GEOMATICS I

Measurement Theory

CONTENTS

The notes on Measurement Theory offer a synthetic view of the main concepts which allow to define the quality of a measurement process mainly oriented to the Geomatic applications

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GEOMATICS I

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1. Introduction

Making observations (measurements), and subsequent computations and analyses using them, are fundamental tasks of surveyors. Good measurements require a combination of human skill and mechanical/electronic equipment applied with the highest judgment.

However, no matter how carefully made, observations are never exact and will always contain errors.

Surveyors (Geomatics), whose work must be performed to existing standards, should therefore thoroughly understand the different kind of errors, their sources and expected magnitudes under varying conditions, and their manner of propagation. Only then can they select instruments and procedures necessary to reduce error sizes to within tolerable limits.

Of equal importance, surveyors must be capable of assessing the magnitudes of errors in their observations so that either their acceptability can be verified or, if necessary, new ones made.

The design of measurement systems is now practiced. Computers and sophisticated software are tools now commonly used by surveyors to plan measurements actions and to investigate and evaluate errors after results have been obtained.

In the following paragraphs the measurement theory will be explained and practical examples will be offered in the field of Geomatics but the same concepts could be applied to each kind of measurements of physical quantities such as material composition, temperature, colours, mechanical properties, seismic actions, etc.

2. Direct and indirect observations

Observations may be made directly or indirectly.

Direct measurements are the ones obtained by comparing the physical quantity to be measured with the unit of measurement and by counting how many of them are contained. This action could be performed by using ad-hoc instruments able to display in some way the number of units of measurement that are present in the quantity to be measured.

Examples of direct observations are: applying a tape to a line, fitting a goniometer to an angle, or turning an angle with a total station.

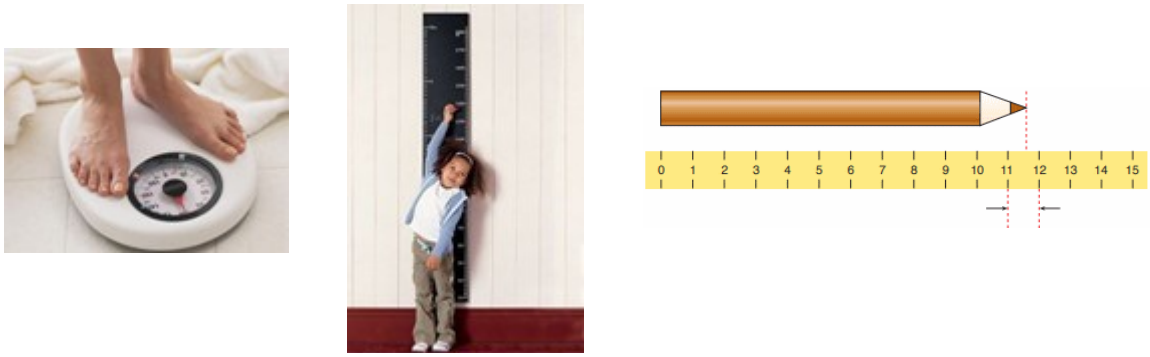

Figure 1 - Direct measurement examples

Each instrument is characterized by its resolution: the resolution of an instrument is the minimum unit of measurement it is able to display. The operator can also try to increase the resolution of the instrument by estimating the portion of a unit of measurement (see the right image in Figure 1).

The resolution must not be confused with the precision of the instrument: this property depends on many other factor that will be explained in the following chapters.

When it is not possible to make a direct measurement (e.g. volume of a building, Cartesian coordinates of a point, etc.) it is necessary to use the indirect measurements.

Indirect measurement are relationships between other physical quantities, which allow the estimation of a physical quantity by using some other physical quantities directly measured.

A simple example is the measurement of the volume of a cube. By measuring directly the length of the side *a* of the cube, the volume *V* could be estimated by using the following relationship

$$V = a^3 \tag{2.1}$$

Many indirect observations are made in surveying, and since all measurements contain errors, it is inevitable that quantities computed from them will also contain errors. The manner by which errors in measurements combine to produce erroneous results is called error propagation.

Before to continue it is necessary to point the attention to some glossary convention we started to use. Because of always-present errors in each kind of direct measurements, the results of direct measurements are called "**observations**" and the results of indirect measurements are called "**estimations**" of the quantity to be measured.

3. Errors in measurements

By definition an error is the difference between an observed/estimated value for a quantity and its true value:

$$E = X - \bar{X} \tag{3.1}$$

where *E* is the error in an observation/estimation, *X* the observed value and $\bar{X}$ its true value.

It can be unconditionally stated that:

- no observation/estimation is exact;
- every observation/estimation contains errors;
- the true value of an observation/estimation is never known;
- the error present is always unknown.

These facts are demonstrated by the following considerations and examples.

When a distance is directly measured with a scale divided into millimetres, the distance can be read only to tenth of millimetre (by interpolation). If a better scale

graduated in tenth of a millimetre is available and read under magnification, however, the same distance can be observed to hundreds of millimetres and so on. Obviously, the resolution of observations depends on the scale's division size, reliability of the equipment used, and human limitations in estimating closer than about one-tenth of a scale division.

As better equipment is developed, observations more closely approach the true value of the measured quantity, but they can never be exact.

Note that observations, not counts (of cars, pennies, marbles, or other objects) are under consideration here.

If one try to measure (both in direct and indirect way) many times the same quantity (length, angle, surface, temperature, etc.) all the results are in general different.

The physical interpretation of this phenomenon is that the measurement is not a number but a physical experiment which is influenced by four factors:

- the man/woman who make the measurements: he/she could be more or less skilled, more or less concentrated;
- the instrument used: could be more or less precise, offering more o less resolution;
- the physical materialization of the quantity to be measured: could be defined in different ways. Figure 2 shows the different meaning of the length of a side of the polygon which has rounded corners (see. Figure 2);
- the environmental conditions where the measurements is performed: could be more or less advantageous to allow the operator to work in a comfortable way, to allow the instrument to run in a more or less correct way.

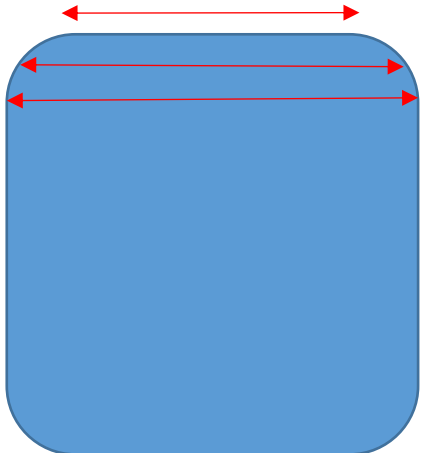

Figure 2 – Materialization of the physical quantity to be measured

3.1 Sources of errors in making observations

The total error *E* (see eq. 3.1) can be interpreted by considering their different origin as follow.

Natural errors are caused by variations in wind, temperature, humidity, atmospheric pressure, atmospheric refraction, gravity, and magnetic declination. An example is a steel tape whose length varies with changes in temperature.

Instrumental errors result from any imperfection in the construction or calibration of instruments and from the movement of individual parts. For example, the graduation

on a scale may not be perfectly spaced or the scale may be warped. The effect of many instrumental errors can be reduced, or even eliminated, by adopting proper surveying procedures or applying computed corrections.

Personal errors arise principally from limitations of the human senses of sight and touch. As an example, a small error occurs in the observed value for a horizontal angle if the vertical crosshair is a total station instrument is not aligned perfectly on the target, or if the target is the top of a rod which is being held slightly out of plumb.

Form a statistical point of view it is not important to discuss about the causes of the errors but the analysis has to be concentrated on the possibility to skip or eliminate them in some way.

By using this second possible classifications the total error *E* (see eq. 3.1) contains gross errors, systematic errors, and random errors.

3.2 Gross errors

These are observer blunders and are usually caused by misunderstanding the problem, carelessness, fatigue, missed communication, or poor judgment.

Examples include: manual transcription of numbers, such as recording 73.96 instead of the correct value of 73.69; reading an angle counterclockwise but indicating it as clockwise angle in the field notes; sighting the wrong target, etc.

They can be avoided by employing expert surveyors, who know how to pay all their attention and expertise during the execution of the measurements.

