_e} (-\bar{D}_{12}) \psi_i^2 \psi_j^3\, dxdy, \qquad i = 1, 2, \ldots, s; \quad j = 1, 2, \ldots, p, \\
K_{ij}^{24} &= 0, \qquad i = 1, 2, \ldots, s; \quad j = 1, 2, \ldots, q, \\
K_{ij}^{33} &= \int_{\Omega_e} (-\bar{D}_{11}) \psi_i^3 \psi_j^3\, dxdy, \qquad i, j = 1, 2, \ldots, p, \\
K_{ij}^{34} &= 0, \qquad i = 1, 2, \ldots, p; \quad j = 1, 2, \ldots, q, \\
K_{ij}^{44} &= \int_{\Omega_e} (D_{66})^{-1} \psi_i^4 \psi_j^4\, dxdy, \qquad i, j = 1, 2, \ldots, q,
\end{aligned} \tag{10.136}$$

$$F_i^1 = \int_{\Omega_e} q\psi_i^1\, dxdy + \oint_{\Gamma_e} Q_n\psi_i^1\, ds, \qquad i = 1, 2, \ldots, r,$$
$$F_i^2 = \oint_{\Gamma_e} \theta_x n_x\ \psi_i^2\, ds, \qquad i = 1, 2, \ldots, s,$$
$$F_i^3 = \oint_{\Gamma_e} \theta_y n_y\ \psi_i^3\, ds, \qquad i = 1, 2, \ldots, p,$$
$$F_i^4 = \oint_{\Gamma_e} (\theta_y n_x + \theta_x n_y)\psi_i^4\, ds, \qquad i = 1, 2, \ldots, q.$$

An examination of the weak forms in Eqs. (10.128)–(10.131) shows that the following minimum continuity conditions are required of the interpolations functions ψ_i^α ($\alpha = 1, 2, 3, 4$) used in Eq. (10.134):

$$\begin{aligned} \psi_i^1 &= \text{linear in } x \text{ and linear in } y, \\ \psi_i^2 &= \text{linear in } x \text{ and constant in } y, \\ \psi_i^3 &= \text{linear in } y \text{ and constant in } x, \\ \psi_i^4 &= \text{linear in } x \text{ and linear in } y. \end{aligned} \tag{10.137}$$

For a rectangular element, the interpolation functions ψ_i^α that meet the minimum requirements are

$$\psi_1^2 = 1 - \frac{\bar{x}}{a}, \qquad \psi_2^2 = \frac{\bar{x}}{a}, \qquad \psi_1^3 = 1 - \frac{\bar{y}}{b}, \qquad \psi_2^3 = \frac{\bar{y}}{b}, \tag{10.138}$$

and ψ_i^1 and ψ_i^4 are the bilinear functions of a rectangular element [see Eq. (9.119)]. The rectangular element associated with this choice of interpolation functions is shown in Fig. 10.1b. The coefficient matrices in Eq. (10.136) can be easily evaluated for this element. For instance, we have (a and b are the side lengths of the

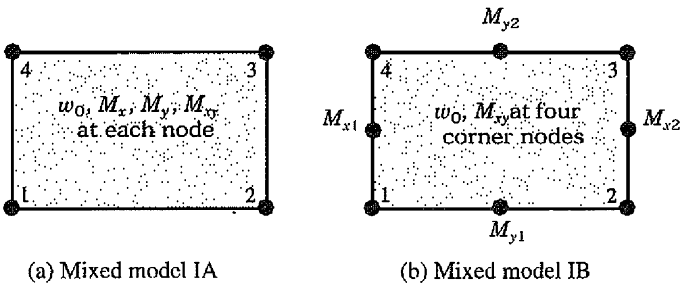

Figure 10.1 Mixed plate-bending elements based on CPT. (a) Rectangular element IA. (b) Rectangular element IB.

rectangular element):

$$[K^{12}] = \frac{b}{2a}\begin{bmatrix} 1 & -1 \\ -1 & 1 \\ -1 & 1 \\ 1 & -1 \end{bmatrix}, \qquad [K^{13}] = \frac{a}{2b}\begin{bmatrix} 1 & -1 \\ 1 & -1 \\ -1 & 1 \\ -1 & 1 \end{bmatrix},$$
$$[K^{22}] = -\bar{D}_{22}\frac{a}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}, \qquad [K^{33}] = -\bar{D}_{11}\frac{b}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}, \tag{10.139}$$

etc. The element stiffness matrix is of the order 12 × 12.

Another choice of ψ_i^α is provided by (see Fig. 10.1a):

$$\psi_i^1 = \psi_i^2 = \psi_i^3 = \psi_i^4 = \text{bilinear functions of a rectangular element.} \tag{10.140}$$

Computation of the coefficient matrices is simple and straightforward. The resulting stiffness matrix is of the order 16 × 16.

10.5.3 Mixed Model II

Governing Equations A simplified mixed model can be derived by eliminating the twisting moment M_{xy} from equations (10.123) and (10.126). We have [cf. Eq. (8.132)]

$$-\left(\frac{\partial^2 M_{xx}}{\partial x^2} - 4D_{66}\frac{\partial^4 w_0}{\partial x^2 \partial y^2} + \frac{\partial^2 M_{yy}}{\partial y^2}\right) - \lambda_v w_0$$
$$+ \lambda_b\left(N_1\frac{\partial^2 w_0}{\partial x^2} + 2N_6\frac{\partial^2 w_0}{\partial x \partial y} + N_2\frac{\partial^2 w_0}{\partial y^2}\right) = q, \tag{10.141a}$$

$$\frac{\partial^2 w_0}{\partial x^2} = -\left(\bar{D}_{22}M_{xx} + \bar{D}_{12}M_{yy}\right),$$
$$\frac{\partial^2 w_0}{\partial y^2} = -\left(\bar{D}_{12}M_{xx} + \bar{D}_{11}M_{yy}\right), \tag{10.141b}$$

where, in the interest of solving the natural vibration frequencies and buckling loads, the contributions of the inertia term (neglecting the rotary inertia) and in-plane compressive and shear forces are added through the parameters λ_v and λ_b, with $(I_0 = \rho h)$

$$\lambda_v = I_0\omega^2, \qquad \lambda_b = \frac{\hat{N}_{xx}}{N_1} = \frac{\hat{N}_{yy}}{N_2} = \frac{\hat{N}_{xy}}{N_6}. \tag{10.141c}$$

Here h denotes the plate thickness and $(\hat{N}_{xx}, \hat{N}_{yy}, \hat{N}_{xy})$ the in-plane compressive and shear forces.

Weak Forms The weak forms of Eqs. (10.141a,b) over a typical element Ω_e are

$$
\begin{aligned}
0 = & \int_{\Omega_e} \left(\frac{\partial \delta w_0}{\partial x} \frac{\partial M_{xx}}{\partial x} + \frac{\partial \delta w_0}{\partial y} \frac{\partial M_{yy}}{\partial y} + 4D_{66} \frac{\partial^2 \delta w_0}{\partial x \partial y} \frac{\partial^2 w_0}{\partial x \partial y} - q\delta w_0 \right) dxdy \\
& - \int_{\Omega_e} \left\{ \lambda_v w_0 \delta w_0 + \lambda_b \left[N_1 \frac{\partial \delta w_0}{\partial x} \frac{\partial w_0}{\partial x} + N_2 \frac{\partial \delta w_0}{\partial y} \frac{\partial w_0}{\partial y} \right.\right. \\
& \left.\left. + N_6 \left(\frac{\partial \delta w_0}{\partial x} \frac{\partial w_0}{\partial y} + \frac{\partial \delta w_0}{\partial y} \frac{\partial w_0}{\partial x} \right) \right] \right\} dxdy \\
& - \oint_{\Gamma_e} \left[\delta w_0 \bar{Q}_n - 2D_{66} \frac{\partial^2 w_0}{\partial x \partial y} (\delta\theta_x n_y + \delta\theta_y n_x) \right] ds, \qquad (10.142)
\end{aligned}
$$

$$
\begin{aligned}
0 = & \int_{\Omega_e} \left[\frac{\partial w_0}{\partial x} \frac{\partial \delta M_{xx}}{\partial x} - \delta M_{xx} \left(\bar{D}_{22} M_{xx} - \bar{D}_{12} M_{yy} \right) \right] dxdy \\
& - \oint_{\Gamma_e} \delta M_{xx} \theta_x n_x \, ds, \qquad (10.143)
\end{aligned}
$$

$$
\begin{aligned}
0 = & \int_{\Omega_e} \left[\frac{\partial w_0}{\partial y} \frac{\partial \delta M_{yy}}{\partial y} - \delta M_{yy} \left(\bar{D}_{11} M_{yy} - \bar{D}_{12} M_{xx} \right) \right] dxdy \\
& - \oint_{\Gamma_e} \delta M_{yy} \theta_y n_y \, ds. \qquad (10.144)
\end{aligned}
$$

The primary and secondary variables of the formulation are

$$
w_0, \quad M_{xx}, \quad M_{yy}, \qquad (10.145a)
$$

$$
V_n, \quad \theta_x n_x \equiv \frac{\partial w_0}{\partial x} n_x, \quad \theta_y n_y \equiv \frac{\partial w_0}{\partial y} n_y, \qquad (10.145b)
$$

where V_n is the effective shear force (Kirchhoff free edge condition) [cf. (8.142)]:

$$
V_n = \bar{Q}_n + \frac{\partial M_{ns}}{\partial s}, \quad \bar{Q}_n = \bar{Q}_x n_x + \bar{Q}_y n_y, \qquad (10.145c)
$$

$$
\begin{aligned}
\bar{Q}_x &= Q_x - \lambda_b \left(N_1 \frac{\partial w_0}{\partial x} + N_6 \frac{\partial w_0}{\partial y} \right), \\
\bar{Q}_y &= Q_y - \lambda_b \left(N_6 \frac{\partial w_0}{\partial x} + N_2 \frac{\partial w_0}{\partial y} \right). \qquad (10.145d)
\end{aligned}
$$

The functional associated with Eqs. (10.142)–(10.144) is

$$
\begin{aligned}
J_2(w_0, M_x, M_y) = & \int_{\Omega_e} \left\{ \frac{1}{2} (-\bar{D}_{22} M_{xx}^2 + 2\bar{D}_{12} M_{xx} M_{yy} - \bar{D}_{11} M_{yy}^2 - \lambda_v w_0^2) \right. \\
& - \frac{1}{2} \lambda_b \left[N_1 \left(\frac{\partial w_0}{\partial x} \right)^2 + N_2 \left(\frac{\partial w_0}{\partial y} \right)^2 + 2N_6 \frac{\partial w_0}{\partial x} \frac{\partial w_0}{\partial y} \right]
\end{aligned}
$$

$$+ \frac{\partial w_0}{\partial x}\frac{\partial M_{xx}}{\partial x} + \frac{\partial w_0}{\partial y}\frac{\partial M_{yy}}{\partial y} + 2D_{66}\left(\frac{\partial^2 w_0}{\partial x \partial y}\right)^2 - q w_0 \Big\} dxdy$$

$$- \oint_{\Gamma_e} (M_{xx}\theta_x n_x + M_{yy}\theta_y n_y + V_n w_0)\, ds. \tag{10.146}$$

Finite Element Model An examination of the weak forms (10.143) and (10.144) reveals that the minimum continuity conditions of the interpolation functions ψ_i^α ($\alpha = 1, 2, 3$) are the same as those listed in Eq. (10.137). Therefore, the interpolation functions of Eq. (10.138) or Eq. (10.140) can be used. The corresponding rectangular elements are shown in Fig. 10.2.

The finite element model of Eqs. (10.142)–(10.144) is obtained by substituting the approximations in Eq. (10.134). We obtain

$$\begin{bmatrix} [K^{11}] & [K^{12}] & [K^{13}] \\ & [K^{22}] & [K^{23}] \\ \text{symm.} & & [K^{33}] \end{bmatrix} \begin{Bmatrix} \{w\} \\ \{M_x\} \\ \{M_y\} \end{Bmatrix}$$

$$= \begin{Bmatrix} \{F^1\} \\ \{F^2\} \\ \{F^3\} \end{Bmatrix} + \lambda_v \begin{Bmatrix} [S]\{w\} \\ \{0\} \\ \{0\} \end{Bmatrix} + \lambda_b \begin{Bmatrix} [G]\{w\} \\ \{0\} \\ \{0\} \end{Bmatrix}, \tag{10.147a}$$

where

$$K_{ij}^{11} = 4D_{66}\int_{\Omega_e} \frac{\partial^2 \psi_i^1}{\partial x \partial y}\frac{\partial^2 \psi_i^1}{\partial x \partial y}\, dxdy, \qquad i, j = 1, 2, \ldots, r,$$

$$K_{ij}^{12} = \int_{\Omega_e} \frac{\partial \psi_i^1}{\partial x}\frac{\partial \psi_j^2}{\partial x}\, dxdy, \qquad i = 1, 2, \ldots, r; \quad j = 1, 2, \ldots, s,$$

$$K_{ij}^{13} = \int_{\Omega_e} \frac{\partial \psi_i^1}{\partial y}\frac{\partial \psi_j^3}{\partial y}\, dxdy, \qquad i = 1, 2, \ldots, r; \quad j = 1, 2, \ldots, p,$$

$$K_{ij}^{22} = \int_{\Omega_e} (-\bar{D}_{22})\psi_i^2\psi_j^2\, dxdy, \qquad i, j = 1, 2, \ldots, s,$$

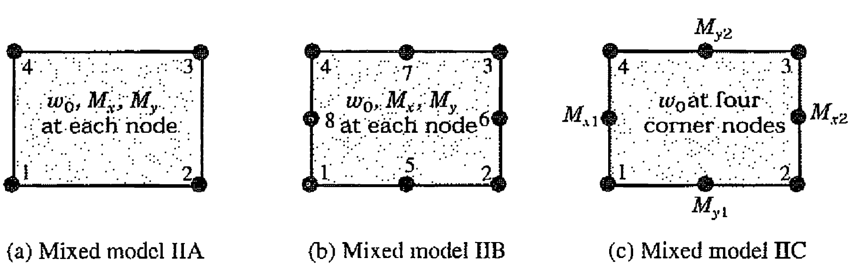

Figure 10.2 Mixed rectanglular plate-bending elements based on CPT. (a) Mixed model IIA. (b) Mixed model IIB. (c) Mixed model IIC.

$$K_{ij}^{23} = \int_{\Omega_e} (-\bar{D}_{12})\psi_i^2\psi_j^3\,dxdy, \qquad i = 1, 2, \ldots, s; \quad j = 1, 2, \ldots, p,$$

$$K_{ij}^{33} = \int_{\Omega_e} (-\bar{D}_{11})\psi_i^3\psi_j^3\,dxdy, \qquad i, j = 1, 2, \ldots, p,$$

$$F_i^1 = \int_{\Omega_e} q\psi_i^1\,dxdy + \oint_{\Gamma_e} V_n\psi_i^1\,ds, \qquad i = 1, 2, \ldots, r,$$

$$F_i^2 = \oint_{\Gamma_e} \theta_x n_x \psi_i^2\,ds, \qquad i = 1, 2, \ldots, s,$$

$$F_i^3 = \oint_{\Gamma_e} \theta_y n_y \psi_i^3\,ds, \qquad i = 1, 2, \ldots, p,$$

$$S_{ij} = \int_{\Omega_e} \psi_i^1\psi_j^1\,dxdy, \qquad i, j = 1, 2, \ldots, r,$$

$$G_{ij} = \int_{\Omega_e} \left[N_1\frac{\partial\psi_i^1}{\partial x}\frac{\partial\psi_j^1}{\partial x} + N_2\frac{\partial\psi_i^1}{\partial y}\frac{\partial\psi_j^1}{\partial y} + N_6\left(\frac{\partial\psi_i^1}{\partial x}\frac{\partial\psi_j^1}{\partial y} + \frac{\partial\psi_i^1}{\partial y}\frac{\partial\psi_j^1}{\partial x}\right)\right]dxdy,$$

$$i, j = 1, 2, \ldots, r. \qquad (10.147b)$$

This completes the development of the mixed finite element models. A number of plate problems are analyzed using the mixed rectangular elements developed in this section, and the results are discussed in the examples presented below.

Example 10.4 The four mixed plate elements discussed above are used to analyze bending of rectangular plates with simply supported and clamped boundary conditions. The bending analysis is carried out using the following types of mixed elements.

Mixed Model I Bilinear approximations are used for deflection w and moments M_x, M_y, and M_{xy} (mixed model IA). Thus the element has four nodes and four degrees of freedom at each node (see Fig. 10.2a), resulting in a 16×16 element stiffness matrix.

Mixed Model II This model is subdivided into three models depending on the type of interpolation used. In mixed model IIA, bilinear interpolation of w_0, M_{xx}, and M_{yy} is used; in mixed model IIB, quadratic (serendipity) interpolation is used (see Fig. 10.2b,c). These elements have stiffness matrices of order 12×12 and 24×24, respectively. In mixed model IIC, bilinear approximations are used for deflection w_0 and partial linear approximations for M_{xx} and M_{yy}. The four corner nodes each have one deflection degree of freedom. Midnodes on the sides perpendicular to the y-axis have one degree of freedom M_{yy}, and midnodes on the sides perpendicular to the x-axis have one degree of freedom M_{xx}. This element results in a 8×8 element stiffness matrix.

Table 10.2 Maximum deflection and bending moment of simply supported, square, isotropic ($\nu = 0.3$) plate under uniformly distributed load

Mesh Size	Mixed Models IA	IIA	IIB	IIC	Herrmann [22]
	Central Deflection, $(w_0 D/q_0 a^4)10^2$ (0.4062)[a]				
1 × 1	0.4613 (16)[b]	0.3906 (12)	0.3867 (24)	0.4943 (8)	0.9018 (6)
2 × 2	0.4237 (36)	0.4082 (27)	0.4053 (63)	0.4289 (21)	0.5127 (15)
4 × 4	0.4106 (100)	0.4069 (75)	0.4062 (195)	0.4117 (65)	0.4316 (45)
6 × 6	0.4082 (196)	0.4066 (147)	0.4062 (399)	0.4087 (133)	0.4174 (101)
	Bending Moment at the Center, $(M_{xx}/q_0 a^2)10$ (0.4789)[c]				
1 × 1	0.7196	0.6094	0.3813	0.3482	0.328
2 × 2	0.5246	0.5049	0.4818	0.4498	0.446
4 × 4	0.4891	0.4849	0.4788	0.4721	0.471
6 × 6	0.4834	0.4851	0.4789	0.4579	0.476

[a]Exact solution from Eq. (8.162a).
[b]Number of degrees of freedom in the mesh of a quarter plate model.
[c]Exact solution from Eq. (8.162b).

Table 10.3 Maximum deflection and bending moment of clamped, square, isotropic ($\nu = 0.3$) plate under uniformly distributed load

Mesh Size	Mixed Models IA	IIA	IIB	IIC	Herrmann [22]
	Central Deflection, $(w_0 D/q_0 a^4)10^2$ (0.1265)[a]				
1 × 1	0.1664 (16)[b]	0.1563 (12)	0.1466 (24)	0.2278 (8)	0.7440 (6)
2 × 2	0.1529 (36)	0.1480 (27)	0.1260 (63)	0.1627 (21)	0.2854 (15)
4 × 4	0.1339 (100)	0.1325 (75)	0.1264 (195)	0.1359 (65)	0.1696 (45)
6 × 6	0.1299 (196)	0.1292 (147)	0.1265 (399)	0.1307 (133)	0.1463 (101)
	Bending Moment at the Center, $(M_{xx}/q_0 a^2)10$ (0.231)[a]				
1 × 1	0.5193	0.4875	0.2056	0.2487	0.208
2 × 2	0.3166	0.2899	0.2248	0.2432	0.242
4 × 4	0.2478	0.2443	0.2287	0.2339	0.235
6 × 6	0.2374	0.2358	0.2290	0.2313	0.232

[a]From Timoshenko and Woinowski-Krieger [23]; also see Example 8.12.
[b]Number of degrees of freedom in the mesh of a quarter plate model.

The mixed finite element solutions are presented in Tables 10.2 and 10.3 for isotropic plates and in Table 10.4 for orthotropic plates. The orthotropic properties used are:

$$\text{Glass–epoxy plates:}\quad E_1 = 7.8 \times 10^6\,\text{psi},\qquad E_2 = 2.6 \times 10^6\,\text{psi},$$
$$\nu_{12} = 0.25,\qquad G_{12} = 1.3 \times 10^6\,\text{psi}.$$

Table 10.4 Maximum deflection and bending moments of simply supported, square, orthotropic plates under uniformly distributed transverse load [$\bar{w} = (w_0 H/q_0 a^4)10^3$, $H = D_{12} + 2D_{66}$; $\bar{M}_{xx} = (M_{xx}/q_0 a^2)10$; $\bar{M}_{yy} = (M_{yy}/q_0 a^2)10^2$]

	IIA			IIB		
Mesh	$\bar{w}$	$\bar{M}_{xx}$	$\bar{M}_{yy}$	$\bar{w}$	$\bar{M}_{xx}$	$\bar{M}_{yy}$
			Glass–Epoxy Plates			
1 × 1	2.9737	0.9435	3.6290	2.9412	0.5718	2.2297
2 × 2	3.1041	0.8078	2.8937	3.3081	0.7687	2.7849
4 × 4	3.0930	0.7737	2.7851	3.0872	0.7627	2.7556
6 × 6	3.0901	0.7676	2.7685	3.0876	0.7628	2.7556
8 × 8	3.0890	0.7654	2.7628	3.0876	0.7628	2.7556
Exact[a]	3.0876	0.7628	2.7556	3.0876	0.7628	2.7556
			Graphite–Epoxy Plates			
1 × 1	0.9348	1.6152	0.6720	0.9207	0.9491	0.3800
2 × 2	0.9346	1.3570	0.2860	0.9224	1.3094	0.3370
4 × 4	0.9220	1.2845	0.3143	0.9198	1.2661	0.3274
6 × 6	0.9208	1.2740	0.3252	0.9199	1.2658	0.3272
8 × 8	0.9204	1.2704	0.3270	0.9200	1.2658	0.3272
Exact[a]	0.9200	1.2659	0.3271	0.9200	1.2659	0.3271

[a]From Reddy [24].

Table 10.5 Natural frequencies ($\lambda_v = \omega^2 \rho h\, a^4/D$) of simply supported (or hinged) and clamped square isotropic ($\nu = 0.3$) plates

	Hinged (Exact: 389.636)		Clamped (Exact: 1295.28)	
Mesh	IIA	IIB	IIA	IIB
1 × 1	576.000	408.439	1440.000	1355.077
2 × 2	431.529	390.461	1361.283	1303.380
3 × 3	407.809	389.792	1325.837	1298.051
4 × 4	399.766	389.685	1312.403	1296.114

Graphite–epoxy plates: $E_1 = 31.8 \times 10^6$ psi, $E_2 = 1.02 \times 10^6$ psi, $\nu_{12} = 0.31$, $G_{12} = 0.96 \times 10^6$ psi.

The present finite element solutions for center deflection and bending moments are compared with various finite element solutions available in the literature. From the numerical results it is clear that mixed models IIA and IIB are the best among the mixed models discussed here.

Example 10.5 Fundamental frequencies and buckling loads of plates are presented for the following cases:

(1) vibration of simply supported and clamped square plates (Table 10.5);

Table 10.6 Frequency parameters ($\lambda_V^{1/2} = \omega a^2\sqrt{\rho h/D}$) of cantilever plates

Mode	Mesh	Present Analysis IIA	Present Analysis IIB	Anderson [25]	Barton [26][a] Ritz	Barton [26][a] Test	Plunkett[a] [27]
1		3.27	3.46	3.99	3.47	3.42	3.50
2		14.46	14.60	15.30	14.93	14.52	14.50
3	2 × 1	22.89	22.49	21.16	21.26	20.86	21.70
4		51.28	49.07	49.47	48.71	46.90	48.10
1		3.41	3.44	3.44	3.47	3.42	3.50
2		14.55	14.76	14.76	14.93	14.52	14.50
3	4 × 2	22.64	21.51	21.60	21.26	20.86	21.70
4		49.93	48.13	48.28	48.71	46.90	48.10

[a]Results are independent of mesh.

Table 10.7 Buckling coefficients (λ_b) for simply supported and clamped, square, isotropic ($\nu = 0.3$) plates under uniform edge loads

Load Case (Exact)[a]	Mesh	Simply Supported IIA	Simply Supported IIB	Simply Supported Carson and Newton [28]	Clamped IIA	Clamped IIB	Clamped Carson and Newton [28]
Uniaxial	2 × 2	4.2095	4.0063	4.0158	11.530	10.2009	—
(S: 4.00)	3 × 3	4.0922	4.0012	4.0032	10.688	10.1107	—
(C: 10.07)	4 × 4	4.0517	4.0004	4.0010	10.412	10.0866	—
	8 × 8	4.0129	4.0000	4.0007	10.156	10.0748	—
Biaxial	2 × 2	2.1048	2.0031	—	6.0592	5.3746	5.3622
(S: 2.00)	3 × 3	2.0461	2.0006	—	5.6339	5.3233	5.3660
(C: 5.30)	4 × 4	2.0258	2.0002	—	5.4868	5.3102	5.3271
	8 × 8	2.0064	2.0000	—	5.3487	5.3040	5.3054
Shear[b]	2 × 2	—	14.4124	10.016	—	29.3967	23.264
(S: 9.34)	3 × 3	14.3220	9.6758	9.418	25.7773	15.4640	15.043
(C: 14.71)	4 × 4	11.1499	9.3816	—	18.5902	14.8361	—
	8 × 8	10.2898	9.3403	—	16.7104	14.7123	—

[a]From Timoshenko and Gere [29] and Reddy [30]; S = simply supported; C = clamped.
[b]Obtained using full plate model.

(2) vibration of rectangular cantilever plate (Table 10.6); and

(3) buckling of simply supported and clamped square plates under various edge loads (Table 10.7).

In summary, the mixed finite element models of plates give accurate results for deflections, stress, buckling loads, and natural frequencies. The mixed models give more accurate results when compared to certain conforming and nonconforming elements. Further, the mixed finite elements are algebraically less complex, and involve less "element forming" efforts.

10.6 CLOSURE

In this chapter, mixed variational formulations of problems of solid mechanics are presented. The construction of functionals associated with the Hellinger–Reissner and Reissner variational principles was developed for simple problems of beams. The use of the variational methods to determine approximate solutions as well as finite element models based on the mixed variational statements is illustrated via beam theories. Alternative mixed variational formulations of the classical plate theory were also developed and their finite element models were derived. Mixed formulations and associated finite element models of the first-order shear deformation theory are not included here, but they can be found in Refs. 31 and 32.

There exists a vast literature on mixed and so-called hybrid formulations (see, for example, [33–35]). Finite element models based on these elements have received mixed success. Finite element models based on mixed formulations in which the strains are treated as independent variables (e.g., see Section 10.4.2) have enjoyed better success.

EXERCISES

Derive the Euler equations of the stationary functionals in Exercises 10.1–10.4.

10.1

$$I(u, \mathbf{v}) = \int_V \left(-\frac{1}{2k}\mathbf{v}\cdot\mathbf{v} + \mathbf{v}\cdot \operatorname{grad} u - fu \right) dV - \int_{S_2} qu\, ds - \int_{S_1} \hat{\mathbf{n}}\cdot\mathbf{v}(u - \hat{u})\, ds.$$

10.2

$$I(w, \phi, \lambda) = \int_0^L \left[\frac{EI}{2}\left(\frac{d\phi}{dx}\right)^2 + fw + \lambda\left(\frac{dw}{dx} + \phi\right)\right] dx - F_0 w(L).$$

10.3

$$I(u_1, u_2, P) = \int_R \left\{ \frac{\mu}{2}(u_{i,j} + u_{j,i})u_{i,j} - Pu_{i,i} \right\} dx_1 dx_2 - \int_{S_2} \hat{t}_i u_i\, ds.$$

10.4

$$I(w, \phi_1, \phi_2, \lambda_1, \lambda_2) = \frac{D}{2}\int_\Omega \left[\left(\frac{\partial \phi_1}{\partial x_1}\right)^2 + \left(\frac{\partial \phi_2}{\partial x_2}\right)^2 + 2\nu \frac{\partial \phi_2}{\partial x_1}\frac{\partial \phi_2}{\partial x_2} + (1-\nu)\left(\frac{\partial \phi_1}{\partial x_2} + \frac{\partial \phi_2}{\partial x_1}\right)^2 \right] \times dx_1 dx_2 + \int_\Omega \left[\lambda_1\left(\frac{\partial w}{\partial x_1} + \phi_1\right) + \lambda_2\left(\frac{\partial w}{\partial x_2} + \phi_2\right) - fw \right] dx_1 dx_2.$$

10.5 Derive the Hellinger–Reissner functional for the bar problem of Section 10.1.2 [i.e., Eqs. (10.2)–(10.4) and (10.13)].

10.6 Derive the stationary functional corresponding to the problem of minimizing the functional

$$I(u_i) = \int_V \left[\tfrac{1}{2}(u_{i,j} + u_{j,i})u_{i,j} - f_i u_i\right] dV - \int_{S_2} \hat{t}_i u_i \, ds$$

subject to the constraint $u_{i,i} = 0$.

10.7 (*Washizu's Principle* [4,7–10]). Derive the following generalized Washizu functional:

$$\Pi_W(u_i, \varepsilon_{ij}, \sigma_{ij}) = \int_V \Big\{\left[\tfrac{1}{2}(u_{i,j} + u_{j,i} + u_{m,i}u_{m,j}) - \varepsilon_{ij}\right]\sigma_{ij}$$

$$+ \rho\Psi - f_i u_i\Big\} \, dV - \int_{S_1} (u_i - \hat{u}_i)t_i \, ds - \int_{S_2} \hat{t}_i u_i \, ds,$$

where ε_{ij}, σ_{ij}, u_i, and t_i are subject to independent variations, and $\Psi = \Psi(\varepsilon_{ij}, T)$ is the Gibb's free-energy function.

10.8 Derive the Euler equations of the Washizu functional by requiring that the actual state of equilibrium of a body subjected to mechanical and thermal loads is associated with a stationary value of the functional Π_W in Exercise 10.7.

10.9 Modify the Hellinger–Reissner variational principle for elastic bodies acted upon by mechanical and thermal loads.

10.10 The mixed variational principle for axisymmetric bending of circular plates requires $\delta\Pi_m = 0$, where

$$\Pi_m(w_0, M_r, M_\theta) = \pi \int_0^a \Bigg[K_s G_{13} h \left(\phi + \frac{dw_0}{dr}\right)^2 + 2M_r \frac{d\phi}{dr} + \frac{2}{r} M_\theta \phi$$

$$+ \bar{D}_{22} M_r^2 - 2\bar{D}_{12} M_r M_\theta + \bar{D}_{11} M_\theta^2 - 2q w_0\Bigg] r \, dr,$$

where a is the radius of the plate, K_s the shear correction factor, and

$$\bar{D}_{ij} = \frac{D_{ij}}{D_0}, \qquad D_0 = D_{12}^2 - D_{11}D_{22}.$$

Obtain the Euler equations of the functional.

10.11 The principle of minimum total potential energy for axisymmetric bending of polar orthotropic plates according to the first-order shear deformation theory requires $\delta\Pi(w_0, \phi) = 0$, where

$$\delta\Pi(w_0,\phi)=\int_b^a \left[\left(D_{11}\frac{d\phi}{dr}+D_{12}\frac{\phi}{r}\right)\frac{d\delta\phi}{dr}+\frac{1}{r}\left(D_{12}\frac{d\phi}{dr}+D_{22}\frac{\phi}{r}\right)\delta\phi \right.$$
$$\left. +A_{55}\left(\phi+\frac{dw_0}{dr}\right)\left(\delta\phi+\frac{d\delta w_0}{dr}\right)-q\delta w_0\right] r\,dr \qquad \text{(a)}$$

where b is the inner radius and a the outer radius. Derive the displacement finite element model of the equations. In particular, show that the finite element model is of the form (i.e., define the matrix coefficients of the following equation):

$$\begin{bmatrix} [K^{11}] & [K^{12}] \\ [K^{12}]^{\mathrm{T}} & [K^{22}] \end{bmatrix} \begin{Bmatrix} \{w\} \\ \{\phi\} \end{Bmatrix} = \begin{Bmatrix} \{F\} \\ \{0\} \end{Bmatrix}. \qquad \text{(b)}$$

10.12 Derive the mixed finite element model for the annular plates using the following variational statement:

$$0 = \int_b^a \left\{A_{55}\left(\phi + \frac{dw_0}{dr}\right)\frac{d\delta w_0}{dr} + M_r\frac{d\delta\phi}{dr} + \left(\frac{1}{r}M_\theta + Q\right)\delta\phi \right.$$
$$+ \left(\frac{d\phi}{dr} + \bar{D}_{22}M_r - \bar{D}_{12}M_\theta\right)\delta M_r$$
$$\left. + \left[\frac{\phi}{r} + \left(\bar{D}_{11}M_\theta - \bar{D}_{12}M_r\right)\right]\delta M_\theta - q\delta w_0\right\} r\,dr,$$

where $Q = A_{55}(\phi+dw_0/dr)$ and $\bar{D}_{ij} = D_{ij}/D_0$, $D_0 = D_{12}D_{12}-D_{11}D_{22}$. Show that the finite element model is of the form

$$\begin{bmatrix} [K^{11}] & [K^{12}] & [0] & [0] \\ & [K^{22}] & [K^{23}] & [K^{24}] \\ \text{symmetric} & & [K^{33}] & [K^{34}] \\ & & & [K^{44}] \end{bmatrix} \begin{Bmatrix} \{w\} \\ \{\phi\} \\ \{M_r\} \\ \{M_\theta\} \end{Bmatrix} = \begin{Bmatrix} \{F\} \\ \{0\} \\ \{0\} \\ \{0\} \end{Bmatrix}.$$

10.13 Derive the functional in Eq. (10.133a) by identifying $B(\Lambda, \delta\Lambda)$ and $l(\delta\Lambda)$, where $\Lambda = (w_0, M_{xx}, M_{yy}, M_{xy})$, and then using Eq. (7.53):

$$J(\Lambda) = \tfrac{1}{2}B(\Lambda, \delta\Lambda) - l(\delta\Lambda).$$

10.14 Verify the element matrices in Eq. (10.139).

10.15 Evaluate the element matrices in Eq. (10.147a,b) for mixed model ΠA.

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ANSWERS/SOLUTIONS TO SELECTED PROBLEMS

Chapter 2

2.1 The required equation is given by $\mathbf{C} \cdot [\mathbf{A} - (\mathbf{A} \cdot \hat{\mathbf{e}}_B)\hat{\mathbf{e}}_B] = 0$ (or any multiple of it).