Gross errors must be detected by careful and systematic checking of all work, and, in case, by repeating some or all of the measurements.

Usually gross errors are easy to be individuated by expert surveyors but, sometimes, their amount is so small that their influence on the observations cannot be detected. In this case robust statistical estimators can be used to find out small gross errors[1].

3.3 Systematic errors

Systematic errors, also known as biases, result from factors that include the "measuring system" and include the environment, the instrument, and the surveyor.

So long as system conditions remain constant, the systematic errors will likewise remain constant. If conditions change, the magnitudes of systematic errors could also change.

Conditions producing systematic errors conform to physical laws that can be modelled mathematically.

Thus if the conditions are known to exist and can be observed, a correction can be computed and applied to observed values.

An example of a constant systematic error is the use of a 10 m steel tape that has been calibrated and found to be 0.01 m too long. It introduces a 0.01 m error each time it is used, but applying a correction readily eliminates the error.

An example of variable systematic error is the change in length of a steel tape resulting from temperature differentials that occur during the period of the tape's use. If the temperature changes are observed, length corrections can be computed by a simple formula and then eliminated from the achieved measure.

[1] *Olive, David J. "Applied robust statistics." Preprint M-02-006 (2008).*

A second possibility sometime used is the adoption of specific operating strategies which allow the elimination of systematic errors also without knowing their amount. Some examples are the Bessel's rule used for the measurement of horizontal directions with total stations, relative positioning in Global Navigation Satellite Systems, etc.

3.4 Random errors

Random errors are those that remain in measured values after gross and systematic errors have been eliminated.

They are caused by factors beyond the control of the observer, and are sometimes called accidental errors.

They are present in all surveying observations. The magnitude and algebraic signs of random errors are matter of chance. There is no absolute way to compute or eliminate them.

Random errors are also known as compensating errors, since they tend to partially cancel themselves in a series of observations. For example a person interpolating to tenths of a millimetres on a tape graduate only to millimetres will presumably estimate too high on some values and too low on others.

However, individual personal characteristics may nullify such partial compensation since some people are inclined to interpolate high, others interpolate low, and many favour certain digits (e.g. 7 instead of 6 or 8, 3 instead of 2 or 4, and particularly 0 instead of 9 or 1.

If a big number of measurements are acquired and if all the gross and systematic errors are not present, the obtained results show small differences and their relative frequencies seems to assume a constant value when the number of observation increases.

Actually each time that a new measurements are added the relative frequencies change but the variations are every time smaller.

This physical phenomenon is known as the "empirical random law" and the values towards the relative frequencies seems to point at, are called "probabilities". This trend is called "stochastic trend" and it is not similar to the well-known deterministic trend of a continuous functions toward its limits.

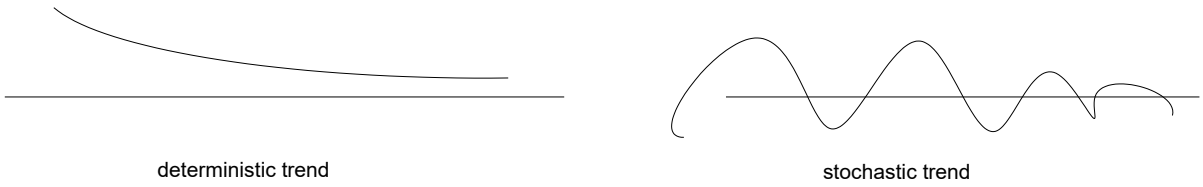

Figure 3 – Deterministic and stochastic trends

But a problem still exists!

Which is the measure of a quantity if every time new values are obtained? In engineering and architecture applications we need a practical solution and just a number which can represent the measure of a quantity.

If we want to solve this problem we have to find a mathematical model able to give us the possibility to manage this phenomenon and to find out a practical solution.

The central theorem of the Statistics says: "*If a phenomenon is affected only by*

random errors and no one of them has a significant difference in magnitude against the others the distribution of the probabilities can be described by the Gauss's distribution of probability".

Let us consider 100 measurements acquired by eliminating all the gross and systematic errors.

Value	Nr. Of obs.	Value	Nr. Of obs.	Value	Nr. Of obs.
19.5	1	23.8	2	26.3	1
20	1	23.9	3	26.5	1
20.5	1	24	5	26.6	3
20.8	1	24.1	3	26.7	1
21.2	1	24.3	1	26.8	2
21.3	1	24.5	2	26.9	1
21.5	1	24.7	3	27	1
22.1	2	24.8	3	27.1	3
22.3	1	24.9	2	27.4	1
22.4	1	25	2	27.5	2
22.5	2	25.1	3	27.6	1
22.6	1	25.2	1	27.7	2
22.8	2	25.4	1	28	1
23	1	25.5	2	28.6	2
23.1	2	25.7	3	28.7	1
23.2	2	25.8	4	29	1
23.3	3	25.9	2	29.4	1
23.6	2	26.1	1	29.7	1
23.7	2	26.2	2	30.8	1
TOTAL NUMEBR OF OBSERVATIONS					**100**

classes	range	Abs.Freq.	Rel.Freq.
A	19.5-20.2	2	0.02
B	20.3-21.0	2	0.02
C	21.1-21.8	3	0.03
D	21.9-22.6	7	0.07
E	22.7-23.4	10	0.10
F	23.5-24.2	17	0.17
G	24.3-25.0	13	0.13
H	25.1-25.8	14	0.14
I	25.9-26.6	10	0.10
L	26.7-27.4	9	0.09
M	27.5-28.2	6	0.06
N	28.3-29.0	4	0.04
O	29.1-29.8	2	0.02
P	29.9-30.8	1	0.01
	Σ	**100**	**1.00**

Table 1 - List of the results of 100 measurements and their classification

We classified all the obtained results by using 14 different ranges of values (indicated above with capital letters).

If we draw a bar diagram we obtain the following graphical description of the measurements.

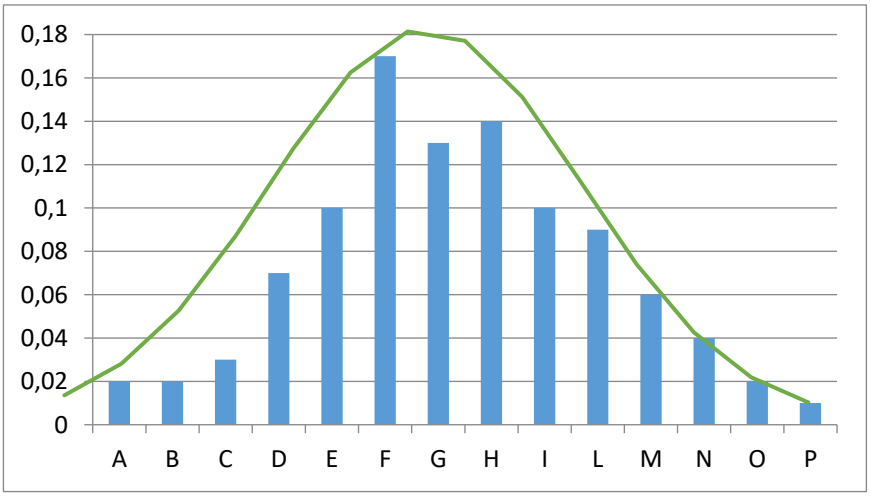

Figure 4 - Graphic representation of the measurements

Also by considering only 100 measurements it is possible to observe that a Guass's probability distribution can be assumed as a mathematical model for the physical phenomenon.

As a conclusion we can state that if we are sure that all the gross and systematic errors are eliminated from our measurements, the random errors which affect our measurements are normally distributed.

It means also that the measurements is conceived, from a mathematical point of view, as a random extraction from an infinite set of possible values.

To be sure that all gross and systematic errors are not present in our measurements we have to take care about the people and instruments involved in the measurement: therefore the surveyor has to know in the best possible way the instrument, its systematic errors, and the strategies to be adopted to eliminate them and has to pay the highest attention during the measurements in order to avoid gross errors.

4. The Guass's probability distribution

The equation of the Gauss's probability distribution is expressed by:

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \cdot e^{-\frac{(x-m)^2}{2\sigma^2}} \tag{4.1}$$

where *m* is the mean and σ^2 is the variance of the population.