2.2 A necessary and sufficient condition for the three vectors to be coplanar is that

$$(\mathbf{A} - \mathbf{B}) \times (\mathbf{B} - \mathbf{C}) \cdot (\mathbf{A} - \mathbf{C}) = 0,$$

which provides an equation for the required plane.

2.3 We begin with the left side of the equality and arrive at the right side:

$$\begin{aligned}\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) &= A_i \hat{\mathbf{e}}_i \times (\hat{\mathbf{e}}_l \varepsilon_{jkl} B_j C_k) = A_i B_j C_k \varepsilon_{jkl} \varepsilon_{pil} \hat{\mathbf{e}}_p \\ &= (\delta_{jp}\delta_{ki} - \delta_{ji}\delta_{kp}) A_i B_j C_k \hat{\mathbf{e}}_p = A_i B_j C_i \hat{\mathbf{e}}_j - A_i B_i C_k \hat{\mathbf{e}}_k \\ &= (\mathbf{A} \cdot \mathbf{C})\mathbf{B} - (\mathbf{A} \cdot \mathbf{B})\mathbf{C}.\end{aligned}$$

2.4 This follows from Exercise 2.3.

2.5 (a) $\delta_{ii} = \delta_{11} + \delta_{22} + \delta_{33} = 1 + 1 + 1 = 3$.

(c) $F_{ij}\delta_{jk} = F_{i1}\delta_{1k} + F_{i2}\delta_{2k} + F_{i3}\delta_{3k}$. Clearly, the first term is zero unless $k = 1$. If $k = 1$, the remaining two terms are zero, giving F_{i1}. Similarly, the second term is zero unless $k = 2$, which makes the first and third terms zero so that the result is F_{i2}. Finally, the third term is zero unless $k = 3$, which makes the first two terms zero, giving F_{i3}. Thus, $F_{ij}\delta_{jk}$ is equal to F_{ik}.

(e) Follows from part (d).

2.6 Note that $\mathbf{B} \times \mathbf{C} \cdot \mathbf{A} = \mathbf{A} \times \mathbf{B} \cdot \mathbf{C}$.

2.8 Let $A = (1, 1, 0, 0)$, $B = (0, 1, 0, 1)$, $C = (0, 0, 1, 1)$ and $D = (1, 1, 1, 1)$. Then the relation

$$\alpha A + \beta B + \gamma C + \mu D = 0$$

implies

$$\alpha + \mu = 0, \qquad \alpha + \beta + \mu = 0, \qquad \gamma + \mu = 0, \qquad \beta + \gamma + \mu = 0$$

whose solution is $\alpha = \gamma = -\mu$. Hence, the vectors are linearly dependent.

2.10 (a) The set is linearly dependent and does not span $\Re^3$.

(b) The set is linearly independent. In particular, the vectors $\hat{\mathbf{e}}_1$, $\hat{\mathbf{e}}_2$, and $\hat{\mathbf{e}}_3$ are represented by the vectors **A**, **B**, and **C** as

$$\hat{\mathbf{e}}_1 = -\tfrac{1}{2}\mathbf{A} + \tfrac{1}{2}\mathbf{B} + \mathbf{C}, \qquad \hat{\mathbf{e}}_2 = \tfrac{1}{2}\mathbf{A} + \tfrac{1}{2}\mathbf{B} - \mathbf{C}, \qquad \hat{\mathbf{e}}_3 = -\tfrac{1}{2}\mathbf{A} + \tfrac{1}{2}\mathbf{B}.$$

Hence the set (**A**, **B**, **C**) spans $\Re^3$.

2.11 Follows as outlined in the problem statement.

2.12 (a) We have $\hat{\mathbf{e}}'_1 = \frac{1}{\sqrt{3}}(\hat{\mathbf{e}}_1 - \hat{\mathbf{e}}_2 + \hat{\mathbf{e}}_3)$; $\hat{\mathbf{e}}'_2 = \hat{\mathbf{n}}$, the unit normal to the plane; finally, the third basis vector in an orthonormal system is related to the other two vectors by $\hat{\mathbf{e}}'_3 = \hat{\mathbf{e}}'_1 \times \hat{\mathbf{e}}'_2$. Thus, the two coordinate systems are related by (note the matrix of direction cosines)

$$\begin{Bmatrix} \hat{\mathbf{e}}'_1 \\ \hat{\mathbf{e}}'_2 \\ \hat{\mathbf{e}}'_3 \end{Bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{2}{\sqrt{14}} & \frac{3}{\sqrt{14}} & \frac{1}{\sqrt{14}} \\ \frac{-4}{\sqrt{42}} & \frac{1}{\sqrt{42}} & \frac{5}{\sqrt{42}} \end{bmatrix} \begin{Bmatrix} \hat{\mathbf{e}}_1 \\ \hat{\mathbf{e}}_2 \\ \hat{\mathbf{e}}_3 \end{Bmatrix}.$$

(b) Transformation matrix relating $(\hat{\mathbf{e}}'_1, \hat{\mathbf{e}}'_2, \hat{\mathbf{e}}'_3)$ to $(\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2, \hat{\mathbf{e}}_3)$ is given by

$$[A] = \begin{bmatrix} \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{2}{\sqrt{6}} \end{bmatrix} \quad (a_{ij} = \hat{\mathbf{e}}'_i \cdot \hat{\mathbf{e}}_j).$$

(c)

$$[A] = \begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} & 0 \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

2.14 Follows from the definition

$$[A] = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{2} & \frac{1}{\sqrt{2}} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{\sqrt{2}} & \frac{1}{2} \end{bmatrix}.$$

2.15 The direction cosines are $(2/3, -2/3, -1/3)$.

2.16 (a) $\mathbf{A} \cdot \mathbf{B} = 6 - 4 = 2$.

(b) The angle between **A** and **B** is $\theta = 82.34°$.

2.18 (a) We have

$$\text{grad}(r) = \hat{\mathbf{e}}_i \frac{\partial}{\partial x_i}(x_j x_j)^{1/2} = \hat{\mathbf{e}}_i \frac{1}{2}(x_j x_j)^{(1/2)-1} 2x_i$$
$$= \hat{\mathbf{e}}_i x_i (x_j x_j)^{-1/2} = \frac{\mathbf{r}}{r}.$$

(b) We have

$$\text{grad}(r^n) = \hat{\mathbf{e}}_i \frac{\partial}{\partial x_i}(x_j x_j)^{n/2} = \hat{\mathbf{e}}_i \frac{n}{2}(x_j x_j)^{(n/2)-1} 2x_i$$
$$= n\hat{\mathbf{e}}_i x_i (x_j x_j)^{(n-2)/2} = nr^{n-2}\mathbf{r}.$$

(c) We have, in view of the result in (b),

$$\nabla^2(r^n) = \frac{\partial^2}{\partial x_i \partial x_i}(r^n) = n(n-2)r^{n-3}\frac{\partial r}{\partial x_i} x_i + nr^{n-2}\delta_{ii}$$
$$= n(n-2)r^{n-3}\frac{x_i}{r} x_i + 3nr^{n-2}$$
$$= [n(n-2) + 3n]r^{n-2} = n(n+1)r^{n-2}.$$

(e)

$$\text{div}(\mathbf{r} \times \mathbf{A}) = \hat{\mathbf{e}}_i \cdot \frac{\partial}{\partial x_i}(\varepsilon_{jkl} x_j A_k \hat{\mathbf{e}}_l) = \varepsilon_{jkl}\delta_{il}\left(\frac{\partial x_j}{\partial x_i} A_k + x_j \frac{\partial A_k}{\partial x_i}\right)$$
$$= (0 + 0) = 0.$$

(g)

$$\text{div}(r\mathbf{A}) = \hat{\mathbf{e}}_i \cdot \frac{\partial}{\partial x_i}(r A_j \hat{\mathbf{e}}_j) = \hat{\mathbf{e}}_i \cdot \hat{\mathbf{e}}_j \left(\frac{\partial r}{\partial x_i} A_j\right) = \frac{x_i}{r} A_i = \frac{1}{r}\mathbf{r} \cdot \mathbf{A}.$$

2.19 (a) Since $\partial^2 F/\partial x_i \partial x_j$ is symmetric in i and j, we obtain $\nabla \times (\nabla F) = \left(\hat{\mathbf{e}}_i \frac{\partial}{\partial x_i}\right) \times \left(\hat{\mathbf{e}}_j \frac{\partial F}{\partial x_j}\right) = \varepsilon_{ijk}\hat{\mathbf{e}}_k \frac{\partial^2 F}{\partial x_i \partial x_j} = 0.$

(d)

$$\operatorname{grad}(FG) = \hat{\mathbf{e}}_i \frac{\partial}{\partial x_i}(FG) = \hat{\mathbf{e}}_i \left(\frac{\partial F}{\partial x_i} G + F \frac{\partial G}{\partial x_i} \right)$$
$$= \nabla F G + F \nabla G.$$

(g) First, show that

$$\nabla(\mathbf{A} \cdot \mathbf{B}) = \nabla \mathbf{A} \cdot \mathbf{B} + \nabla \mathbf{B} \cdot \mathbf{A} \tag{1}$$

and

$$\mathbf{A} \times \operatorname{curl} \mathbf{B} = \nabla \mathbf{B} \cdot \mathbf{A} - \mathbf{A} \cdot \nabla \mathbf{B}. \tag{2}$$

From Eqs. (1) and (2) the required vector identity follows.

(h)

$$\operatorname{div}(\mathbf{A} \times \mathbf{B}) = \hat{\mathbf{e}}_i \frac{\partial}{\partial x_i}(\varepsilon_{jkl} A_j B_k \hat{\mathbf{e}}_l) = \varepsilon_{ijk} \left(\frac{\partial A_j}{\partial x_i} B_k + A_j \frac{\partial B_k}{\partial x_i} \right)$$
$$= \operatorname{curl} \mathbf{A} \cdot \mathbf{B} - \operatorname{curl} \mathbf{B} \cdot \mathbf{A}.$$

(j)

$$(\nabla \times \mathbf{A}) \times \mathbf{A} = \varepsilon_{ijk} \frac{\partial A_j}{\partial x_i} \hat{\mathbf{e}}_k \times (A_p \hat{\mathbf{e}}_p)$$
$$= \varepsilon_{ijk} \varepsilon_{kpq} \frac{\partial A_j}{\partial x_i} A_p \hat{\mathbf{e}}_q = (\delta_{ip}\delta_{jq} - \delta_{iq}\delta_{jp}) \frac{\partial A_j}{\partial x_i} A_p \hat{\mathbf{e}}_q$$
$$= \frac{\partial A_j}{\partial x_i} A_i \hat{\mathbf{e}}_j - \frac{\partial A_j}{\partial x_i} A_j \hat{\mathbf{e}}_i = \mathbf{A} \cdot \nabla \mathbf{A} - \nabla \mathbf{A} \cdot \mathbf{A}.$$

2.20 Follows from the gradient theorem, Eq. (2.54a).

2.21 Note that $\partial r / \partial x_i = x_i / r$ and $\operatorname{grad}(r^2) = 2\mathbf{r}$. Then use the divergence theorem to obtain the result.

2.23 The integral relations are obvious. For example, the identity in (a) is obtained by substituting $A = \phi \nabla \psi$ for **A** into Eq. (2.54b). The identity in (b) follows directly from (a) by interchanging ϕ and ψ and subtracting the resulting identity from the one in (a). Finally, identity (c) follows from (b) by replacing ψ with $\nabla^2 \psi$.

2.24 See Exercise 2.12(a) for the base vectors of the barred coordinate system in terms of the unbarred system; the matrix of direction cosines $[A]$ is given there. Then the components of the dyad in the barred coordinate system are $[\bar{T}] = [A][T][A]^{\mathrm{T}}$.

2.25 First establish the identity that the determinant of 3×3 matrix $[A]$ is given by

$$|A| = \varepsilon_{ijk} a_{1i} a_{2j} a_{3k} = \tfrac{1}{6} \varepsilon_{ijk} \varepsilon_{rst} a_{ir} a_{js} a_{kt}. \tag{1}$$

Then write the characteristic polynomial associated with the three-dimensional stress matrix $[\sigma]$ as $|[\sigma - \lambda I]| = 0$, and expand the determinant as in (1).

2.26 (a) Clearly, $\lambda_1 = 3$ is an eigenvalue of the matrix. The remaining two eigenvalues are $\lambda_2 = 2(1 + \sqrt{5})$, $\lambda_3 = 2(1 - \sqrt{5})$. The eigenvector associated with $\lambda_1 = 3$ is $\hat{\mathbf{A}}^{(1)} = \pm(0, 0, 1)$. Similarly, $\hat{\mathbf{A}}^{(2)} = \pm(-0.851, 0.526, 0)$ and $\hat{\mathbf{A}}^{(3)} = \pm(0.526, 0.851, 0)$.

(c) The eigenvalues are $\lambda_1 = 4$, $\lambda_2 = 2$, $\lambda_3 = 1$. The eigenvectors are

$$\hat{\mathbf{A}}^{(1)} = \pm\frac{1}{\sqrt{2}}(0, 1, -1), \qquad \hat{\mathbf{A}}^{(2)} = \pm\frac{1}{\sqrt{2}}(0, 1, 1),$$
$$\hat{\mathbf{A}}^{(3)} = \pm\frac{1}{\sqrt{2}}(1, 0, 0).$$

(e) The eigenvalues are $\lambda_1 = 11.824$, $\lambda_2 = 1.285$, $\lambda_3 = -7.109$. The eigenvector associated with λ_1 is

$$\hat{\mathbf{A}}^{(1)} = \pm(0.5239, 0.2462, 0.4396).$$

(f) The eigenvalues are

$$\lambda_1 = 3.247, \qquad \lambda_2 = 1.555, \qquad \lambda_3 = 0.198.$$

2.27 The invariants of a matrix $[A]$ in terms of its coefficients a_{ij} as well as its eigenvalues are

$$I_1 = a_{ii} = a_{11} + a_{22} + a_{33}, \qquad I_2 = -\tfrac{1}{2}(a_{ii}a_{jj} - a_{ij}a_{ji}), \qquad I_3 = |A|,$$
$$I_1 = \lambda_1 + \lambda_2 + \lambda_3, \qquad I_2 = -\tfrac{1}{2}\left[(\lambda_1 + \lambda_2 + \lambda_3)^2 - \lambda_1^2 - \lambda_2^2 - \lambda_3^2\right],$$
$$I_3 = \lambda_1 \lambda_2 \lambda_3.$$

(a) Using the actual matrix coefficients, we obtain

$$\begin{aligned} I_1 &= 4 + 0 + 3 = 7, \\ I_2 &= -\tfrac{1}{2}\left[(7)(7) - 2a_{12}^2 - 2a_{13}^2 - 2a_{23}^2 - a_{11}^2 - a_{22}^2 - a_{33}^2\right] \\ &= -\tfrac{1}{2}\left[49 - 2 \times 16 - (4)^2 - (3)^2\right] = 4, \\ I_3 &= -(-4)\left[(-4)3 - 0\right] = -48. \end{aligned}$$

(c) Using the actual matrix coefficients, we obtain

$$I_1 = 1 + 3 + 3 = 7,$$
$$I_2 = -\tfrac{1}{2}\left[49 - 2(-1)^2 - (1)^2 - (3)^2 - (3)^2\right] = -14,$$
$$I_3 = (1)\,[(3)(3) - (-1)(-1)] = 8.$$

(e) Using the matrix coefficients, we obtain

$$I_1 = 3 + 1 + 2 = 6,$$
$$I_2 = -\tfrac{1}{2}\left[(6)^2 - 2(5)^2 - 2(8)^2 - (3)^2 - (1)^2 - (2)^2\right] = -78,$$
$$I_3 = 3\,[(1)(2) - 0] - 5\,[(5)(2) - 0] + 8\,[0 - (8)(1)] = -108.$$

Using the eigenvalues, we obtain

$$I_1 = 11.824 + 1.285 - 7.109 = 6,$$
$$I_2 = -\tfrac{1}{2}\left[(6)^2 - (11.824)^2 - (1.285)^2 - (-7.109)^2\right] = -78,$$
$$I_3 = (11.824)(1.285)(-7.109) = -108.$$

2.28 (i) We have $\mathbf{t}_{\hat{\mathbf{n}}} = 2(\hat{\mathbf{e}}_1 + 7\hat{\mathbf{e}}_2 + \hat{\mathbf{e}}_3)$. $\sigma_n = 7.33$ ksi and $\sigma_s = 12.26$ ksi.
(iii) The stress vector is given by $\mathbf{t}_{\hat{\mathbf{n}}} = (7+\sqrt{2})\hat{\mathbf{e}}_1 - (8+\sqrt{2})\hat{\mathbf{e}}_2 + 4(1+\sqrt{2})\hat{\mathbf{e}}_3$.

2.30 (c) We have $(\overset{\leftrightarrow}{\phi} \times \mathbf{A})^{\mathrm{T}} = \phi_{ij} A_k \varepsilon_{jkl} (\hat{\mathbf{e}}_i \hat{\mathbf{e}}_l)^{\mathrm{T}} = \phi_{ij} A_k \varepsilon_{jkl} \hat{\mathbf{e}}_l \hat{\mathbf{e}}_i = -(A_k \hat{\mathbf{e}}_k) \times (\phi_{ij} \hat{\mathbf{e}}_j \hat{\mathbf{e}}_i) = \mathbf{A} \times (\overset{\leftrightarrow}{\Phi})^{\mathrm{T}}$.

Chapter 3

Exercises 3.1–3.14 in this chapter are designed to review general principles of continuum mechanics. Many of them have specific directions to carry out the proofs. Therefore, the answers are already given in the problem statements. The proofs are easier to establish if one uses the index notation in rectangular Cartesian coordinates.

3.1 The result follows from subtracting Eq. (b) from Eq. (a) of the problem statement.

3.5 Let $Q = 1$ in Eq. (b) of Exercise 3.1 and obtain

$$\frac{D}{Dt}\int_R dV = \oint_S \mathbf{v} \cdot \hat{\mathbf{n}}\, dS = \int_R \operatorname{div} \mathbf{v}\, dV.$$

By shrinking the volume to an infinitesimal volume dV, we have

$$\frac{D}{Dt}(dV) = \operatorname{div} \mathbf{v}\, dV \qquad \text{or} \qquad \operatorname{div} \mathbf{v} = \lim_{dV \to 0} \frac{1}{dV} \frac{D}{Dt}(dV).$$

3.6 Show that $\text{grad}(v^2/2) - \mathbf{v} \times \text{curl}\,\mathbf{v} = \mathbf{v} \cdot \text{grad}\,\mathbf{v}$.

3.7 Follows from the product rule of differentiation.

3.8 This is an interesting (perhaps difficult) exercise. It requires the establishment of the following identities [see Exercises 2.19(b), 2.19(i), and 2.19(j)]:

$$\nabla \cdot (\nabla \times \mathbf{A}) = 0, \tag{1}$$

$$(\nabla \times \mathbf{A}) \times \mathbf{A} = \mathbf{A} \cdot \nabla\mathbf{A} - \nabla\mathbf{A} \cdot \mathbf{A}, \tag{2}$$

$$\nabla \times (\mathbf{A} \times \mathbf{B}) = \mathbf{B} \cdot \nabla\mathbf{A} - \mathbf{A} \cdot \nabla\mathbf{B} + \mathbf{A}\,\text{div}\,\mathbf{B} - \mathbf{B}\,\text{div}\,\mathbf{A}. \tag{3}$$

Next, begin with the definition of the material derivative of the velocity vector to establish the required identity.

3.9 Follows directly from Eq. (c) of Exercise 3.1 by replacing Q with $\rho\mathbf{v}$.

3.10 Follows directly from Eq. (a) of Exercise 3.4 and the divergence theorem.

3.12 Follows directly from Exercises 3.7 and 3.10.

3.14 We have

$$\begin{aligned}
\text{div}\left(\overleftrightarrow{\sigma} \cdot \mathbf{v}\right) &= \hat{\mathbf{e}}_i \frac{\partial}{\partial x_i} \cdot (\sigma_{jk}\hat{\mathbf{e}}_j\hat{\mathbf{e}}_k \cdot v_m\hat{\mathbf{e}}_m) \\
&= \frac{\partial}{\partial x_i}(\sigma_{ik}v_k) = v_k\frac{\partial \sigma_{ik}}{\partial x_i} + \sigma_{ik}\frac{\partial v_k}{\partial x_i} \\
&= \text{div}\,\overleftrightarrow{\sigma} \cdot \mathbf{v} + \overleftrightarrow{\sigma} : (\nabla\mathbf{v})^{\mathrm{T}}.
\end{aligned}$$

3.16 Check the equilibrium equations for $f_1 = f_2 = f_3 = 0$. (a) Not possible. (b) Not possible. (c) Possible.

3.17 The body force components are $f_1 = 0$, $f_2 = 0$, and $f_3 = -4$.

3.18 The stresses are

$$\sigma_{12} = -\frac{F_0}{2I_3}(h^2 - x_2^2), \qquad \sigma_{22} = 0 \quad \left(I_3 = \frac{2bh^3}{3}\right).$$

3.19 The stresses are

$$\sigma_{12} = -\frac{q_0 x_1}{2I_3}(h^2 - x_2^2), \qquad \sigma_{22} = -\frac{q_0 x_2^3}{6I_3} + \frac{q_0 x_2 h^2}{2I_3} - \frac{q_0}{2b}.$$

3.20

$$\begin{Bmatrix} t_1 \\ t_2 \\ t_3 \end{Bmatrix}_{(1,1,3)} = \frac{1}{\sqrt{3}}\begin{Bmatrix} -3 \\ -14 \\ 7 \end{Bmatrix}, \qquad t_n = -\frac{10}{3}\ \text{psi}, \qquad t_s = 8.576\ \text{psi}.$$

3.21 The eigenvalues are

$$\lambda_1 = 6.856, \qquad \lambda_2 = -10.533, \qquad \lambda_3 = -3.323.$$

The eigenvectors are

$$\hat{\mathbf{A}}^{(1)} = \pm(0.42, 0.0498, -0.905), \qquad \hat{\mathbf{A}}^{(2)} = \pm(0.257, -0.964, 0.066),$$
$$\hat{\mathbf{A}}^{(3)} = \pm(0.870, 0.261, 0.418).$$

3.22 The eigenvalues (or principal stresses) are

$$\lambda_1 = 11.824 \text{ (psi)}, \qquad \lambda_2 = -7.109 \text{ (psi)}, \qquad \lambda_3 = 1.285 \text{ (psi)}.$$

The maximum stress is $\sigma_1 = 11.824$ psi and the plane of maximum stress is given by $\hat{\mathbf{n}}^{(1)} = \pm(0.730, 0.337, 0.594)$.

3.23 The linear strains are

$$e_{11} = 2(1-x_1)x_2; \qquad 2e_{12} = (3c_2+c_3)x_2^2 - 2c_1; \qquad e_{22} = -2c_3(1-x_1)x_2.$$

3.25 The displacements are given by

$$u_1 = x_1 - X_1 = \frac{e_0}{b}X_2, \qquad u_2 = x_2 - X_2 = 0,$$

from which the strains can be computed.

3.26 The displacements are given by

$$u_1 = x_1 - X_1 = \left(\frac{e_0}{b^2}\right)X_2^2, \qquad u_2 = 0.$$

3.28 Check for compatibility of strains: (a) yes. (b) no.

3.29 Use the strain transformation equations to obtain $e'_{11}(=e'_n) = e_0/2b$, and $e'_{12}(=e'_s) = 0$.

3.30 The principal strains computed using the eigenvalue approach are $\lambda_1 = 0$ and $\lambda_2 = 10 \times 10^{-4}$ (in./in.). The principal directions are $\hat{\mathbf{A}}^{(1)} = \frac{1}{\sqrt{5}}(2, 1)$ and $\theta = 26.57°$ (clockwise from the x-axis).

3.31 See answer to Exercise 3.29.

3.32 To express the vector form of the equilibrium equations

$$\nabla \cdot \overset{\leftrightarrow}{\sigma} + \rho_0 \mathbf{f} = \rho_0 \frac{\partial \mathbf{u}}{\partial t^2} \tag{1}$$

in cylindrical coordinates (r, θ, z), express $\overset{\leftrightarrow}{\sigma}$ in nonion form, and the del operator ∇, the body force vector $\mathbf{f}$, and the displacement vector $\mathbf{u}$ in the cylindrical coordinate system. Next, carry out the operations indicated in the equilibrium equation (i.e., take the divergence of the stress tensor, noting the dot product between the basis vectors) and collect the coefficients of various base vectors to obtain the required result.

3.33 The linear strain tensor in vector form is defined by

$$\overleftrightarrow{\varepsilon} = \tfrac{1}{2}\left[\nabla\mathbf{u} + (\nabla\mathbf{u})^T\right] \tag{1}$$

where

$$\nabla = \hat{\mathbf{e}}_r\frac{\partial}{\partial r} + \frac{1}{r}\hat{\mathbf{e}}_\theta\frac{\partial}{\partial\theta} + \hat{\mathbf{e}}_z\frac{\partial}{\partial z}, \qquad \frac{\partial\hat{\mathbf{e}}_r}{\partial\theta} = \hat{\mathbf{e}}_\theta, \qquad \frac{\partial\hat{\mathbf{e}}_\theta}{\partial\theta} = -\hat{\mathbf{e}}_r. \tag{2}$$

Using Eq. (2) and the displacement vector

$$\mathbf{u} = u_r\hat{\mathbf{e}}_r + u_z\hat{\mathbf{e}}_z, \tag{3}$$

evaluate the expressions needed in the vector definition of the strain tensor.

3.34 A special case of Exercise 3.33.

3.35 Begin with the requirement that

$$\frac{\partial^2 u_i}{\partial x_j\partial x_k} = \frac{\partial^2 u_i}{\partial x_k\partial x_j}.$$

Thus, $u_{i,jk}$ is symmetric in j and k. Hence,

$$0 = \frac{\partial^2 u_i}{\partial x_j\partial x_k}\varepsilon_{jkl} = \varepsilon_{jkl}\frac{\partial}{\partial x_k}\left(\frac{\partial u_i}{\partial x_j}\right) = \varepsilon_{jkl}\frac{\partial}{\partial x_k}(e_{ij} + \omega_{ij}).$$

Differentiate with respect to x_m and multiply with ε_{imn} to obtain

$$0 = \varepsilon_{jkl}\varepsilon_{imn}(e_{ij,km} + \omega_{ij,km}) = \varepsilon_{jkl}\varepsilon_{imn}e_{ij,km},$$

which is the component form the required identity. By multiplying with $\hat{\mathbf{e}}_n$ and $\hat{\mathbf{e}}_l$, one can obtain the vector identity.

Chapter 4

4.1 $V = (m_1 + m_2)gL_1(1 - \cos\theta_1) + m_2 g L_2(1 - \cos\theta_2)$.

4.2 $V = (WL/4)(3\theta_1^2 + 5\theta_2^2 + \theta_3^2) + \frac{1}{2}[k_1\theta_1^2 + k_2(\theta_2 - \theta_1)^2 + k_3(\theta_3 - \theta_2)^2]$.

4.3 The increment of complementary work done is given by $\Delta W^* = (u + v\theta)\times \Delta F_x + [v - (L + u)\theta]\Delta F_y$ (assuming small θ).

4.5 The strain energy and work done by external applied loads are

$$U = \int_0^L \frac{EI}{2}\left(\frac{d^2w_0}{dx^2}\right)^2 dx + \tfrac{1}{2}k[w_0(L)]^2, \qquad V = -\int_0^L q_0 w_0\, dx.$$

4.6 The total complementary strain energy is ($R_A = 8$ kN) $W^* = 74.9 + 0.244 f_s$ (N-m). Clearly, the strain energy due to transverse shear is very small compared to that due to bending.

4.8 The complementary strain energy is given by

$$U^* = \frac{1}{2EI}\left(\frac{P^2b^3}{3} + \frac{q_0^2b^5}{20} + \frac{Pq_0b^4}{4}\right) + \frac{1}{2EI}(P + q_0b)^2\frac{a^3}{3}$$
$$+ \frac{1}{2GJ}\left(Pb + \frac{q_0b^2}{2}\right)^2 a.$$

4.9 The total complementary strain energy of the structure is the sum of the complementary strain energies of the members:

$$U^* = \frac{A}{3K^2}\left[2 \times 5 \times \left(\frac{5P}{8A}\right)^3 + 2 \times 3 \times \left(\frac{3P}{8A}\right)^3 + 4\left(\frac{P}{A}\right)^3\right].$$

4.10 The strain energy and complementary strain energy are given by

$$U = \frac{19}{25}AEv\sqrt{v}, \qquad U^* = \left(\frac{50}{57}\right)^2 \frac{P^3}{3E^2A^2}.$$

4.11 The admissible virtual displacements are

$$\delta w_0 = 0, \qquad \frac{d\delta w_0}{dx} = 0 \quad \text{at } x = 0.$$

4.13 There are several possibilities. For example, if δP is applied at $x = L$ (much as P in the figure), then it should be in equilibrium such that

$$\delta P - \delta R_A - \delta R_c = 0,$$

where δR_A is the virtual tensile force at the left end and δR_c is the virtual compressive force at the right end.

4.15 One possibility is to apply a vertical force (downward) of δP at point D, and $0.5\delta P$ (upward) each at points A and B.

4.16 The total virtual work expression for the problem is

$$\delta W = \int_0^L EA\frac{d\delta u_0}{dx}\frac{du_0}{dx}dx + ku_0(L)\delta u_0(L) - \left[\int_0^L f(x)\delta u_0\, dx + P\delta u_0(L)\right].$$

4.18 The total virtual work is $\delta W^* = (v - \theta L \cos\theta)\delta F_y + (u + \theta L \sin\theta)\delta F_x$. For small θ, we have $v = \theta L$ and $u = 0$.

4.19 The total complementary virtual work expression for the problem is

$$\delta W^* = \int_0^L \frac{N}{EA}\delta N\, dx - \delta P_B u_0(L) + \frac{1}{k}\delta P_B F_s,$$

where it is assumed that a virtual force δP_B is applied to the right at $x = L$.

4.21 For this problem, we have $\delta W^* = \delta W_I^* + \delta W_E^*$, where $\delta W_I^* = \delta U^*$ and U^* is given in Exercise 4.8. The virtual work done by a virtual force δP applied at point A is $\delta W_E^* = -w_0^A \delta P$, where w_0^A is the vertical deflection at point A in the direction of the virtual load δP.

4.23 The total virtual work done is $\delta W^* = L_1(F_y \cos\theta_1 - F_x \sin\theta_1)\delta\theta_1 + L_2(F_y \cos\theta_2 - F_x \sin\theta_2)\delta\theta_2$.

4.24 Denote the forces and extensions in the three springs by (F_1, F_2, F_3) and (v_1, v_2, v_3), respectively, with similar notation for the virtual forces. Then the total virtual work done is

$$\delta W^* = \delta F_3 \left[(v_1 - v_2)\frac{b}{a} + v_3 - v_2\right] + \delta F(v_2 - v) + \delta M \left[\frac{v_2 - v_1}{a} - \theta\right].$$

4.25 The virtual strain energy is

$$\delta U = \int_0^L \left[N_{xx}\left(\frac{d\delta u_0}{dx} + \frac{dw_0}{dx}\frac{d\delta w_0}{dx}\right) - M_{xx}\frac{d^2\delta w_0}{dx^2}\right] dx.$$

4.27 The linear strain–displacement relations in cylindrical coordinates, for the axisymmetric case, are (see Section 8.2.1 for complete developments):

$$\begin{aligned} &\varepsilon_{rr} = \frac{\partial u_r}{\partial r}, \qquad \varepsilon_{\theta\theta} = \frac{u_r}{r}, \qquad \varepsilon_{zz} = \frac{\partial u_z}{\partial z}, \\ &\varepsilon_{rz} = \frac{1}{2}\left(\frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r}\right), \qquad \varepsilon_{r\theta} = 0, \qquad \varepsilon_{z\theta} = 0. \end{aligned} \tag{c}$$

4.29

$$\delta I = \int_a^b \left[u\frac{u'}{\sqrt{1 + (u')^2}}\delta u' + \sqrt{1 + (u')^2}\,\delta u\right] dx.$$

4.31

$$\delta I = \int_a^b (u''\delta u'' + \delta u'' u' + u''\delta u' + u'\delta u' - \delta u)dx.$$

4.33

$$\delta I = \int_\Omega \big(u_x \delta u_x + \delta u_x v_x + u_x \delta v_x + \delta u_y v_y + u_y \delta v_y + v_y \delta v_y + f\delta u + g\delta v\big) dx dy.$$

4.35

$$\delta I = \int_\Omega \big[\mu\left(\delta u_{i,j} u_{i,j} + u_{j,i}\delta u_{i,j}\right) - P\delta u_{i,i} - \delta P u_{i,i}\big] dx_1 dx_2 - \int_\Omega f_i \delta u_i \, dx_1 dx_2 - \int_{S_2} \hat{t}_i \delta u_i \, ds.$$

4.37 The total time taken by a particle in going from point A to point B is given by

$$T = \frac{1}{\sqrt{2g}} \int_0^b \sqrt{\frac{1 + (du/dx)^2}{u}} dx \equiv I(u).$$

4.38 The problem consists of finding the curve of minimum length joining the two given points A: (a, y_a) and B: (b, y_b). Mathematically, this amounts to minimizing the integral

$$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx \equiv I(y).$$

4.40 The Euler equations are

$$u'''' - u'' - 1 = 0, \quad \text{in } a < x < b,$$
$$u' - u''' = 0, \qquad u'' + u' = 0, \quad \text{at } x = a, b.$$