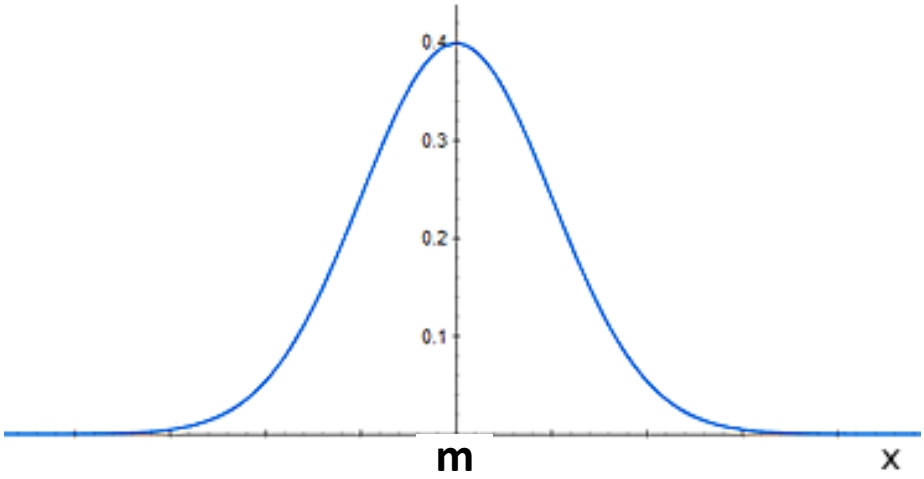

Figure 5 – The Gauss's distribution of probability

The Gauss's distribution of probability (also called normal distribution) is symmetrical with respect to the mean value *m* and therefore the mean *m* is the most probable value of the distribution itself.

The normal distribution has an inflection point which distance from the axe of the symmetry is equal to the square root of the variance: this parameter is called mean square error (m.s.e.)

The area defined by the function between two specific values represent the

probability that a random extraction described by the normal distribution could be contained inside the considered range. As a consequence:

$$\int_{-\infty}^{+\infty} p(x) = \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi\sigma^2}} \cdot e^{-\frac{(x-m)^2}{2\sigma^2}} = 1 \qquad (4.2)$$

Figure 6 shows the probability of some interesting ranges for a normal distribution.

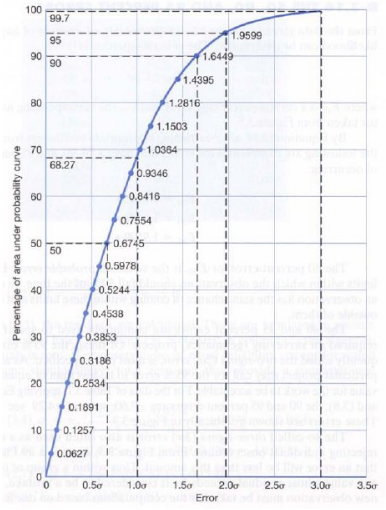

Figure 6 - Relation between error and percentage of area under normal distribution curve

By observing Figure 6 we can state:

- the probability of a random extraction from a normal distribution inside the range m ± σ is equal to 68.26%
- the probability of a random extraction from a normal distribution inside the range m ± 2σ is equal to 95.40%
- the probability of a random extraction from a normal distribution inside the range m ± 3σ is equal to 99.74%

As a consequence we can state that if a measurement phenomenon is described by a normal distribution of given *m* and σ, the probability to obtain a result which differs from the mean less than σ is of about 68.26% while the probability to obtain a result with a difference lower than 2σ is of about 95.4%. Finally the probability to obtain a result with a difference lower than 3σ is of about the 99.7%.

From a practical point of view, if a measurement is completely free of gross and systematic errors every measure gives a value that is far from the mean less than 3σ.

This conclusion satisfy the apparent incongruence between a real measurement, which values are discreet due to the resolution of the used instrument and limited in value by the true dimension of the measured quantity, and the normal distribution values which can range between -∞ and + ∞ in a continuous range.

The m.s.e. indicates in which way the possible values are distributed around the mean: a big value of m.s.e. indicates that the obtained values are far from their mean.

Therefore, the m.s.e. can be assumed as an indicator of the so called "precision" of a

measurement experiment: the lower the m.s.e. the higher the precision (see Figure 7). In fact the lower the m.s.e the higher is the probability that a random extraction is close to the mean.

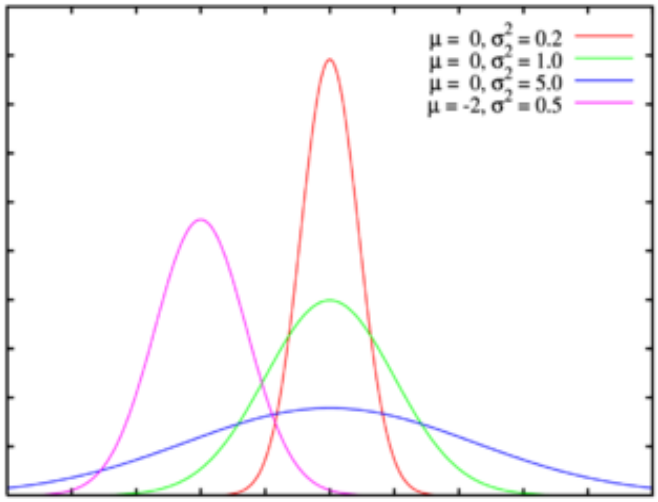

Figure 7 – Gauss's distribution of probability with different m.s.e.

5. Precision and accuracy

A discrepancy (or residual) is the difference between two observed values (measurements) of the same quantity.

A small discrepancy indicates there are probably no gross errors and random errors are small. However, small discrepancies do not preclude the presence of systematic errors.

Precision refers to the degree of refinement or consistency of a group of observations and is evaluated on the basis of discrepancy size.

If multiple observations are made of the same quantity and small discrepancies result, this indicates high precision.

The degree of attainable precision is dependent on equipment resolution and surveyor skill.

The m.s.e. of the normal distribution which describes the measurement experiment, is the parameter mostly used to define the precision: a small m.s.e. means high precision and, on the contrary, high values of m.s.e. indicate low precisions.

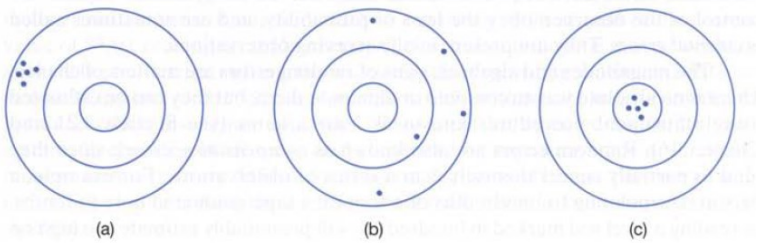

Figure 8 - Examples of precision and accuracy: a) Results are precise but not accurate. b) Results are neither precise nor accurate. c) Results are both precise and accurate.

The term "accuracy" is defined as the absolute nearness of measured quantities to their true values.

It is measured by means of "tolerance": the highest the tolerance the lowest the accuracy of a measurement.

While m.s.e. can be estimated starting from a set of observations (as we will see in the following paragraphs), the tolerance of a set of measurements can be estimated only by comparing the obtained results with a set of measurements characterized by an higher precision (e.g. to estimate the tolerance of a set of measurements with a precision of about 1 cm, a set of measurements with a precision of almost 1 mm has to be used to estimate the accuracy).

The difference between "precision" and "accuracy" is best illustrated by Figure 8.

By considering the definitions of precision and accuracy, it seems that a relationship between the two properties of the measurement do not exist but in practical surveys it is not true.

Remembering that a set of observations free from gross and systematic errors give results at a certain probability inside a known range (see par. 4) we can say that by accepting a probability of about 95% the accuracy of a set of observations is two times the m.s.e. of the measurement, and by accepting a probability of about 99% the accuracy of a set of observations id three times the m.s.e. of the measurement.

The main goal of a surveyor is to obtain measurements that are at the same time precise as much as needed to guarantee the required accuracy.

The accuracy is usually fixed by the final user of the survey (e.g. engineer, architect, geologists, etc.); knowing it the surveyor know that he has to obtain measurement with a minimum precision defined by the above described conventions.