4.41 The Euler equations are

$$-\frac{d}{dx}\left\{EA\left[\frac{du}{dx} + \frac{1}{2}\left(\frac{dw}{dx}\right)^2\right]\right\} = 0, \qquad 0 < x < L,$$

$$-\frac{d}{dx}\left\{EA\frac{dw}{dx}\left[\frac{du}{dx} + \frac{1}{2}\left(\frac{dw}{dx}\right)^2\right]\right\} + \frac{d^2}{dx^2}\left(EI\frac{d^2w}{dx^2}\right) = q, \qquad 0 < x < L,$$

$$\left\{EA\left[\frac{du}{dx} + \frac{1}{2}\left(\frac{dw}{dx}\right)^2\right]\right\}\Bigg|_{x=L} - P = 0,$$

$$\left\{EA\frac{dw}{dx}\left[\frac{du}{dx} + \frac{1}{2}\left(\frac{dw}{dx}\right)^2\right] - \frac{d}{dx}\left(EI\frac{d^2w}{dx^2}\right)\right\}\Bigg|_{x=L} = 0,$$

$$\left(EI\frac{d^2w}{dx^2}\right)\Bigg|_{x=L} - M_0 = 0.$$

4.42 The Euler equations are

$$-u_{xx} - v_{xx} - v_{yy} + f = 0, \qquad -u_{xx} - u_{yy} - v_{yy} + g = 0 \quad \text{in } \Omega,$$
$$(u_x + v_x)n_x + v_y n_y = 0, \qquad u_x n_x + (u_y + v_y)n_y = 0 \quad \text{on } \Gamma.$$

4.43 The Euler equations are

$$\mu\left(-u_{i,jj} - u_{j,ij}\right) - P_{,i} - f_i = 0 \quad \text{in } \Omega,$$
$$\mu\left(u_{i,j} + u_{j,i}\right)n_j - Pn_i - \hat{t}_i = 0 \quad \text{on } S_2.$$

4.44 The Euler equation and the natural boundary conditions are given by

$$\delta w:\quad D\left(\frac{\partial^4 w}{\partial x^4} + \frac{\partial^4 w}{\partial y^4} + 2\frac{\partial^4 w}{\partial x^2 \partial y^2}\right) + kw - q = 0, \tag{i}$$

$$\frac{\partial \delta w}{\partial x}:\quad D\left(\frac{\partial^2 w}{\partial x^2} + \nu\frac{\partial^2 w}{\partial y^2}\right) = 0 \quad \text{at } x = 0, a \text{ for any } y, \tag{ii}$$

$$\frac{\partial \delta w}{\partial y}:\quad (1-\nu)D\left(\frac{\partial^2 w}{\partial x \partial y}\right) = 0 \quad \text{at } x = 0, a \text{ for any } y, \tag{iii}$$

$$\frac{\partial \delta w}{\partial y}:\quad D\left(\frac{\partial^2 w}{\partial y^2} + \nu\frac{\partial^2 w}{\partial x^2}\right) = 0 \quad \text{at } y = 0, b \text{ for any } x, \tag{iv}$$

$$\frac{\partial \delta w}{\partial x}:\quad (1-\nu)D\left(\frac{\partial^2 w}{\partial x \partial y}\right) = 0 \quad \text{at } y = 0, b \text{ for any } x. \tag{v}$$

4.45 The Euler equation and the natural boundary conditions are

$$\frac{d^2}{dr^2}\left(D_{11} r \frac{d^2 w_0}{dr^2}\right) - \frac{d}{dr}\left(D_{22}\frac{1}{r}\frac{dw_0}{dr}\right) = 0 \quad \text{in } r_i < r < r_0, \tag{1}$$

$$\left(D_{11} r \frac{d^2 w_0}{dr^2} + D_{12}\frac{dw_0}{dr}\right) = 0 \quad \text{at } r = r_i \text{ and } r = r_0, \tag{2}$$

$$-\left[D_{11}\frac{d}{dr}\left(r\frac{d^2 w_0}{dr^2}\right) - D_{22}\frac{1}{r}\frac{dw_0}{dr}\right] = 0 \quad \text{at } r = r_i. \tag{3}$$

4.46 The Euler equations are

$$\delta w_0:\quad \frac{\partial \lambda_x}{\partial x} + \frac{\partial \lambda_y}{\partial y} - kw_0 + q = 0, \tag{1}$$

$$\delta \phi_x:\quad \frac{\partial^2 \phi_x}{\partial x^2} + \nu\frac{\partial^2 \phi_y}{\partial x \partial y} + (1-\nu)\frac{\partial}{\partial y}\left(\frac{\partial \phi_x}{\partial y} + \frac{\partial \phi_y}{\partial x}\right) - \lambda_x = 0, \tag{2}$$

$$\delta \phi_y:\quad \frac{\partial^2 \phi_y}{\partial y^2} + \nu\frac{\partial^2 \phi_x}{\partial x \partial y} + (1-\nu)\frac{\partial}{\partial x}\left(\frac{\partial \phi_x}{\partial y} + \frac{\partial \phi_y}{\partial x}\right) - \lambda_y = 0, \tag{3}$$

$$\delta\lambda_x:\quad \frac{\partial w_0}{\partial x}+\phi_x=0,\qquad \delta\lambda_y:\quad \frac{\partial w_0}{\partial y}+\phi_y=0, \tag{4}$$

and the natural boundary conditions are

$$\delta\phi_x:\quad D\left(\frac{\partial\phi_x}{\partial x}+\nu\frac{\partial\phi_y}{\partial y}\right)=0\quad \text{at } x=0,a \text{ for any } y, \tag{5}$$

$$\delta\phi_y:\quad D(1-\nu)\left(\frac{\partial\phi_x}{\partial y}+\frac{\partial\phi_y}{\partial x}\right)=0\quad \text{at } x=0,a \text{ for any } y, \tag{6}$$

$$\delta\phi_y:\quad D\left(\frac{\partial\phi_y}{\partial y}+\nu\frac{\partial\phi_x}{\partial x}\right)=0\quad \text{at } y=0,b \text{ for any } x, \tag{7}$$

$$\delta\phi_x:\quad D(1-\nu)\left(\frac{\partial\phi_x}{\partial y}+\frac{\partial\phi_y}{\partial x}\right)=0\quad \text{at } y=0,b \text{ for any } x. \tag{8}$$

4.47 The Euler equations are

$$\delta w_0:\quad \gamma_x\frac{\partial}{\partial x}\left(\frac{\partial w_0}{\partial x}+\phi_x\right)+\gamma_y\frac{\partial}{\partial y}\left(\frac{\partial w_0}{\partial y}+\phi_y\right)-kw_0+q=0,$$

$$\delta\phi_x:\quad \frac{\partial^2\phi_x}{\partial x^2}+\nu\frac{\partial^2\phi_y}{\partial x\partial y}+(1-\nu)\frac{\partial}{\partial y}\left(\frac{\partial\phi_x}{\partial y}+\frac{\partial\phi_y}{\partial x}\right)-\gamma_x\left(\frac{\partial w_0}{\partial x}+\phi_x\right)=0,$$

$$\delta\phi_y:\quad \frac{\partial^2\phi_y}{\partial y^2}+\nu\frac{\partial^2\phi_x}{\partial x\partial y}+(1-\nu)\frac{\partial}{\partial x}\left(\frac{\partial\phi_x}{\partial y}+\frac{\partial\phi_y}{\partial x}\right)-\gamma_y\left(\frac{\partial w_0}{\partial y}+\phi_y\right)=0.$$

4.48

(a) The Euler equation is

$$1+\gamma\frac{d}{dx}\left\{\left[\int_a^b\sqrt{1+\left(\frac{du}{dx}\right)^2}\,dx-L\right]\frac{\frac{du}{dx}}{\sqrt{1+\left(\frac{du}{dx}\right)^2}}\right\}=0. \tag{1}$$

(b) The Euler equations are

$$\delta u:\qquad 1+\frac{d}{dx}\left[\frac{\lambda\frac{du}{dx}}{\sqrt{1+\left(\frac{du}{dx}\right)^2}}\right]=0, \tag{2}$$

$$\delta\lambda:\qquad \int_a^b\sqrt{1+\left(\frac{du}{dx}\right)^2}\,dx-L=0. \tag{3}$$

Clearly, a comparison of Eq. (2) with (1) shows that

$$\lambda = \gamma \left[\int_a^b \sqrt{1 + \left(\frac{du}{dx}\right)^2}\, dx - L \right]. \tag{4}$$

4.50 The Euler equations are

$$\delta\varepsilon_{xx}\colon\ EI\kappa_{xx} + \lambda = 0; \quad \delta\lambda\colon\ \kappa_{xx} - \frac{d^2w_0}{dx^2} = 0; \quad \delta w_0\colon\ -\frac{d^2\lambda}{dx^2} - q = 0.$$

Chapter 5

5.1 The Euler equations are

$$-\frac{d}{dx}\left(EA\frac{du_0}{dx}\right) - f(x) = 0, \qquad 0 < x < L,$$

$$\left[EA\frac{du_0}{dx}\right]_{x=L} + ku_0(L) - P = 0.$$

5.4 The Euler equations are

$$-\frac{dN}{dx} = f, \qquad -\frac{dQ}{dx} - q = 0, \qquad -\frac{dM}{dx} + Q = 0.$$

The boundary expressions indicate that u_0, w_0, and ϕ are the *primary variables* and N, M, and Q are the *secondary variables* of the problem. Thus the natural boundary conditions involve specifying N, M, and Q.

5.5 $v = (50P/57AE)^2$.

5.6 The (tensile) forces in the wires are

$$P_B = \frac{3M_0}{11L}, \qquad P_C = \frac{4M_0}{11L}.$$

5.7 The slope at $x = L$ and reactions are

$$u_4 = \frac{L}{4EI}\left(\frac{q_0L^2}{12} - M_0\right), \qquad F_1 = -\frac{3}{L}\left(\frac{q_0L^2}{12} - M_0\right) - \frac{q_0L}{2},$$

$$F_2 = \frac{1}{2}\left(\frac{q_0L^2}{12} - M_0\right) + \frac{q_0L^2}{12}, \qquad F_3 = \frac{3}{2L}\left(\frac{q_0L^2}{12} - M_0\right) - \frac{q_0L}{2}.$$

5.8 $v = (2595/32)(P^2/A^2E^2)$, where $L_1 = 60$ in., $L_3 = 48$ in., and $L_4 = 36$ in. (with the understanding that A and $K = E$ are in sq. in. and psi, respectively).

5.9 $w_c = (5q_0L^4/384EI)$.

5.10 $\theta_0 = -(M_0L/4EI) + (q_0L^3/48EI)$.

5.11

$$w_c - \frac{w_e}{2} = -\left(\frac{25F_0L^3 + q_0L^4}{768EI}\right).$$

Note that this is one of many relations between w_c and w_e that can be obtained depending on the choice of the virtual forces. It can be verified that the above relationship is exact.

5.12 $w_B = (5F_0a^3/6EI)$.

5.13 The force F_C in the cable is

$$F_C = \left(\frac{L_c}{E_cA_c} + \frac{L_b}{E_bA_b}\cos^2\alpha + \frac{L_b^3}{3E_bI_b}\sin^2\alpha\right)\left[\frac{5L_b^3}{48E_bI_b}\sin\alpha\right]^{-1} F_0.$$

5.14 The slope at B is $\theta_B = (170.67/EI)$ clockwise.

5.15 The structure has bending and torsional deformations (see Example 5.14):

$$w_A = \frac{1}{EI}\left[\frac{Qa^3}{3} + \frac{(P+Q)b^3}{3}\right] + \frac{1}{GJ}(Qa^2b).$$

5.16 The displacements are

$$v = \frac{1}{6EI}\left[P_v\left(2b^3 + 6ab^2\right) + 3P_ha^2b\right] + \frac{P_va}{EA},$$

$$u = \frac{1}{6EI}\left(3P_va^2b + 2P_ha^3\right) + \frac{P_hb}{EA}.$$

5.17 We have $u = 0$ and $v = (2595/32)(P^2/A^2E^2)$, where $L_1 = 60$ in., $L_3 = 48$ in., and $L_4 = 36$ in. (with the understanding that A and E are in sq. in. and psi, respectively).

5.18 $(P_C = 4M_0/11L)$.

5.19

$$w(L) = \frac{q_0L^4}{30EI}\left(\frac{1}{1 + (kL^3/3EI)}\right).$$

As a special case, the deflection at the free end of the beam when not supported by a spring is obtained by setting $k = 0$.

5.20 The displacements are

$$w = \frac{PR^3\pi}{4EI}, \qquad u = \frac{PR^3}{2EI}.$$

5.21 The displacements are

$$u_B = \frac{FR^3\pi}{2EI}, \qquad w_B = \frac{2FR^3}{EI}.$$

5.22 The displacements are

$$u = -\frac{2pR^4}{EI}, \qquad v = -\frac{3\pi pR^4}{2EI}.$$

5.23 The displacement is

$$u_B = \frac{FR\pi}{2EA} + \frac{FR^3\pi}{2EI} + \frac{f_s FR\pi}{2GA}.$$

5.24 The deflection (up) and the slope in clockwise direction at point A are

$$w_A = -\frac{q_0L^4}{8EI}\left(1 + \frac{kL^3}{3EI}\right)^{-1}, \qquad \theta_A = -\frac{q_0L^3}{6EI}\left[\frac{3 - (kL^3/8EI)}{3 + (kL^3/EI)}\right].$$

5.25 The deflection is

$$w_A = \frac{1}{EI}\left(\frac{F_0L^3}{3} + \frac{q_0L^4}{8}\right) \quad \text{(without shear)},$$

$$w_A = \frac{1}{EI}\left(\frac{F_0L^3}{3} + \frac{q_0L^4}{8}\right) + \frac{f_s}{GA}\left(F_0L + \frac{q_0L^2}{2}\right) \quad \text{(with shear)}.$$

5.26 $w_C = w_0(L) = (\partial U^*/\partial F_0) = (F_0 b^2 L/3EI)$.

5.27 $(\theta_A = F_0 ab/6EI)$.

5.28 The vertical and horizontal displacements are given by

$$v_C = 11.6\frac{PL}{EA}, \qquad u_C = 2\frac{PL}{EA}.$$

5.29 The reactions are $D_y = 0.44P$ and $C_y = 0.56P$.

5.31 The work done by F_0 in moving through the displacement w_{BA} is

$$W_{BA} = F_0 w_{BA} = \frac{F_0 M_0 L^2}{16EI}.$$

The work done by M_0 in moving through the rotation θ_{AB} is

$$W_{AB} = M_0\theta_{AB} = \frac{M_0 F_0 L^2}{16EI}.$$

Thus, $W_{BA} = W_{AB}$.