Let us consider an example: the final user ask for an accuracy (maximum difference between the metric survey and the real surveyed object) of 5 cm at 95% of probability. The surveyor knows that he/she have to obtain the final measurements with a m.s.e. not greater than 2.5 cm. Considering those values the surveyor is able to select in which way he/she has to make the measurement (survey's schemes, instruments to be used, etc.).

As it was stated many times in previous paragraphs, the surveyor, to be able to guarantee that at the end of all the measurement acquisitions and treatments he/she will reach the foreseen precision, has to be skilled, concentrated on his work and able to use the ad-hoc instruments in the best possible way to be sure that no gross and systematic errors will affect his/her measurements.

6. Mean and m.s.e. estimation

The true values of the mean and m.s.e. of the normal distribution which describes a measurement can be exactly known only if an infinite number of measurements could be done.

Therefore, in practice, the surveyor do not has the possibility to know exactly the values of the parameters of the normal distribution because only a limited number of measurements can be done in reality.

In addition we have to consider that in real surveys the surveyor do not have time and money to waste, but, at the same time, they need to know in the best possible way the fundamental parameters of the normal distribution associated to their

measurements: the mean is the value that will be used in the procedures that follow the measurements acquisition (3D modelling) and the m.s.e. will certify the quality of the measurements and the achievement of the foreseen tolerances.

After all this considerations practical procedures which allow to estimate in the best possible way the mean and the m.s.e of a normal distribution have to be defined by using only a small amount of observations.

The mean and the m.s.e of the Gauss's distribution of probability which describes a measurement can be "estimated" in the best possible way by using the so called "estimators". The estimators are equations which, by considering the results of the achieved observations (O_1, O_2,…,O_n) are able to give back the best estimation of the statistical parameter (e.g. mean or m.s.e.).

The estimators have to satisfy some conditions to be useful for the estimation of the Gauss's distribution parameters.

The estimators must be:

- consistent
- efficient
- not biased

The Statistics affirms that if each observation could be interpreted as a random extraction from a normal distribution, also the estimators make a random extraction form a normal distribution.

The normal distribution of the estimators has its own mean (m_e) and m.s.e. ($m.s.e._e$).

An estimator is consistent if, by considering the possibility to use an infinite number of measurements, it will give the theoretical value of the estimated parameter.

An estimator is efficient if the $m.s.e._e$ of the normal distribution of the estimator has the minimum value.

An estimator is not biased if the m_e of the normal distribution of the estimator coincides with the theoretical value of the estimated parameter (e.g. the mean of the normal distribution of the mean's estimator must coincides with the theoretical value of the mean of the observations).

6.1 The maximum likelihood principle

To define a possible estimator the maximum likelihood principle is normally used. This principle says that the measurements that are used to estimate a parameter have the maximum probability.

Let us consider the probability *f(x)* (see eq. 4.2) of the observations x_1, x_2, …, x_n randomly extracted from a Gauss's distribution. The probability of each value is expressed by the following:

$$dp_i = f(x_i) \cdot dx \tag{6.1}$$

The probability of all the *n* observations is expressed by:

$$P = f(x_1) \cdot f(x_2) \cdot f(x_3) \cdot \cdots \cdot f(x_n) \cdot dx^n \tag{6.2}$$

If the theoretical parameters are known the total probability will assume a unique

value *P**, but actually those parameters are not known so the total probability of a set of observations changes its value each time a new set of observations is considered.

Since the equation (6.2), it is possible to look for the maximum value of *P* and considering the possible estimator as the one which assumes that the obtained observations are "the best possible ones" (e.g. they have as a whole the maximum probability).

If we assume as distribution parameter estimations the ones who maximize the probability of the set of observations used to estimate the parameters themselves, it means that the set of *n* observations is considered the most probable among all the possible sets of *n* extraction.

If the f(x) is the one of the normal distribution:

$$f(x_i) = \frac{1}{\sqrt{2\pi\sigma^2}} \cdot e^{-\frac{(x_i - m)^2}{2\sigma^2}} \tag{6.3}$$

the probability of a set of *n* random extraction from the normal distribution defined in (6.3) is:

$$P(x_1, x_2, \cdots, x_n, m, \sigma^2) = \frac{1}{(2\pi\sigma^2)^{n/2}} e^{-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_1 - m)^2} \cdot dx^n \tag{6.4}$$

The maximum of the probability *P* coincides with the minimum of the exponent of the second term. Therefore the maximum of *P* is reached when:

$$\sum_{1=1}^{n}(x_i - m)^2 = \sum_{1=1}^{n} v^2 = minimum \tag{6.5}$$

The (6.5) is the analytical expression of the so called "least square principle".

The least squares principle says that the maximum of the probability of a set of *n* observations is reached when the sum of the squares of the differences (residuals) between each observed values and the theoretical mean of the Guass's distribution of probability from where the observations have been extracted, is minimum.

The respect of the least square principles do not guarantee that the obtained equation is a good estimator because it is not always true that the obtained equation is also consistent, efficient, and not biased.

The equation obtained by applying the least square principle could be considered as a first tentative to obtain an estimator but the main properties of the estimators must be verified case by case.

Using the least square principle (that is the application of the maximum likelihood principle to a Gauss's distribution of probability) means that the survey has to guarantee that his/her observations are extractions from a Gauss's distribution of probability. Therefore all the observations have to be free from gross and systematic errors.

In the following paragraphs we will define the best estimators (consistent, efficient and not biased) for the mean and the variance of the normal distribution of different kind of measurements (e.g. direct and indirect).

6.2 Direct observations with same precision[2]

A surveyor make *n* direct observations of the same physical quantity (e.g. a length) in the same operative conditions, by using the same instrument, with the same materialisation of the measured physical quantity: $x_1, x_2, \ldots, x_n$ are the results of the obtained observations.

From a statistical point of view if all the gross and systematic errors have been eliminated and if all the random errors are of the same levels, each measurement can be interpreted as a random extraction of *n* values from a normal distribution.

Now, the mean and the variance of the normal distribution have to be estimated.

In order to find out an estimator for the mean, let us use the maximum likelihood principle (e.g. the least squares principle): in mathematical terms we have to find the value $\bar{m}$ which make minimum the (6.5):

$$\sum_{1=1}^{n} (x_i - m)^2 = min \tag{6.6}$$

As known the minimum of a function is where the its first derivative is equal to zero and the second derivative is positive.

Looking for the point where the first derivative of (6.6) is null it is possible to define the possible estimator for the mean of the normal distribution of a set of observations with same precision:

$$-2\sum_{1=1}^{n}(x_i - \bar{m}) = 0 \quad \rightarrow \quad \sum_{1=1}^{n} x_i - n\bar{m} = 0 \quad \rightarrow \quad \bar{m} = \frac{\sum_{i=1}^{n} x_i}{n} \tag{6.7}$$

which is the well-known arithmetic mean of a series of numbers.

To assume the arithmetic mean as the estimator of the mean of a set of observations with same precision, the consistency, efficiency, and not biased properties have to be verified.

Let us introduce the operator *M* which, applied to a probability distribution, extracts the theoretical mean.

Applying the operator M to the (6.7):

[2] If one of the conditions listed in par.3 (surveyor, instrument, environmental conditions, and materialization of the quantity to be measured) changes, the observations have be considered as random extractions from different Gauss's distribution of probability. The observations obtained by changing also just one of the measurement conditions are considered observations of different precisions (e.g. extracted from different Gauss's distribution of probability).

$$M(\bar{m}) = \frac{1}{n}\{M[x_1] + M[x_2] + \cdots + M[x_n]\} = \frac{n \cdot m}{n} = m \tag{6.8}$$

Therefore we can say that the arithmetic mean, if the number of observations is infinite, will give us the theoretical mean and it is not biased because the mean of the estimator is equal to the mean of the observations.

It is also possible to demonstrate that the arithmetic mean is efficient (e.g. its variance is the minimum among all the possible estimators of the mean) and finally we can conclude by saying that: the best estimator for the mean of the normal distribution which describes the physical experiment of direct measurements performed by a unique operator, with the same instrument and in the same operative conditions, is the arithmetic mean.

By considering the rules of the combination of many normal distribution we can state that the variance of the distribution of the means of n values is:

$$\sigma_{\bar{m}}^2 = \frac{\sigma^2 + \sigma^2 + \ldots + \sigma^2}{n^2} = \frac{n\sigma^2}{n^2} = \frac{\sigma^2}{n} \tag{6.9}$$

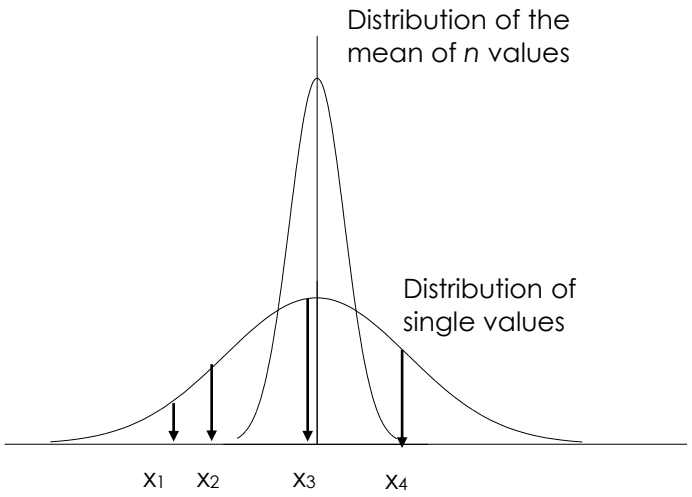

Figure 9 – Normal distributions of direct measurements and of the arithmetic mean

This demonstrates that making *n* observations is equivalent to make one observation with a variance that is in a ratio of 1/n. Therefore the arithmetic mean, is an indirect measurement, and it is more precise than the extraction of a single value.

To estimate the variance we assume as possible estimator the following equation extracted from (6.9)

$$s^2 = \frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{m})^2 \tag{6.10}$$

If in (6.10) the theoretical mean *m* is added and subtracted:

$$s^2 = \frac{1}{n}\sum_{i=1}^{n}(x_i - m + m - \bar{m})^2 = \frac{1}{n}\sum_{i=1}^{n}(x_i - m)^2 + \frac{1}{n}\sum_{i=1}^{n}(m - \bar{m})^2 + \frac{2}{n}\sum_{i=1}^{n}(x_i - m)\cdot(m - \bar{m})$$

But:

$$\frac{1}{n}\sum_{i=1}^{n}(m - \bar{m})^2 = (m - \bar{m})^2$$

and:

$$\frac{2}{n}\sum_{i=1}^{n}(x_i - m)\cdot(m - \bar{m}) = \frac{2}{n}(m - \bar{m})\sum_{i=1}^{n}(x_i - m) = \frac{2}{n}(m - \bar{m})\cdot\left(\sum_{i=1}^{n}x_i - nm\right)$$
$$= \frac{2}{n}(m - \bar{m})\cdot(n\bar{m} - nm) = -2(\bar{m} - m)^2$$

Therefore:

$$s^2 = \frac{1}{n}\sum_{i=1}^{n}(x_i - m)^2 - (\bar{m} - m)^2 \qquad (6.11)$$

Inserting the operator *M* it is possible to verify the mean of s^2 is equal to the theoretical variance of the measurement's distribution:

$$M[s^2] = \frac{1}{n}\sum_{i=1}^{n}M[(x_i - m)^2] - M[(\bar{m} - m)^2] = \frac{1}{n}n\sigma^2 - \frac{\sigma^2}{n} = \sigma^2\frac{n-1}{n} \qquad (6.12)$$

It is possible to observe that the theoretical mean of s^2 do not coincides with σ^2 therefore the adopted estimator (6.10) is biased.

To obtain a not biased estimator of the variance the (6.10) must be modified

$$\overline{\sigma^2} = \frac{\sum_{i=1}^{n}(x_i - \bar{m})^2}{n-1} \qquad (6.13)$$

It is possible to demonstrate that this solution is consistent and efficient.

In the same way we can state that the variance of the mean of *n* measurement is expressed in the following way:

$$\overline{\sigma^2} = \frac{\sum_{i=1}^{n}(x_i - \bar{m})^2}{n(n-1)} \qquad (6.14)$$

Looking at (6.13) and (6.14) a variance near to 0 can happen if the extracted values are very close (the variance depends on the discrepancies between each measurements and the estimated mean) or when the number *n* of measurements used is very high.

The last case is not true!

Each instrument is characterized by means of a calibration certificate which define which is the minimum m.s.e. that it is possible to reach by using in the best possible way that instrument: the so called instrumental m.s.e.

If the estimated variance computed using the (613) and (6.14) is lower than the instrumental variance, the instrumental m.s.e. is assigned to the set of observations.

By concluding to directly measure a quantity the surveyor has to make *n* observations and the result of the measurement is expressed by

$$L = \bar{m} \pm \bar{\sigma}_{\bar{m}} \qquad (6.14)$$

The symbol "±" has not a specific meaning: it just separates the estimated mean form the estimated m.s.e. of the mean: the maximum possible variation of the estimated mean (with a probability of 99.7%) is $\pm 3\bar{\sigma}_{\bar{m}}$ and the maximum possible variation (with the 99.7% of probability) of a single measurement is $\pm 3\bar{\sigma}$.

If only one observation is achieved, the obtained value is an estimation of the mean but it is not possible to estimate the m.s.e.

Sometimes this solution is adopted (e.g. in detail survey) but only when the surveyor has the possibility to check (by its experience and skill) that no gross or systematic errors can be present.

6.3 Direct observations with different precision

A set of direct measurements can be conceived as a random extraction from a unique distribution of probability if the surveyor, the instruments, and the operative conditions do not change.

The observations made in different conditions are interpreted a random extractions from different normal distributions which have the same theoretical mean because they refer to the same physical quantity.

When a set of direct measurements of the same quantity are judged coming from different distributions, an alternative approach to the one previously described must be adopted.

Let as suppose that the measurement of a quantity has been performed *n* times with *n* different instruments and/or in *n* different operative conditions and O_1, O_2 …, O_n are the results of these observations.

From a statistical point of view it is possible to say that each observation comes from a distribution which has the same mean but different m.s.e. and from each of them a set (or just one) random extraction have been done.

In case of many extractions from a unique distribution, a first estimation of the variance σ_i^2 is possible while in case of only one observation from a unique normal distribution it is not possible.

Each observation give a first estimation of the mean of all the distributions involved in

the measurement process but obviously, they offer, in general, different estimations of the mean.

To solve this problem the estimations of the mean and of the variance, by considering all the obtained results, have to be pursued.

These estimations have to consider the different precision of each observation: the bigger the precision of a single observation the bigger will be the influence of the observation itself inside the estimation process.

In practical cases, two different possibilities can rise up:

- the variance of each observation is known
- the ratio between the different precisions is know

In the last case, the knowledge of the ratios between the precisions of the observations can be solved in numerical terms by adopting the concept of "weight".

The weight of an observation is the ratio between the variance of an ideal distribution with a weight equal to 1, and the variance of the considered distribution:

$$w_i = \frac{\sigma_0^2}{\sigma_i^2} \tag{6.15}$$

where σ_0^2 is an unknown arbitrary constant and represent the variance of an ideal distribution to which a weight equal to 1 has been assigned: it is called the unit weight variance.

It is possible to fix the weights just by considering instruments and operative conditions of each of the observations.

As an example, if two observations have been realized by using two different instruments and the first one has an instrumental m.s.e., which is twice the instrumental m.s.e. of the second instrument, a weight 1 is assigned to the first observation (the one coming from the first instrument) and a weight 2 to the second observation. In this case, the second observation, coming from the best instrument, will influence more the final estimation than the first one.

In the following, the best estimators for mean and variance in this particular case are described.

As in previous paragraph, the maximum likelihood principle is used to obtain a possible estimator for the mean.