5.33 The deflection at the midspan is

$$w_0\left(\frac{L}{2}\right) = -\left(\frac{5F_0L^3}{48EI} + \frac{17q_0L^4}{384EI}\right).$$

5.34 From Maxwell's theorem, we have

$$F_B \cdot w_{BA} + F_C \cdot w_{CA} = F_A \cdot w_{AB} + F_A \cdot w_{AC} = F_A \cdot w_A,$$

$$w_A = w_{AB} + w_{AC}, \quad F_A = 4{,}000 \text{ lb}, \quad F_B = 4{,}500 \text{ lb}, \quad F_C = 2{,}000 \text{ lb}.$$

Chapter 6

6.1 The Lagrangian L of the system is

$$L = -U = -\frac{1}{2}\left[k_1x_1^2 + (k_2 + k_3)(x_2 - x_1)^2 + k_4(x_3 - x_2)^2\right].$$

6.3 The rate of the complementary Lagrangian function is

$$\delta \dot{L}^* = -\dot{x}_1\delta F_s - \dot{x}_2\delta F_d + \dot{x}\delta F = \left(-\frac{\dot{F}}{k_1} - \frac{F}{\eta} + x\right)\delta F.$$

6.4 The Euler–Lagrange equations are given by

$$\begin{aligned}
\delta\theta_1: \quad & -m_2L_1L_2\dot{\theta}_1\dot{\theta}_2\sin(\theta_1 - \theta_2) - (m_1 + m_2)gL_1\sin\theta_1 \\
& -\frac{d}{dt}\left[(m_1 + m_2)l_1^2\dot{\theta}_1 + m_2L_1L_2\cos(\theta_1 - \theta_2)\dot{\theta}_2\right] = 0, \\
\delta\theta_2: \quad & m_2L_1L_2\dot{\theta}_1\dot{\theta}_2\sin(\theta_1 - \theta_2) - m_2gL_2\sin\theta_2 \\
& -m_2\frac{d}{dt}\left[l_2^2\dot{\theta}_2 + L_1L_2\cos(\theta_1 - \theta_2)\dot{\theta}_1\right] = 0.
\end{aligned}$$

6.5 The total kinetic energy of the rigid bar assemblage is

$$K = \frac{WL^2}{6g}\left(2\dot{\theta}_1^2 + 4\dot{\theta}_2^2 + \dot{\theta}_3^2 + 9\dot{\theta}_1\dot{\theta}_2 + 3\dot{\theta}_2\dot{\theta}_3 + 3\dot{\theta}_1\dot{\theta}_3\right).$$

6.6 The Lagrangian function is

$$\begin{aligned}
L = & \tfrac{1}{2}m_1\left[l^2\dot{\theta}^2 + \dot{x}^2 - 2\dot{x}l\dot{\theta}\sin\theta\right] + \tfrac{1}{2}m_2\dot{x}^2 \\
& + m_1g(x - l\cos\theta) + m_2gx + \tfrac{1}{2}k(x + h)^2,
\end{aligned}$$

where h is the elongation in the spring due to the masses:

$$h = \frac{g}{k}(m_1 + m_2).$$

6.7 The Lagrangian is given by

$$L = \tfrac{1}{2}m_1\dot{u}_1^2 + \tfrac{1}{2}m_2\left(\dot{u}_1^2 + \dot{u}_2^2 - \sqrt{2}\dot{u}_1\dot{u}_2\right) + \tfrac{1}{\sqrt{2}}m_2 g u_2.$$

6.9 The equation of motion is

$$\ddot{x}_1 = \left(\frac{m_1 - m_2}{m_1 + m_2}\right) g.$$

6.10 The equation of motion is

$$\ddot{x}_1 = \left(\frac{m_1 - m_2 - m_3}{m_1 + m_2 + m_3}\right) g.$$

6.11 $\ddot{x}_1 = g/23.$

6.12 The equations of motion are

$$m(\ddot{x} + l\ddot{\theta}\cos\theta - l\dot{\theta}^2\sin\theta) + kx = F,$$

$$m\left[l\ddot{x}\cos\theta + (l^2 + \Omega^2)\ddot{\theta}\right] + mgl\sin\theta = 2aF\cos\theta.$$

6.13 The natural boundary conditions become

$$\rho I\frac{\partial^3 w_0}{\partial x\partial t^2} + m\frac{\partial^2 w_0}{\partial t^2} - \frac{\partial}{\partial x}\left(EI\frac{\partial^2 w_0}{\partial x^2}\right) = 0, \qquad \text{at } x = L,$$

$$EI\frac{\partial^2 w_0}{\partial x^2} + J\frac{\partial^3 w_0}{\partial x\partial t^2} = 0, \qquad \text{at } x = L.$$

6.14 The Euler–Lagrange equations are

$$\delta u_0: \quad -\frac{\partial N_{xx}}{\partial x} - f + \frac{\partial}{\partial t}\left(m_0\frac{\partial u_0}{\partial t}\right) = 0,$$

$$\delta w_0: \quad -\frac{\partial Q_x}{\partial x} - q + \frac{\partial}{\partial t}\left(m_0\frac{\partial w_0}{\partial t}\right) = 0,$$

$$\delta\phi: \quad -\frac{\partial M_{xx}}{\partial x} + Q_x + \frac{\partial}{\partial t}\left(m_2\frac{\partial\phi}{\partial t}\right) = 0.$$

The boundary expressions indicate that u_0, w_0, and ϕ are the *primary variables* and N_{xx}, M_{xx}, and Q_x are the *secondary variables* of the problem. Thus the natural boundary conditions involve specifying M_{xx} and Q_x.

6.15 The stress resultants are

$$M_{xx} = EI\frac{\partial\phi}{\partial x}, \qquad Q_x = K_s GA\left(\frac{\partial w_0}{\partial x} + \phi\right).$$

6.17 The equation of motion is

$$\frac{\partial^2}{\partial x \partial t}\left(\rho I \frac{\partial^2 w}{\partial x \partial t}\right) - \frac{\partial}{\partial t}\left(\rho A \frac{\partial w}{\partial t}\right) + \frac{\partial^2 M_{xx}}{\partial x^2}$$
$$- \frac{\partial^2}{\partial x^2}\left(I c_s \frac{\partial^3 w}{\partial x^2 \partial t}\right) - c\frac{\partial w}{\partial t} + q = 0.$$

6.18 The Euler–Lagrange equations are given by

$$\delta u_0: \quad -\rho A \ddot{u}_0 + \frac{\partial N_{xx}}{\partial x} = 0, \tag{a}$$

$$\delta w_0: \quad -\rho A \ddot{w}_0 + \rho I \frac{\partial^2 \ddot{w}_0}{\partial x^2} + \frac{\partial}{\partial x}\left(N_{xx}\frac{\partial w_0}{\partial x}\right) + \frac{\partial^2 M_{xx}}{\partial x^2} + q = 0. \tag{b}$$

6.19 The Euler–Lagrange equations are

$$\delta u: \quad -\rho A \frac{\partial^2 u}{\partial t^2} + \frac{\partial N_x}{\partial x} + f = 0,$$

$$\delta w: \quad c_1 \rho I \left(c_1 \frac{\partial^4 w}{\partial x^2 \partial t^2} + c_2 \frac{\partial^3 \phi}{\partial x \partial t^2}\right) - \rho A \frac{\partial^2 w}{\partial t^2} - c_1 \frac{\partial^2 M_x}{\partial x^2}$$
$$+ (1 + c_1)\frac{\partial Q_x}{\partial x} + q = 0,$$

$$\delta \phi: \quad -c_2 \rho I \left(c_1 \frac{\partial^3 w}{\partial x \partial t^2} + c_2 \frac{\partial^2 \phi}{\partial t^2}\right) + c_2 \frac{\partial M_x}{\partial x} - c_2 Q_x = 0.$$

6.20 The Euler–Lagrange equations are

$$\delta u_0: \quad -\frac{\partial N_{xx}}{\partial x} - f + \frac{\partial}{\partial t}\left(m_0 \frac{\partial u_0}{\partial t}\right) = 0,$$

$$\delta w_0: \quad -\frac{\partial Q_x}{\partial x} - \frac{\partial}{\partial x}\left(\frac{\partial w_0}{\partial x} N_{xx}\right) - q + \frac{\partial}{\partial t}\left(m_0 \frac{\partial w_0}{\partial t}\right) = 0,$$

$$\delta \phi: \quad -\frac{\partial M_{xx}}{\partial x} + Q_x + \frac{\partial}{\partial t}\left(m_2 \frac{\partial \phi}{\partial t}\right) = 0.$$

6.21 The equation of motion is

$$-\frac{\partial}{\partial x}\left(T_0 \frac{\partial u}{\partial x}\right) - \frac{\partial}{\partial y}\left(T_0 \frac{\partial u}{\partial y}\right) + \frac{\partial}{\partial t}\left(\rho \frac{\partial u}{\partial t}\right) = f(x, y, t).$$

6.22 $\omega_1 = (9.8766/L^2)\sqrt{EI/\rho A}$.

6.23 $\omega_1 = (15.45/L^2)\sqrt{EI/\rho A}$.

6.24 $\omega_1 = (22.45/L^2)\sqrt{(EI/\rho A)}$.

Chapter 7

7.1 (a) A subspace. (b) Not a subspace. (c) A subspace. (d) A subspace.

7.2 (a) Linearly dependent. (b) Linearly independent. (c) Linearly independent.

7.3 (a) Linearly independent. (b) Linearly independent. (c) Linearly dependent; same as Exercise 7.2(c).

7.4 (a) $\|u\|_0 = \sqrt{\frac{5}{6} - \frac{2}{\pi}}$, sup-norm $= 1$.

(c) $\|u\|_0 = \sqrt{\frac{5}{6} - \frac{8}{\pi^2}}$, sup-norm $= 0.207$.

(e) $\|u\|_0 = \sqrt{50}$, sup-norm $= 100$.

7.6 (a) Qualifies as an inner product. (b) Qualifies as an inner product (note that it is symmetric and positive-definite).

7.7 (a) $(u, v)_0 = 4\pi^3$, $(u, v)_1 = \frac{4}{\pi}\left(1 + \frac{1}{\pi^2}\right)$.
(c) $(u, v)_0 = 0$, $(u, v)_1 = 0$.
(e) $(u, v)_0 = \frac{1}{\pi}\left(-1 + \frac{2}{\pi} + \frac{4}{\pi^2} + \frac{8}{\pi^3}\right)$, $(u, v)_1 = (u, v)_0 + \frac{16}{\pi^2}$.

7.8 $\|u - v\|_0^2 = (33/35)$.

7.9 The functions are *not* orthogonal.

7.10 The functions are *not* orthogonal.

7.11 $a = 1037/270$ and $b = 362/135$.

7.12 (a) $C = 1/3$. (b) $C = (-7 \pm \sqrt{50})/2$.

7.14 (a) Linear. (b) Nonlinear. (c) Linear.

7.15 (a) $(T_1 + T_2)(\mathbf{x}) = (x_2 - x_3, 3x_1 - x_2 - x_3, x_1 + x_2 + x_3)$.
(c) $(T_2 T_1)(\mathbf{x}) = (x_1 - x_2 - 2x_3, x_3 - x_1, 0)$.

7.16 The transformation matrix is

$$\begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 1 & 0 & 1 & 0 \end{bmatrix}.$$

7.17 (a) Bilinear; it is a linear functional in v for fixed u. (b) Linear functional in v for fixed u.

7.18 (a) A bilinear form. (b) A bilinear form.

7.20 (a) δI is linear in u because $I(u)$ is quadratic in u. The first variation of any functional is always linear in δu; thus, δI is linear in both u and δu.

(b) δI is always linear in δu. Since $I(u)$ is a functional, its first variation is also functional, linear in δu.

7.21 $\phi_0 = (h/L)x$, $\phi_1 = x(L-x)$, and $\phi_2 = x^2(L-x)$. The trigonometric functions are

$$\phi_0 = h\sin\frac{\pi x}{L}, \qquad \phi_n = \sin\frac{n\pi x}{L}.$$

7.23 $\phi_1 = x(L-x)$ and $\phi_2(x) = x^2(\frac{3}{4}L - x)$ or $x(\frac{3}{4}L^2 - x^2)$.

7.25 See Exercise 7.23.

7.27 $\phi_1 = (x-a)(y-b)$, which violates the biaxial symmetry, $\phi_2 = (x^2-a^2)\times(y^2-b^2)$.

7.28 The solution becomes

$$U_2(x) = -\frac{f_0}{2T}x(L-x) + \frac{hx}{L}.$$

7.29 The solution is $W_2(x) = (q_0L^4/24EI)(5(x^2/L^2) - 2(x^3/L^3))$.

7.30 The Ritz solution is

$$W_2 = \frac{q_0L^4}{96EI}\left(4\frac{x}{L} - \frac{x^2}{L^2} - 4\frac{x^3}{L^4}\right).$$

7.31 The Ritz solution is

$$W_2 = \frac{F_0L^3}{64EI}\left(7\frac{x^2}{L^2} - 12\frac{x^3}{L^3} + 5\frac{x^4}{L^4}\right).$$

7.32 The Ritz solution is

$$W_2 = \frac{q_0L^4}{96EI}\left(1+\frac{128EI}{3kL^3}\right)\left(\frac{x}{L}-\frac{x^2}{L^2}\right)+\frac{q_0L^4}{24EI}\left(1+\frac{30EI}{kL^3}\right)\left(\frac{3x}{4L}-\frac{x^3}{L^3}\right).$$

7.33 See Example 7.16.

7.34 The one-parameter solution is given by

$$U_1 = \tfrac{3}{8}f_0(x-a)(y-a).$$

The two-parameter solution is given by

$$U_2 = -\frac{16f_0}{274}\phi_1 + \frac{95f_0}{274a^2}\phi_2, \quad \phi_1 = (x-a)(y-a), \quad \phi_2 = (x^2-a^2)(y^2-a^2).$$

The one-parameter solution with a trigonometric function is

$$U_1 = \frac{32f_0}{\pi^4}\cos\frac{\pi x}{2}\cos\frac{\pi y}{2}.$$

7.36 (a) Let $\phi_1 = x(L-x),\ \phi_2 = x^2(L-x)$. The solution becomes

$$c_1 = \frac{q_0 L^3}{12\Delta}\left(6EIL^3 + \frac{kL^7}{420}\right), \qquad c_2 = 0,$$

$$\Delta = 12(EI)^2L^4 + \frac{k^2L^{12}}{25200} + \frac{11}{105}EIkL^8.$$

7.38 The critical buckling load is $\hat{N}_{cr} = 9.87(EI/L^2)$.

7.39 The critical buckling load is

$$N_{cr} = \frac{\pi^2 D}{L^2}\left(1 - \frac{\pi^2 D/L^2}{(\pi^2 D/L^2) + S}\right).$$

7.40 For a beam simply supported at both ends and subjected to a suddenly applied uniformly distributed load $q = q_0 H(t)$, the solution is

$$w_0(x,t) = \frac{4q_0}{\pi}\sum_{n=1}^{\infty}\frac{1}{nK_n}(1-\cos\lambda_n t)\sin\frac{n\pi x}{L} \tag{7}$$

for n odd.

7.41 The two-parameter Ritz solution is

$$u_2(x,t) = c_1(t)x + c_2(t)x^2,$$

where

$$c_1(t) = 4\left[\frac{A_1}{\beta^2}(1-\cos\beta t) + \frac{B_1}{\gamma^2}(1-\cos\gamma t)\right],$$

$$c_2(t) = -\frac{10}{3}\left[\frac{A_2}{\beta^2}(1-\cos\beta t) + \frac{B_2}{\gamma^2}(1-\cos\gamma t)\right],$$

where

$$\alpha_1^2 = 30, \qquad \alpha_2^2 = 24, \qquad \beta^2 = 4.155, \qquad \gamma^2 = 38.512,$$

$$A_1 = \frac{\alpha_1^2 - \beta^2}{\gamma^2 - \beta^2}, \qquad B_1 = \frac{\gamma^2 - \alpha_1^2}{\gamma^2 - \beta^2}, \qquad A_2 = \frac{\alpha_2^2 - \beta^2}{\gamma^2 - \beta^2}, \qquad B_2 = \frac{\gamma^2 - \alpha_2^2}{\gamma^2 - \beta^2}.$$

7.43 The one-parameter solution is

$$W_1(x) = \frac{q_0 L^4}{3(4EI + kL^3)}\left(\frac{x}{L}\right)^2.$$

7.44 (a) For one-parameter approximation $u(r) \approx U_1(r) = c_1 \cos(\pi r/2a)$, one has $\lambda = 5.832/a^2$.

(b) For two-parameter approximation $U_2(r) = c_1 \cos(\pi r/2a) + c_2 \cos(3\pi r/2a)$, the smaller root is $\lambda = 5.792/a^2$.

7.45 The solution is

$$U_2(x) = x(1 - x)\left(\frac{71}{369} + \frac{7}{41}x\right).$$

7.46 The solution is

$$W_1(x) = 0.8111(1 - x)(2 - x) \rightarrow U_1(x) = 0.8111(1 - x)(2 - x) + x.$$

7.47 This is a special case of Exercise 7.34.

7.49 For the choice $U_2(x) = c_1(2x - x^2) + c_2(3x - x^3)$, one obtains

$$c_1 = \frac{516 - 126\pi}{433\pi}, \qquad c_2 = -\frac{120 - 130\pi}{433\pi}.$$

7.50 (b) For two-parameter Galerkin approximation with

$$\phi_1 = (x - L)^2(x^2 + 2Lx + 3L^2), \quad \phi_2(x) = (x - L)^3(3x^2 + 4Lx + 3L^2),$$

one obtains $c_1 = 0.01374$ and $c_2 = 0.00228$.

7.51 Approximation of the type $U_2 = c_1\phi_1 + c_2\phi_2 + \phi_0$ with

$$\phi_0 = (1 - y)\sin \pi x, \qquad \phi_1 = \sin \pi x \sin \pi y, \qquad \phi_2 = \sin 2\pi x \sin 2\pi y.$$

yields $c_1 = -\pi/8$ and $c_2 = \pi/24$.

7.52 (a) The solution is

$$U_1 = \frac{\sqrt{2}}{4}(3 + x^2).$$

(b) For $\psi_1 = x$, we obtain two values of c_1. The choice $c_1 = \sqrt{2} - \sqrt{3}$ yields the smaller value of $\mathcal{E} = \int_0^1 \mathcal{R}\, dx$.

7.53 The eigenvalues are $\lambda_1 = 4.212$, $\lambda_2 = 34.188$.

7.57 The solution is given by

$$U_2(x) = 0.1763(2x - x^2) - 0.15(3x - x^3).$$

7.58 Let $u = \sqrt{2} + (1 - x^2)c_1$, and obtain

$$U_1 = \sqrt{2} - 0.326 + 0.326x^2.$$

7.59 The eigenvalues are $\lambda_1 = 4.212$ and $\lambda_2 = 34.188$.

7.60 See Exercise 7.54.

7.61 Let $W_3(x) = c_1x^2 + c_2x^3 + c_3x^4$, and obtain $c_3 = 0$ and

$$c_1 = \frac{f_0L^2}{4a_0} \quad \text{and} \quad c_2 = -\frac{f_0L}{36a_0}.$$

7.62 The solution is

$$U_1 = (-2\sqrt{2} + \sqrt{6})(1 - x^2) + \sqrt{2} = 1.0355 + 0.3789x^2.$$

7.63 See Exercises 7.53 and 7.59. The eigenvalues are $\lambda_1 = 3.785$ and $\lambda_2 = 19.753$.

7.64 The one-parameter approximation $U_1 = c_{11} \sin \pi x \sin \pi y$ yields $c_{11} = (f_0/2\pi^2)$, which is about half the value of the first term in Exercise 7.54 $(c_{11} = 8f_0/\pi^4)$.