The probability density for a single observation, by considering its weight is:

$$f(O_i) = \frac{1}{\sqrt{2\pi\sigma_i^2}} \cdot e^{-\frac{(O_i-m)^2}{2\sigma_i^2}} = \frac{1}{\sqrt{2\pi\sigma_i^2}} \cdot w_i^{1/2} \cdot e^{\frac{w_i}{2\sigma_0^2}\cdot(O_i-m)^2} \tag{6.16}$$

Therefore, the probability of all the measurements is

$$P(O_1, O_2, \cdots, O_n) = \frac{(w_1, w_2, \cdots, w_n)^{1/2}}{(2\pi\sigma_0^2)^{n/2}} \cdot e^{-\frac{1}{2\sigma_0^2}\sum_{i=1}^{n} w_i(O_i - m)^2} \cdot dx^n \qquad (6.17)$$

Independently form the value of σ_0^2, the maximum of the probability of all the measurements involved in the estimation procedure is obtained when

$$\sum_{i=1}^{n} w_i(O_i - m)^2 = min \qquad (6.18)$$

Therefore, a possible estimator of the mean $\bar{m}_w$ (weighted mean)is:

$$-2\sum_{i=1}^{n} w_i(O_i - \bar{m}_w) = 0 \rightarrow \bar{m}_w = \frac{w_1 O_1 + w_2 O_2 + \cdots + w_n O_n}{w_1 + w_2 + \cdots + w_n} \qquad (6.19)$$

The (6.19) is a real estimator because it is possible to demonstrate that it is consistent, efficient and not biased.

The estimated mean is called "weighted mean".

If the weights are multiplied by a constant value, the weighted mean do not change: this is a corroboration to the fact that σ_0^2can assume any arbitrary value if the variances of the observations are not known at the beginning of the estimation process.

By changing the set of observations O_1, O_2 …, O_n the weighted mean is a single random extraction from a distribution which has *m* as theoretical mean.

To find out a possible estimator for the variance in this particular case, let us apply the composition rule of the random variables to the (6.19).

$$\bar{\sigma}^2_{\bar{m}_w} = \left(\frac{w_1}{\sum_{i=1}^{n} w_i}\right)^2 \cdot \bar{\sigma}_1^2 + \left(\frac{w_2}{\sum_{i=1}^{n} w_i}\right)^2 \cdot \bar{\sigma}_2^2 + \cdots + \left(\frac{w_n}{\sum_{i=1}^{n} w_i}\right)^2 \cdot \bar{\sigma}_n^2 = \frac{\sum_{i=1}^{n} w_i^2 \cdot \bar{\sigma}_i^2}{\left(\sum_{i=1}^{n} w_i\right)^2} \qquad (6.20)$$

The (6.20) is consistent, efficient and not biased and solve the problem of the weighted mean variance estimation if the variances of the single observations are known since the beginning (e.g. because they are the results of a set of direct observations with same precision).

In the second case, when only weights are known and assigned to each observations by considering the different conditions under which the observations have been obtained,, it can be noticed that

$$\sigma_i^2 = \frac{\sigma_0^2}{w_i}$$

therefore:

$$\bar{\sigma}^2_{\bar{m}_w} = \frac{\sum_{i=1}^{n} w_i \cdot \sigma_o^2}{\left(\sum_{i=1}^{n} w_i\right)^2} = \sigma_o^2 \cdot \frac{\sum_{i=1}^{n} w_i}{\left(\sum_{i=1}^{n} w_i\right)^2} = \frac{\sigma_o^2}{\sum_{i=1}^{n} w_i} \tag{6.21}$$

The σ_0^2 has to be estimated in some way: this is possible by using the same set of observations used to estimate the weighted mean. It must be noticed that by changing the set of *n* observations, also the estimation of the σ_0^2 will change therefore σ_0^2 is a random variable described by a normal probability distribution.

To extract a possible estimator of σ_0^2 the maximum likelihood principle is used.

In this case, it is convenient to introduce the natural logarithm of the probability of the whole set of observations (6.17):

$$P(O_1, O_2, \cdots, O_n) = \frac{(w_1, w_2, \cdots, w_n)^{1/2}}{(2\pi\sigma_0^2)^{n/2}} \cdot e^{-\frac{1}{2\sigma_0^2}\sum_{i=1}^{n} w_i (O_i - m)^2} \cdot dx^n$$

$$lnP = \frac{1}{2} ln(w_1 \cdot w_2 \cdot \cdots \cdot w_n) - \frac{n}{2} \cdot ln2\pi - \frac{n}{2} ln\sigma_0^2 - \frac{1}{2\sigma_0^2} \sum_{i=1}^{n} w_i (O_i - m)^2 \tag{6.22}$$

This last expression has its maximum where the first derivative of the last equation in (6.22) has a null point:

$$\frac{\partial lnP}{\partial \sigma_0^2} = -\frac{n}{2\sigma_0^2} + \frac{1}{2\sigma_0^4} \sum_{i=1}^{n} w_i (O_i - m)^2 = 0$$

Introducing the weighted mean a possible estimator of the variance of the distribution with weight equal to 1 can be obtained:

$$\bar{s}_0^2 = \frac{\sum_{i=1}^{n} w_i (O_i - \bar{m}_w)^2}{n} \tag{6.23}$$

The (6.23) is consistent and efficient. To check if the (6.23) is or not biased the operator *M* (the operator that applied to a distribution of probability is able to extract the theoretical mean of it) is applied to the possible estimator by using the same approach adopted in paragraph 6.2

$$M[\bar{\sigma}_0^2] = \frac{n-1}{n} \sigma_0^2$$

Finally the best estimator for σ_0^2 is

$$\bar{\sigma}_0^2 = \frac{\sum_{i=1}^{n} w_i (O_i - \bar{m}_w)^2}{n-1} \tag{6.24}$$

6.4 Indirect measurement of a quantity by using direct or indirect measurements

Let us consider a quantity *X* and *n* quantities D_1, D_2 …, D_n; between *X* and the D_i quantities a relation exists:

$$X = f(D_1, D_2, \cdots, D_n) \tag{6.25}$$

This expression defines the *X* as a random variable which parameters (mean and variance) depends on the parameters of the random variables of the D_i quantities.

The D_i quantities can be direct measurements (stochastically independent), and in this case an estimation of the mean and of the variance are known (estimated by one of the two processes previously described).

If D quantities are indirect measurements, generally, they are stochastically correlated and they are characterized by the estimated means and by a variance-covariance matrix of the random distribution with *n* dimensions

$$\mathrm{U} = \begin{vmatrix} \sigma_1^2 & \sigma_{12} & \cdots & \sigma_{1n} \\ \sigma_{12} & \sigma_2^2 & \cdots & \sigma_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ \sigma_{1n} & \sigma_{2n} & \cdots & \sigma_n^2 \end{vmatrix}$$

From a statistical point of view, the indirect measurement of a quantity *X* means the definition of a probability distribution of all the possible measurements of *X* starting from the probability distribution, which defines the n direct measurements Di, or from the n-dimensions probability distributions, which globally defines the indirect measurements of the D_i quantities.

The problem can be easily solved if the relationship between *X* and the D_i is a linear combination:

$$X = a_1 D_1 + a_2 D_2 + \cdots + a_n D_n$$

If the D_i quantities are not correlated, the following estimators for the mean and the variance of *X* can be used:

$$\begin{aligned} \bar{m}_X &= a_1 \bar{m}_{D_1} + a_2 \bar{m}_{D_2} + \cdots + a_n \bar{m}_{D_n} \\ \bar{\sigma}_X{}^2 &= a_1^2 \bar{\sigma}_{D_1}^2 + a_2^2 \bar{\sigma}_{D_2}^2 + \cdots + a_n^2 \bar{\sigma}_{D_n}^2 \end{aligned} \tag{6.26}$$

If the D_i quantities are correlated

$$\begin{aligned} \bar{m}_X &= a_1 \bar{m}_{D_1} + a_2 \bar{m}_{D_2} + \cdots + a_n \bar{m}_{D_n} \\ \bar{\sigma}_X{}^2 &= a_1^2 \bar{\sigma}_{D_1}^2 + a_2^2 \bar{\sigma}_{D_2}^2 + \cdots + a_n^2 \bar{\sigma}_{D_n}^2 + 2a_1 a_2 \sigma_{1,2} + 2a_1 a_3 \sigma_{1,3} + \cdots .2a_{n-1} a_n \sigma_{(n-1),n} \end{aligned} \tag{6.27}$$

If the *X* quantity has a general relationship *f* with the observed quantities *D*, the following procedure can be used.