7.65 The solution is $U_1(x, y) = e^{-\sqrt{10}y}(x - x^2)$.

7.67 The solution is given by

$$W_1(x, t) = c_1(0) \cos at \,(1 - \cos 2\pi x)\,.$$

7.68 For solution of the form $U_1 = c_1(x)(y^2 - b^2)$, one obtains

$$\lambda_1 = \frac{\pi^2}{(2a)^2} + \frac{10}{(2b)^2},$$

whereas the exact value is

$$\lambda = \frac{\pi^2}{(2a)^2} + \frac{\pi^2}{(2b)^2}.$$

7.69 For the two-parameter approximation of the form

$$U_2 = \left[c_1(x) + c_2(x)y^2\right](y^2 - b^2),$$

one obtains

$$\lambda_1 = \frac{\pi^2}{(2a)^2} + \frac{9.871}{(2b)^2}.$$

7.70 A two-parameter approximation of the form (which is more complete than the one-parameter approximation, although it has no effect on the derivative of the solution)

$$\Psi_2 = c_1 + c_2(x^4 - 6x^2y^2 + y^4)$$

yields

$$c_1 = \frac{53}{180} f_0 a^2, \qquad c_2 = -\frac{7}{144} \frac{f_0}{a^2}.$$

Chapter 8

8.1 The Euler equation and the natural boundary conditions at the inner edge are given by

$$\frac{1}{r}\frac{d^2}{dr^2}\left(D_{11} r \frac{d^2 w_0}{dr^2}\right) - \frac{1}{r}\frac{d}{dr}\left(\frac{D_{22}}{r}\frac{dw_0}{dr}\right) - q = 0, \qquad b < r < a,$$

$$-\frac{d}{dr}\left(r D_{11}\frac{d^2 w_0}{dr^2}\right) + \frac{D_{22}}{r}\frac{dw_0}{dr} = 0, \qquad \text{at } r = b,$$

$$r D_{11}\frac{d^2 w_0}{dr^2} + D_{22}\frac{dw_0}{dr} = 0, \qquad \text{at } r = b.$$

Note that the terms involving D_{12} cancel out.

8.3 The Euler equation is

$$D\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right]\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial w_0}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 w_0}{\partial \theta^2}\right] + k w_0 = q.$$

8.4 The deflection due to the applied edge moment M_a at $r = a$ is

$$w_0(r) = \frac{M_a a^2}{2(1+\nu)D}\left(1 - \frac{r^2}{a^2}\right).$$

8.5 The solution is given by

$$w_0 = \frac{q_0 a^4}{64D}\left\{\frac{(5+\nu)D + \beta a}{(1+\nu)D + \beta a} - 2\left[\frac{(3+\nu)D + \beta a}{(1+\nu)D + \beta a}\right]\left(\frac{r}{a}\right)^2 + \left(\frac{r}{a}\right)^4\right\},$$

$$M_{rr} = \frac{q_0 a^2}{16}\left[(1+\nu)\frac{(3+\nu)D + \beta a}{(1+\nu)D + \beta a} - (3+\nu)\left(\frac{r}{a}\right)^2\right],$$

$$M_{\theta\theta} = \frac{q_0 a^2}{16}\left[(1+\nu)\frac{(3+\nu)D + \beta a}{(1+\nu)D + \beta a} - (1+3\nu)\left(\frac{r}{a}\right)^2\right].$$

8.6 This is a special case of the loading shown in Fig. 8.30 (set $q_1 = 0$). For a linearly distributed load of the type

$$q(r) = q_0\left(1 - \frac{r}{a}\right)$$

one obtains

$$c_2 = -\frac{2}{(1+\nu)}\left(F''(a) + \frac{\nu}{a}F'(a)\right) = -\frac{q_0 a^2}{(1+\nu)}\left(\frac{71+29\nu}{360}\right),$$

$$c_4 = -\left(F(a) + \frac{a^2}{4}c_2\right) = \frac{q_0 a^4}{(1+\nu)}\left(\frac{183+43\nu}{4800}\right).$$

8.7 This is a special case of the loading shown in Fig. 8.30 (set $q_0 = 0$). One obtains

$$c_2 = -\frac{2q_1 a^2}{(1+\nu)}\left(\frac{4+\nu}{45}\right), \qquad c_4 = \frac{q_1 a^4}{(1+\nu)}\left(\frac{6+\nu}{150}\right).$$

8.8 $c_2 = -(12q_0a^2/576)$ and $c_4 = (2q_0a^4/576)$.

8.9 $c_2 = -(29q_0a^2/360)$ and $c_4 = (129q_0a^4/14400)$.

8.10 $c_2 = -(2q_1a^2/45)$ and $c_4 = (3q_1a^4/450)$.

8.11 The solution is

$$W_1(r) = \frac{q_1 a^4}{30(1+\nu)D}\left(1 - \frac{r^2}{a^2}\right).$$

8.12 The solution is

$$W_1(r) = -\frac{9q_1 a^4}{160D}\left(1 - \frac{r^2}{a^2}\right).$$

Clearly, the solution is *not* acceptable.

8.13 Assume solution of the form (for $n = 0$)

$$W_1(r) = c_1\phi_1(r,\theta) = c_1 f_1(r), \qquad f_1(r) = 1 - \frac{r^2}{a^2},$$

and obtain

$$\omega = \frac{4.899}{a^2}\sqrt{\frac{(1+\nu)D}{\rho h}}.$$

8.14 See Example 8.7. The fundamental frequency is

$$\lambda_1 = \frac{\rho h \omega^2}{D} = \frac{104.39}{a^4}.$$

The eigenvector is

$$U_2(r) = \left(1 - \frac{r^2}{a^2}\right)^2 + 0.325\left(1 - \frac{r^2}{a^2}\right)^3.$$

8.15 The center deflection of a simply supported plate under asymmetric load is

$$w_c = \frac{q_0 a^2}{8D}\left[\left(\frac{3+\nu}{1+\nu}\right)\frac{a^2}{4} - \frac{a^2}{8}\right] = \frac{q_0 a^4}{64D}\left(\frac{5+\nu}{1+\nu}\right).$$

8.16 The deflection is

$$w_c = \frac{q_0 a^4}{(1+\nu)D}\left(\frac{5+\nu}{64} - \frac{6+\nu}{150}\right).$$

8.17 The deflection is

$$w_c = \frac{43}{4800}\frac{q_0 a^4}{D}.$$

The solution coincides with the solution at $r = 0$ given in Exercise 8.9.

8.20 The solution is given by

$$c_1 = \frac{F_0}{\pi a^2 k}\left[2 + \frac{1}{8}\left(\frac{32D(1+\nu)}{ka^4} + \frac{1}{3}\right)^{-1}\right],$$

$$c_2 = -\frac{Q_0}{\pi a^4 k}\left(\frac{1}{12} + \frac{8D(1+\nu)}{ka^4}\right)^{-1}.$$

8.22 The boundary conditions are

$$\text{at } y = 0:\quad w_0 = 0,\quad \frac{\partial w_0}{\partial y} = 0; \qquad \text{at } y = b:\quad w_0 = 0,\quad \frac{\partial^2 w_0}{\partial y^2} = 0.$$

In view of the boundary conditions, the constants A_m, B_m, C_m, and D_m are found to be

$$A_m = -\frac{4q_0 a^4}{m^5\pi^5 D}, \qquad B_m = -\frac{m\pi}{a}C_m,$$

$$C_m = \frac{2q_0 a^4}{m^5\pi^5 D}\,\frac{2\cosh^2\beta_m - 2\cosh\beta_m - \beta_m\sinh\beta_m}{\cosh\beta_m\sinh\beta_m - \beta_m},$$

$$D_m = -\frac{A_m}{2}\,\frac{2(\frac{m\pi}{a})\sinh\beta_m\cosh\beta_m - (\frac{m\pi}{a})\sinh\beta_m - (\frac{m\pi}{a})^2 b\cosh\beta_m}{\cosh\beta_m\sinh\beta_m - \beta_m},$$

where $\beta_m = (m\pi b/a)$. In the case of a square plate the maximum deflection and maximum bending moment are found to be

$$w_0(a/2, b/2) \approx 0.0028\frac{q_0 a^4}{D}; \qquad M_{xx}(a/2, 0) \approx 0.08 q_0 a^2.$$

8.24 This is a CCCS plate. Hence we choose

$$X_i(x) = \left(\frac{x}{a}\right)^{i+1} - 2\left(\frac{x}{a}\right)^{i+2} + \left(\frac{x}{a}\right)^{i+3}, \qquad Y_i(y) = \left(\frac{y}{b}\right)^{i+1}\left[1 - \left(\frac{y}{b}\right)\right].$$

8.26 This is a CCCF plate. Hence, we choose

$$X_i(x) = \left(\frac{x}{a}\right)^{i+1} - 2\left(\frac{x}{a}\right)^{i+2} + \left(\frac{x}{a}\right)^{i+3}, \qquad Y_i(y) = \left(\frac{y}{b}\right)^{i+1}.$$

8.28 The choice

$$W_1(x, y) = c_{11}\left(1 - \cos\frac{2\pi x}{a}\right)\sin\frac{n\pi y}{b}$$

gives

$$c_{11} = \frac{8q_0 a^4}{\left[16D_{11} + 8\left(D_{12} + 2D_{66}\right)s^2 n^2 + 3D_{22}s^4 n^4\right] n\pi^5},$$

where $s = a/b$.

8.30 We obtain

$$c_{11} = \frac{q_0 a^4}{4\pi^4\left[3D_{11} + 2\left(D_{12} + 2D_{66}\right)s^2 + 3D_{22}s^4\right]}, \qquad s = \frac{a}{b}.$$

For isotropic case, it reduces to

$$c_{11} = \frac{q_0 a^4}{4\pi^4 D\left(3 + 2s^2 + 3s^4\right)}, \qquad s = \frac{a}{b}.$$

8.31 The only difference between this exercise and Exercise 8.30 is in the load. One obtains

$$c_{11} = \frac{Q_0 a^4}{ab\pi^4\left[3D_{11} + 2\left(D_{12} + 2D_{66}\right)s^2 + 3D_{22}s^4\right]}, \qquad s = \frac{a}{b}.$$

8.32 One obtains $c_{11} = (q_0 a^4)/(27.984\pi^4 D_0)$ and $w_{max} = (q_0 a^4)/(6.996\pi^4 D_0) = 0.00147 q_0 a^4/D_0$.

8.33 One has $(A_0 = a^2/\sqrt{3})$ $c_1 = Q_0 a^2/(36\sqrt{3}D)$.

8.34 Evaluate

$$F_1 = \oint_\Gamma M_0 \frac{\partial\varphi_1}{\partial n}\, ds, \qquad \frac{\partial\varphi_1}{\partial n} = n_x\frac{\partial\varphi_1}{\partial x} + n_y\frac{\partial\varphi_1}{\partial y},$$

and the direction cosines n_x and n_y on BC, CA, and AB are known from

$$\hat{\mathbf{n}} = \begin{cases} -\hat{\mathbf{e}}_x & \text{on BC} \\ \frac{1}{2}\hat{\mathbf{e}}_x + \frac{\sqrt{3}}{2}\hat{\mathbf{e}}_y & \text{on CA} \\ \frac{1}{2}\hat{\mathbf{e}}_x - \frac{\sqrt{3}}{2}\hat{\mathbf{e}}_y & \text{on AB.} \end{cases}$$

Thus

$$F_1 = \frac{4M_0}{3\sqrt{3}}$$

on AB, BC, and CA. Hence, $d_1 = M_0 a^2/4D$ ($A_0 = a^2/\sqrt{3}D$).

8.35 Then Galerkin's method yields

$$c_1 = \frac{q_0}{24\left[5(D_{11}/a^4) + 2(\hat{D}_{12}/a^2b^2) + (D_{22}/b^4)\right]}.$$

For the isotropic case, the solution is ($s = a/b$)

$$w_0(x, y) = \frac{q_0 a^4 (x/a)}{24D(s^4 + 2s^2 + 5)}\left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}\right)^2.$$

8.36 For the isotropic case, the one-parameter Ritz solution is given by ($s = a/b$)

$$w_0(x, y) = \frac{q_0 a^4}{8D(1 + 2\nu s^2 + s^4)}\left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}\right).$$

8.40 The resulting equation is

$$\left(\nabla^2 - \frac{I_0}{K_s G h}\frac{\partial^2}{\partial t^2}\right)\left(D\nabla^2 - I_2\frac{\partial^2}{\partial t^2}\right) w_0 + I_0\frac{\partial^2 w_0}{\partial t^2}$$
$$= \left(1 - \frac{D}{K_s G h}\nabla^2 + \frac{I_2}{K_s G h}\frac{\partial^2}{\partial t^2}\right) q.$$

Chapter 9

9.1 The FE solution is

$$U_2 = \tfrac{45}{59}, \qquad U_3 = \tfrac{35}{59}, \qquad P_1^{(1)} = 2.5U_1 - 2.5U_2 = \tfrac{35}{59}.$$

The exact solution is

$$u_0(x) = 1 - \frac{\log(1 + x)}{1 + \log 2}, \qquad u_0(0.5) = 0.7605, \qquad u_0(1.0) = 0.5906.$$

9.3 The solution is

$$U_2 = 0.2 \times 10^{-3}\text{ in.},\quad U_3 = -0.3333 \times 10^{-3}\text{ in.},\quad U_4 = -0.8667 \times 10^{-3}\text{ in.}$$

The forces in each member can be computed from the element equations:

$$P_2^{(1)} = 3{,}000\text{ lb},\qquad P_2^{(2)} = -2{,}000\text{ lb},\qquad P_2^{(2)} = -2{,}000\text{ lb}.$$

9.4 The compression is $U_2 = -0.002996 \approx 0.003$ in. The element forces $P_i^{(b)}$ and $P_i^{(s)}$ are $P_2^{(b)} = -551.59$ lb and $P_2^{(s)} = -778.41$ lb. The stresses in steel and brass are $\sigma_s = 22.47$ ksi (compressive), $\sigma_b = 11.24$ ksi (compressive).

9.5 The displacements are $U_2 = 0.4444$ in. and $U_3 = 1.9444$ in.

9.6 The aluminum member is elongated and the steel member is compressed by the amount $U_2 = 0.0134$ in. The reaction forces and stresses are

$$P_2^{(1)} = -21{,}052.6\text{ lb},\qquad \sigma^{(1)} = 6701.25\text{ psi},$$
$$P_2^{(2)} = -78{,}947\text{ lb},\qquad \sigma^{(2)} = -44{,}675.1\text{ psi}.$$

9.7 The displacement is $U_2 = 0.3$ mm. The forces and stresses in steel and aluminum pipes are

$$P_2^{(1)} = 36.364\text{ kN},\qquad \sigma_s = 606.06\text{ MPa};$$
$$P_2^{(2)} = -63.636\text{ kN},\qquad \sigma_a = -106.06\text{ MPa}.$$

9.8 First we must find the force acting at point B to be $P = 5000$ lb downward. The solution is

$$U_2 = u_B = 0.01167\text{ in.},\qquad U_3 = u_C = 0.01967\text{ in.}$$

9.9 The solution is

$$U_2 = u_B = 0.01163\text{ in.},\qquad U_3 = u_C = 0.01956\text{ in.}$$

Note that if we had taken $k = 10^5$ lb/in., we would have obtained

$$U_2 = u_B = 0.00957\text{ in.},\qquad U_3 = u_C = 0.01255\text{ in.}$$

Thus one may use the value of k to simulate elastic restraint at point C.

9.10 Exploit the symmetry about the middle of the beam and use two beam elements to analyze the problem. The generalized displacements are

$$U_2 = -0.00312,\quad U_3 = 0.01077\text{ in.},\quad U_4 = -0.001833,\quad U_5 = 0.01673\text{ in.}$$

The reaction force at the left support and the internal bending moment at the center are

$$Q_1^{(1)} = -1200 \text{ lb (up)}, \qquad Q_4^{(2)} = 8400 \text{ lb-in. (counterclockwise)}.$$

9.11 The solution is

$$U_3 = -0.03529 \text{ cm}, \qquad U_4 = 0.01438, \qquad U_6 = 0.06376.$$

9.12 The solution is

$$U_2 = -0.001792, \quad U_4 = 0.000512, \quad U_5 = 0.03686 \text{in.}, \quad U_6 = -0.001408.$$

9.13 One-element mesh can be used. The solution is given by

$$U_3 = \frac{q_0 L^4}{8EI\left(1 + \dfrac{kL^3}{3EI}\right)}, \qquad U_4 = -\frac{q_0 L^3}{6EI\left(1 + \dfrac{kL^3}{3EI}\right)}.$$

9.14 This problem can be modeled with four elements with $h_1 = h_2 = h_3 = h_4 = 5'$. The main objective here is to represent the applied loads appropriately. The global node 2 will have a downward load of 1,000 lb and bending moment of 1,000 ft-lb (CCW). The total size of the assembled global stiffness matrix is 10×10. The solution is given by

$$\bar{U}_2 = 0.12187, \quad \bar{U}_3 = -0.45660 \text{ft}, \quad \bar{U}_4 = 0.03021, \quad \bar{U}_5 = -0.32986 \text{ft},$$
$$\bar{U}_6 = -0.06979, \quad \bar{U}_8 = 0.001042, \quad \bar{U}_9 = -0.39583 \text{ ft}, \quad \bar{U}_{10} = 0.10521,$$

where $\bar{U}_i = U_i(EI \times 10^{-5})$. The bending moment at $x = 2.5$ ft, for example, is given by

$$M^c = EI \left.\frac{d^2 w}{dx^2}\right|_{x=2.5} = EI \sum_{i=1}^{4} u_i^1 \left.\frac{d^2\phi_i^1}{dx^2}\right|_{x=2.5h},$$
$$= EI\left.\left(U_2\frac{d^2\phi_2^1}{dx^2} + U_3\frac{d^2\phi_3^1}{dx^2} + U_4\frac{d^2\phi_4^1}{dx^2}\right)\right|_{x=2.5} = 1833.33 \text{ lb-ft}.$$

The shear force at $x = 2.5$ ft is given by $V^c = 733.33$ lb.

9.15 The primary objective of this problem is to compute the force vector for element 1. One obtains ($q_0 = 500$ and $h = 5$)

$$\{q^{(1)}\} = \frac{q_0 h}{60}\begin{Bmatrix} 9 \\ -2h \\ 21 \\ 3h \end{Bmatrix} = \begin{Bmatrix} 375.00 \\ -416.67 \\ 875.00 \\ 625.00 \end{Bmatrix}.$$

The solution is given by

$$\bar{U}_2 = -0.24826, \quad \bar{U}_3 = 0.99537, \quad \bar{U}_4 = -0.11111, \quad \bar{U}_5 = 0.98380,$$
$$\bar{U}_6 = 0.11806, \quad \bar{U}_8 = 0.23611,$$

where $\bar{U}_i = U_i(EI \times 10^{-5})$.

9.16 The components of the force vector due to the distributed load are

$$\{q^{(1)}\} = \frac{q_0 h_1}{60}\begin{Bmatrix} 26 \\ -4h_1 \\ 14 \\ 3h_1 \end{Bmatrix} = \begin{Bmatrix} 3,466.7 \\ -4,266.7 \\ 1,866.7 \\ 3,200.0 \end{Bmatrix}.$$

The solution is

$$\bar{U}_3 = -0.94151, \quad \bar{U}_4 = 0.19413, \quad \bar{U}_5 = -2.2503, \quad \bar{U}_6 = 0.23013,$$

where $\bar{U}_i = U_i(EI \times 10^{-6})$. The bending moment and shear force at $x = 3$ ft, for example, are $M^c = -28,133$ ft-lb and $V^c = 3,866.7$ lb. The values at $x = 0$ are $M(0) = -39,733$ ft-lb and $V(0) = 3,866.7$ lb.

9.17 The global stiffness matrix is 6×6, which can be expressed as

$$\frac{EI}{L^3}\begin{bmatrix} 24 + \frac{k_1 L^3}{EI} & 0 & -12 & -6L & -\frac{k_1 L^3}{EI} & 0 \\ 0 & 8L^2 & 6L & 2L^2 & 0 & 0 \\ -12 & 6L & 12 + \frac{k_2 L^3}{EI} & 6L & 0 & 0 \\ -6L & 2L^2 & 6L & 4L^2 & 0 & 0 \\ -\frac{k_1 L^3}{EI} & 0 & 0 & 0 & 12 + \frac{k_1 L^3}{EI} & 6L \\ 0 & 0 & 0 & 0 & 6L & 4^2 \end{bmatrix}.$$

9.18 The finite element model is

$$[K^e]\{\Delta^e\} - \lambda[G^e]\{\Delta^e\} = \{Q^e\},$$

where $\{\Delta^e\}$ and $\{Q^e\}$ are the columns of nodal generalized displacement and force degrees of freedom of the Euler–Bernoulli beam element. The matrices $[K^e]$ and $[G^e]$, known as the stiffness and stability matrices, are defined by

$$K_{ij}^e = \int_{x_a}^{x_b} EI \frac{d^2\phi_i^e}{dx^2}\frac{d^2\phi_j^e}{dx^2}\,dx, \qquad G_{ij}^e = \int_{x_a}^{x_b} \rho A \frac{d\phi_i^e}{dx}\frac{\phi_j^e}{dx}\,dx, \tag{5}$$

and ϕ_i^e are the Hermite cubic interpolation functions.

9.19 The numerical form of the stiffness matrices $[K^e]$ is given in Eq. (9.61). The expression for the stability matrix $[G^e]$ for the Hermite cubic interpolation is given by

$$[G^e] = \frac{a}{30h}\begin{bmatrix} 36 & -3h & -36 & -3h \\ -3h & 4h^2 & 3h & -h^2 \\ -36 & 3h & 36 & 3h \\ -3h & -h^2 & 3h & 4h^2 \end{bmatrix}.$$

9.20 The finite element model of the equation is of the form

$$[K^e]\{\Delta^e\} - \hat{N}_{cr}[G^e]\{\Delta^e\} = \{Q^e\}.$$

Using one element in half beam, the condensed form of the above equation for the case at hand is

$$\left(\frac{2EI}{h^3}\begin{bmatrix} 2h^2 & 3h \\ 3h & 6 \end{bmatrix} - \hat{N}_{cr}\frac{1}{30h}\begin{bmatrix} 4h^2 & 3h \\ 3h & 36 \end{bmatrix}\right)\begin{Bmatrix} U_2 \\ U_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}.$$

Then, $\hat{N}_{cr}$ is given by the smallest eigenvalue $\hat{N}_{cr} = 9.9439(EI/L^2)$.

9.21 The finite element model is given by

$$[K^e]\{u^e\} = \{f^e\} + \{Q^e\},$$

where

$$K_{ij}^e = \int_{r_a}^{r_b}\left[D_{11}\frac{d^2\phi_i}{dr^2}\frac{d^2\phi_j}{dr^2} + D_{12}\frac{1}{r}\left(\frac{d\phi_i}{dr}\frac{d^2\phi_j}{dr^2} + \frac{d^2\phi_i}{dr^2}\frac{d\phi_j}{dr}\right),\right.$$
$$\left. + D_{22}\frac{1}{r^2}\frac{d\phi_i}{dr}\frac{d\phi_j}{dr}\right] r\,dr,$$

$$q_i^e = \int_{r_a}^{r_b} rq\phi_i\,dr.$$

Set $D_{11} = D_{22} = D$ and $D_{12} = \nu D$ for the isotropic case.

9.22 The finite element model is given by substituting the approximation

$$w_0(r) \approx \sum_{j=1}^{m} W_j^e\phi_j^e(r), \qquad \phi(r) \approx \sum_{j=1}^{n} X_j^e\phi_j^e(r),$$

and $v_1 = \psi_i$ and $v_2 = \phi_i$ into the weak forms. We obtain

$$\begin{bmatrix} [K^{11}] & [K^{12}] \\ [K^{21}] & [K^{22}] \end{bmatrix}\begin{Bmatrix} \{W\} \\ \{X\} \end{Bmatrix} = \begin{Bmatrix} \{F^1\} \\ \{F^2\} \end{Bmatrix},$$

where

$$K_{ij}^{11} = \int_{r_b}^{r_a} r A_{55} \frac{d\psi_i}{dr}\frac{d\psi_j}{dr}\, dr, \qquad K_{ij}^{12} = \int_{r_b}^{r_a} r A_{55} \frac{d\psi_i}{dr}\phi_j\, dr = K_{ji}^{21},$$

$$K_{ij}^{22} = \int_{r_b}^{r_a} \left[r D_{11} \frac{d\phi_i}{dr}\frac{d\phi_j}{dr} + D_{12}\left(\frac{d\phi_i}{dr}\phi_j + \phi_i \frac{d\phi_j}{dr}\right) + \left(\frac{D_{22}}{r} + r A_{55}\right)\psi_i\psi_j \right] dr,$$

$$F_i^1 = \int_{r_a}^{r_b} r f \psi_i\, dr + \psi_i(r_a) Q_1 + \psi_i(r_b) Q_3,$$

$$F_i^2 = \phi_i(r_a) Q_2 + \phi_i(r_b) Q_4.$$

Note that both ψ_i and ϕ_i are the Lagrange interpolation functions.

9.23 Let $w_0(\bar{x}) \approx c_1 + c_2\bar{x} + c_3\bar{x}^2 + c_4\bar{x}^3 + c_5\bar{x}^4 + c_6\bar{x}^5$, where $\bar{x}$ is the local coordinate with the origin at node 1. Evaluating $w_0, \theta \equiv dw_0/dx$, and $\kappa \equiv d^2w_0/dx^2$ at nodes 1 and 2 (i.e., at $x = 0$ and $x = h$), we obtain

$$\begin{Bmatrix} w_1 \\ \theta_1 \\ \kappa_1 \\ w_2 \\ \theta_2 \\ \kappa_2 \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 \\ 1 & h & h^2 & h^3 & h^4 & h^5 \\ 0 & 1 & 2h & 3h^2 & 4h^3 & 5h^4 \\ 0 & 0 & 2 & 6h & 12h^2 & 20h^3 \end{bmatrix} \begin{Bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \\ c_5 \\ c_6 \end{Bmatrix}.$$

Invert the equations to solve for c_i and substitute into $w_0(\bar{x}) \approx c_1 + c_2\bar{x} + c_3\bar{x}^2 + c_4\bar{x}^3 + c_5\bar{x}^4 + c_6\bar{x}^5$ to obtain the result.

9.24 The stiffness matrix is

$$[K^e] = \frac{EI}{70h^3} \begin{bmatrix} 1200 & 600h & 30h^2 & -1200 & 600h & -30h^2 \\ 600h & 384h^2 & 22h^3 & -600h & 216h^2 & -8h^3 \\ 30h^2 & 22h^3 & 6h^4 & -30h^2 & 8h^3 & h^4 \\ -1200 & -600h & -30h^2 & 1200 & -600h & 30h^2 \\ 600h & 216h^2 & 8h^3 & -600h & 384h^2 & -22h^3 \\ -30h^2 & -8h^3 & h^4 & 30h^2 & -22h^3 & 6h^4 \end{bmatrix}.$$

9.25 The finite element model is

$$\sum_{j=1}^{n} K_{ij}^e u_j^e = f_i^e + Q_i^e,$$

where

$$K_{ij}^e = \int_{\Omega^e} \left(c_{11} \frac{\partial \psi_i}{\partial x} \frac{\partial \psi_j}{\partial x} + c_{12} \frac{\partial \psi_i}{\partial x} \frac{\partial \psi_j}{\partial y} + c_{21} \frac{\partial \psi_i}{\partial y} \frac{\partial \psi_j}{\partial x} + c_{22} \frac{\partial \psi_i}{\partial y} \frac{\partial \psi_j}{\partial y} + c_0 \psi_i \psi_j \right) dxdy,$$

$$f_i^e = \int_{\Omega^e} \psi_i^e f \, dxdy, \qquad Q_i^e = \oint_{\Gamma^e} \psi_i^e q_n^e \, ds.$$

Chapter 10

10.1 The Euler equations are

$$\begin{aligned}
&\delta \mathbf{v}: \quad -\frac{1}{k}\mathbf{v} + \text{grad}\, u = 0, \\
&\delta u: \quad -(\nabla \cdot \mathbf{v} + \mathbf{f}) = 0, \qquad \text{in } V, \\
&\delta u: \quad \hat{\mathbf{n}} \cdot \mathbf{v} - q = 0 \qquad \text{on } S_2, \\
&\delta \mathbf{v}: \quad u - \hat{u} = 0, \qquad \text{on } S_1.
\end{aligned}$$

10.2 The Euler equations are

$$\begin{aligned}
&\delta \phi: \quad -\frac{d}{dx}\left(EI \frac{d\phi}{dx} \right) + \lambda = 0, \qquad 0 < x < L, \\
&\delta w: \quad -\frac{d\lambda}{dx} + f = 0, \qquad 0 < x < L, \\
&\delta \lambda: \quad \phi + \frac{dw}{dx} = 0, \qquad 0 < x < L, \\
&\delta \phi: \quad EI \frac{d\phi}{dx} = 0 \qquad \text{at } x = 0, L,
\end{aligned}$$

$$\delta w: \quad \lambda(0) = 0 \quad \text{and} \quad \lambda(L) - V_0 = 0.$$

10.3 The natural boundary conditions are (since δu_i is arbitrary on $S = S_1 + S_2$)

$$\begin{aligned}
&\mu(u_{i,j} + u_{j,i})n_j - Pn_i = 0 \qquad \text{on } S_1, \\
&\mu(u_{i,j} + u_{j,i})n_j - Pn_i - \hat{t}_i = 0 \qquad \text{on } S_2.
\end{aligned}$$

10.4 See answer to Exercise 4.46.

10.5 The Euler equations of this functional are

$$
\begin{aligned}
\delta e_{11}&: \quad EA\varepsilon_{xx} + \lambda = 0, && \text{in } 0 < x < L,\\
\delta\lambda&: \quad \varepsilon_{xx} - \frac{du}{dx} = 0, && \text{in } 0 < x < L,\\
\delta u&: \quad \frac{d\lambda}{dx} = f, && \text{in } 0 < x < L,\\
\delta u&: \quad -\lambda(L) - P = 0,\\
\delta u&: \quad u(0) - u_0 = 0.
\end{aligned}
$$

10.7 Follows from the discussion presented in Section 10.2.2, where

$$\Pi(u_i, \varepsilon_{ij}, T) = \int_V [\rho\Psi(\varepsilon_{ij}, T) - f_i u_i]\, dV - \int_{S_2} \hat{t}_i u_i\, ds.$$

Note that U_0 in Eqs. (10.28) is replaced by $\rho\Psi$.

10.9 The functional in Eq. (10.28) should be modified to include the heat conduction equations and U_0 should be replaced by $\rho\Psi$. The functional is given by

$$\Pi_{HW}(u_i, \varepsilon_{ij}, \sigma_{ij}, T) = \Pi_W(u_i, \varepsilon_{ij}, \sigma_{ij}, T) + \int_V \left[\frac{1}{2}k_{ij}\frac{\partial T}{\partial x_i}\frac{\partial T}{\partial x_j} - QT\right] dV,$$

where dissipation terms are neglected.

10.10 The Euler equations are

$$
\begin{aligned}
\delta w_0&: \quad -K_s G_{13} h \frac{d}{dr}\left[r\left(\phi + \frac{dw_0}{dr}\right)\right] - qr = 0,\\
\delta\phi&: \quad K_s G_{13} h\left(\phi + \frac{dw_0}{dr}\right) r - \frac{d}{dr}(rM_r) + M_\theta = 0,\\
\delta M_r&: \quad \frac{d\phi}{dr} + \bar{D}_{22} M_r - \bar{D}_{12} M_\theta = 0,\\
\delta M_\theta&: \quad \phi + r\left(\bar{D}_{11} M_\theta - \bar{D}_{12} M_r\right) = 0.
\end{aligned}
$$

10.11 The matrix coefficients are

$$K_{ij}^{11} = \int_{r_a}^{r_b} A_{55}\frac{d\psi_i}{dr}\frac{d\psi_j}{dr}\, r\, dr, \qquad K_{ij}^{12} = \int_{r_a}^{r_b} A_{55}\frac{d\psi_i}{dr}\psi_j\, r\, dr,$$

$$K_{ij}^{22} = \int_{r_a}^{r_b}\left[A_{55}\psi_i\psi_j + D_{11}\frac{d\psi_i}{dr}\frac{d\psi_j}{dr} + \frac{1}{r}D_{12}\left(\psi_i\frac{d\psi_j}{dr} + \frac{d\psi_i}{dr}\psi_j\right)\right.$$
$$\left. + \frac{1}{r^2}D_{22}\psi_i\psi_j\right] r\, dr,$$

$$f_i = \int_{r_a}^{r_b} q\psi_i \, r \, dr.$$

10.12 The coefficients $K_{ij}^{\alpha\beta}$ $(\alpha, \beta = 1, 2)$ are defined in Exercise 10.11; additional coefficients are defined as follows:

$$K_{ij}^{23} = \int_{r_a}^{r_b} \frac{d\psi_i}{dr} \psi_j \, r \, dr, \qquad K_{ij}^{24} = \int_{r_a}^{r_b} \psi_i \psi_j \, dr,$$

$$K_{ij}^{33} = \int_{r_a}^{r_b} \bar{D}_{22} r \psi_i \psi_j \, dr, \qquad K_{ij}^{34} = -\int_{r_a}^{r_b} \bar{D}_{12} \psi_i \psi_j \, r \, dr,$$

$$K_{ij}^{44} = \int_{r_a}^{r_b} \bar{D}_{11} \psi_i \psi_j \, r \, dr, \qquad K_{ij}^{22} = \int_{r_a}^{r_b} A_{55} \psi_i \psi_j \, r \, dr.$$

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