Let us consider a set O_1, O_2 …, O_n randomly extracted from D_1, D_2 …, D_n.

Each O_i value is the result of a random extraction from the distribution, which has the theoretical mean D_{im}.

It is possible to express the O_i in the following way

$$O_1 = D_{1m} + v_1 \quad O_2 = D_{2m} + v_2 \qquad \cdots O_n = D_{nm} + v_n \tag{6.28}$$

If the discrepancies v_i are small and their second powers v_i^2 can be negligible, it is possible to linearize (6.25) by using a Taylor series around the point defined by the theoretical means D_{im}.

$$\begin{aligned} f(O_1, O_2, \cdots O_n) &= f(\ D_{1m} + v_1, + \cdots + D_{nm} + v_n\ ) \\ &= f(D_{1m}, \cdots D_{nm}) + \left(\frac{\partial f}{\partial D_1}\right)_m \cdot v_1 + \cdots + \left(\frac{\partial f}{\partial D_n}\right)_m \cdot v_n \end{aligned} \tag{6.29}$$

By applying the operator *M* we obtain

$$M[f(O_1, O_2, \cdots O_n)] - f(D_{1m}, \cdots D_{nm}) = \left(\frac{\partial f}{\partial D_1}\right)_m \cdot M[v_1] + \cdots + \left(\frac{\partial f}{\partial D_n}\right)_m \cdot M[v_n] \tag{6.30}$$

The mean of the first term is equal to 0 because the second term is a linear combination of discrepancy distributions, therefore the mean of $f(O_1, O_2,\ldots,O_n)$ is equal to the mean of $f(D_{1m}, D_{2m},\ldots,D_{nm})$

$$X_m = f(D_{1m}, \cdots D_{nm}) \tag{6.31}$$

By concluding, if the discrepancies v_i are smaller enough to neglect their second order terms (this is true if the O_i values are derived by using a correct estimation of the means of the distribution D_i) the mean of the *X* quantity can be estimated by introducing the estimated means of the D_i quantities.

To estimate the variance we can operate in the following way.

Given

$$\left(\frac{\partial f}{\partial D_1}\right) = a_1, \cdots, \left(\frac{\partial f}{\partial D_n}\right) = a_n$$

and remembering

$$\sigma_X^2 = M[\{f(O_1, O_2, \cdots, O_n) - X_m\}^2] = M[(a_1 v_1 + \cdots + a_n v_n)^2]$$

in case the D_i quantities are not correlated, the estimator of the variance of the X quantity is:

$$\bar{\sigma}_X^2 = \left(\frac{\partial f}{\partial D_1}\right)_{\bar{m}}^2 \cdot \bar{\sigma}_1^2 + \cdots + \left(\frac{\partial f}{\partial D_n}\right)_{\bar{m}}^2 \cdot \bar{\sigma}_n^2 \qquad (6.32)$$

If D_i quantities are correlated, the (6.32) assumes the more general form

$$\bar{\sigma}_X^2 = \left(\frac{\partial f}{\partial D_1}\right)_{\bar{m}}^2 \cdot \bar{\sigma}_1^2 + \cdots + \left(\frac{\partial f}{\partial D_n}\right)_{\bar{m}}^2 \cdot \bar{\sigma}_n^2 + 2\sum_{i<j}\left(\frac{\partial f}{\partial D_i}\right)_{\bar{m}}\left(\frac{\partial f}{\partial D_j}\right)_{\bar{m}} \bar{\sigma}_{ij} \qquad (6.33)$$

The (6.32) and (6.33) are the analytic expressions of the "variance propagation law".

Looking at (6.32) and (6.33) it must be noted that every time measurements are used to generate new measurements the error is always greater that the one of the single measurement involved in the process. A simple sum of the measurements generate a new measurement with higher error and so lower precision.

This simple observation can suggest to limit to the minimum possible the use of measurements manipulation to generate new products.

6.5 Indirect measurements of *m* quantities with directly measured *n* quantities (*m*<*n*)

The last case is the more general approach to indirect measurement estimation and the most used in all metric survey applications (terrestrial survey, GNSS, photogrammetry).

Let us consider a simple example: the forward intersection (see Figure 10).

A generic unknown point *C* is visible from two known points A and B.

To find the coordinates of the point C, a possible solution is to measure two angles and to find out the location of the point C by intersecting the two lines AC and BC.

From a geometric point of view, this problem has always a solution: the only critical case is when the two measured angles are of equal amplitude (in this case the point C is at infinite!).

However, in this way if a gross error occurs it is not possible to find it out and mainly, by just using the minimum geometric constraints, it is not possible to estimate the reached precision in the point P location.

Therefore, by using measurements, simple solutions offered by geometry are not useful and statistical approach must be used.

From a statistical point of view the problem can be described as follow: we want to estimate the means and the variance-covariance matrix of a *m*-dimension probability distribution, depending on *m* indirect measurements $X_1, X_2, \ldots, X_m$ starting from a set of *n* direct measurements described by *n* not correlated probability distributions with known mean and variance $D_1, D_2, \ldots, D_n$ where $m < n$.

The result we are looking for will depend on the set of direct measurements and it will be the result of an adjustment of the executed direct measurements.

The indirect quantities X_i are connected to the direct quantities D_i by means of generic

functions, which depend on geometric and/or physical conditions.

Coming back to the simple problem of Figure 10, adding the measurement of the side *AC* it will allow to check if some gross errors exist, and the estimation of the variances of the two coordinates of the point P.

Adding the measurement of the side *BP* a more affordable estimation of the means and variances of the X and Y coordinates of the point P is possible.

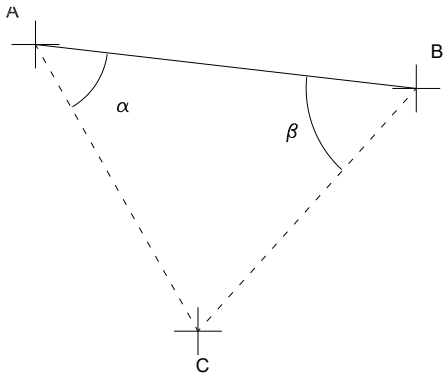

Figure 10 - Forward intersection

As a practical rule, if a problem has *m* unknowns the best number of independent measurements to obtain a correct estimation of the means and variances of the indirect quantities is *2m*.

Let us consider a system of n linear equations which join *m<n* unknown quantities X_1, $X_2,\ldots,X_m$ to the direct measured quantities D_1, $D_2,\ldots,D_n$ with the constraint that each equation contains one and only one direct measured quantity.

$$\begin{cases} a_{11}X_1 + \cdots a_{1m}X_m = D_1 \\ \cdots\cdots\cdots\cdots\cdots \\ a_{n1}X_1 + \cdots a_{nm}X_m = D_n \end{cases} \tag{6.34}$$

where a_{ij} are known coefficients.

In our system, the known and unknown quantities are indicated by using the theoretical values of the means.

In this theoretical case, each set of m equations can give a unique solution to the system, which satisfy also the remaining n-m equations (geometric solution).

Actually, it is not possible to obtain the theoretical means of the direct measurements: in the best case, only the correct estimation of $\bar{D}_i$ could be obtained. The estimated means of the direct measurements differ from the theoretical value D_i of a small residual

$$D_i = \bar{D}_i + v_i$$

Therefore, the (6.34) has to re-written:

$$\begin{cases} a_{11}X_1 + \cdots a_{1m}X_m = \bar{D}_1 + v_1 \\ \cdots\cdots\cdots\cdots\cdots \\ a_{11}X_1 + \cdots a_{1m}X_m = \bar{D}_n + v_n \end{cases} \tag{6.35}$$

The v_i represent the discrepancies between the estimated mean and their theoretical values, therefore they are also unknown quantities.

Therefore, the system has *n* equations but *m+n* unknowns and it cannot be solved by using mathematical approaches.

A possible solution is given by the Statistics.

In (6.35) the theoretical means of the indirect quantities are still indicated. More realistically we can think to be able to look for the best estimation of the means of the indirect quantities $\bar{X}_i$ and the (6.35) becomes:

$$\begin{cases} a_{11}\bar{X}_1 + \cdots a_{1m}\bar{X}_m = \bar{D}_1 + v_1 \\ \cdots\cdots\cdots\cdots\cdots \\ a_{11}\bar{X}_1 + \cdots a_{1m}\bar{X}_m = \bar{D}_n + v_n \end{cases} \tag{6.36}$$

To extract the estimator of the means the maximum likelihood principle is used by fixing that the best estimations of the means of the indirect measurements are the ones which minimize the sum of the squares of the residuals v_i.

To demonstrate a general solution a simple example is used.

To estimate two quantities X_1 and X_2, three direct measurements D_1, D_2, D_3 are considered linked to the indirect measurements by means of linear combinations:

$$\begin{cases} a_{11}X_1 + a_{12}X_2 = D_1 \\ a_{21}X_1 + a_{22}X_2 = D_2 \\ a_{31}X_1 + a_{32}X_2 = D_3 \end{cases}$$

This system has to be re-written in the following way

$$\begin{cases} a_{11}\bar{X}_1 + a_{12}\bar{X}_2 = \bar{D}_1 + v_1 \\ a_{21}\bar{X}_1 + a_{22}\bar{X}_2 = \bar{D}_2 + v_2 \\ a_{31}\bar{X}_1 + a_{32}\bar{X}_2 = \bar{D}_3 + v_3 \end{cases}$$

or, more simply, by using the matrix notation,

$$\boldsymbol{A} \cdot \boldsymbol{X} = \boldsymbol{T} + \boldsymbol{V} \tag{6.37}$$

where:

$$\boldsymbol{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix} \qquad \boldsymbol{X} = \begin{bmatrix} \bar{X}_1 \\ \bar{X}_2 \end{bmatrix} \qquad \boldsymbol{T} = \begin{bmatrix} \bar{D}_1 \\ \bar{D}_2 \\ \bar{D}_3 \end{bmatrix} \qquad \boldsymbol{V} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$$

A is the drawing matrix, ***X*** is the unknown vector, ***T*** the known terms vector and ***V*** the residual's vector.

The (6-37) can be expressed in the following way:

$$\boldsymbol{A}\cdot\boldsymbol{X}-\boldsymbol{T}=\boldsymbol{V}$$
$$\begin{cases} a_{11}\bar{X}_1+a_{12}\bar{X}_2-\bar{D}_1=v_1 \\ a_{21}\bar{X}_1+a_{22}\bar{X}_2-\bar{D}_2=v_2 \\ a_{31}\bar{X}_1+a_{32}\bar{X}_2-\bar{D}_3=v_3 \end{cases} \tag{6.38}$$

The sum of the squared residuals is

$$\sum_{i=1}^{3} v_i^2 = a_{11}^2\bar{X}_1^2+a_{12}^2\bar{X}_2^2+\bar{D}_1^2+2a_{11}a_{12}\bar{X}_1\bar{X}_2-2a_{11}\bar{X}_1\bar{D}_1-2a_{12}\bar{X}_2\bar{D}_1+$$
$$+a_{21}^2\bar{X}_1^2+a_{22}^2\bar{X}_2^2+\bar{D}_2^2+2a_{21}a_{22}\bar{X}_1\bar{X}_2-2a_{21}\bar{X}_1\bar{D}_2-2a_{22}\bar{X}_2\bar{D}_2+$$
$$+a_{31}^2\bar{X}_1^2+a_{32}^2\bar{X}_2^2+\bar{D}_3^2+2a_{31}a_{32}\bar{X}_1\bar{X}_2-2a_{31}\bar{X}_1\bar{D}_3-2a_{32}\bar{X}_2\bar{D}_3$$

Applying the least square principle:

$$\frac{\partial\sum v^2}{\partial\bar{X}_1}=\bar{X}_1(2a_{11}^2+2a_{21}^2+2a_{31}^2)+\bar{X}_2(2a_{11}a_{12}+2a_{21}a_{22}+2a_{31}a_{32})$$
$$-(2a_{11}\bar{D}_1+2a_{21}\bar{D}_2+2a_{31}\bar{D}_3)=0$$
$$\frac{\partial\sum v^2}{\partial\bar{X}_2}=\bar{X}_1(2a_{11}a_{12}+2a_{21}a_{22}+2a_{31}a_{32})+\bar{X}_2(2a_{12}^2+2a_{22}^2+2a_{32}^2)$$
$$-(2a_{12}\bar{D}_1+2a_{22}\bar{D}_2+2a_{32}\bar{D}_3)=0 \tag{6.39}$$

The (6.39) is a linear system with two equations and two unknowns and it can be re-written by using the matrix notation

$$\boldsymbol{N}\cdot\boldsymbol{X}=\boldsymbol{T_N} \tag{6.40}$$

where:

$$\boldsymbol{N}=\begin{bmatrix} a_{11}^2+a_{21}^2+a_{31}^2 & a_{11}a_{12}+a_{21}a_{22}+a_{31}a_{32} \\ a_{11}a_{12}+a_{21}a_{22}+a_{31}a_{32} & a_{12}^2+a_{22}^2+a_{32}^2 \end{bmatrix}$$
$$\boldsymbol{X}=\begin{bmatrix}\bar{X}_1\\ \bar{X}_2\end{bmatrix} \qquad \boldsymbol{T_N}=\begin{bmatrix} a_{11}\bar{D}_1+a_{21}\bar{D}_2+a_{31}\bar{D}_3 \\ a_{12}\bar{D}_1+a_{22}\bar{D}_2+a_{32}\bar{D}_3 \end{bmatrix} \tag{6.41}$$

The (6.41) is called the normal system and it solves the least squares adjustment of the direct measurements in order to estimate the indirect measurements.

The normal matrix ***N*** is a symmetric square matrix which dimensions coincide with the number of the unknowns, and its terms depends only on the known coefficient of the

initial system.

Also the normalized known term vector $\boldsymbol{T_N}$ depends on the known coefficient of the original system and on the estimated means of the direct measurements.

The estimation of the indirect quantity means is obtained by solving the normal system

$$\boldsymbol{X} = \boldsymbol{N}^{-1} \cdot \boldsymbol{T_N}$$

By considering the properties of the symmetric matrices, the inverse of the normal matrix will be also a symmetric matrix.

Considering the (6.41), the normal matrix $\boldsymbol{N}$ and the normalized known term $\boldsymbol{T_N}$ vector depend only on the known coefficients of the initial system.

Computing the product between the transposed drawing matrix $\boldsymbol{A^T}$ and the drawing matrix $\boldsymbol{A}$ it is possible to verify that the normal matrix $\boldsymbol{N}$ can be written without applying the long procedure above described:

$$\boldsymbol{A^T} \cdot \boldsymbol{A} = \begin{bmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \end{bmatrix} \cdot \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix} = \begin{bmatrix} a_{11}^2 + a_{21}^2 + a_{31}^2 & a_{11}a_{12} + a_{21}a_{22} + a_{31}a_{32} \\ a_{11}a_{12} + a_{21}a_{22} + a_{31}a_{32} & a_{12}^2 + a_{22}^2 + a_{32}^2 \end{bmatrix} = \boldsymbol{N}$$

Computing the product between the transposed drawing matrix $\boldsymbol{A^T}$ and the known term vector $\boldsymbol{T}$ it is possible to verify that the normalized known term vector $\boldsymbol{T_N}$ can be computed in an analogous easy way

$$\boldsymbol{A^T} \cdot \boldsymbol{T} = \begin{bmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \end{bmatrix} \cdot \begin{bmatrix} \overline{D}_1 \\ \overline{D}_2 \\ \overline{D}_3 \end{bmatrix} = \begin{bmatrix} a_{11}\overline{D}_1 + a_{21}\overline{D}_2 + a_{31}\overline{D}_3 \\ a_{12}\overline{D}_1 + a_{22}\overline{D}_2 + a_{32}\overline{D}_3 \end{bmatrix} = \boldsymbol{T_N}$$

The relationships just demonstrated in a small example are valid for each values of *n* and *m* and can therefore be used to solve adjustments where the number of measurements and unknows reach big values (e.g. in photogrammetry adjustment of thousands unknowns and thousands direct measurements are regular cases). In fact the matrix algebra can be easily solved using a software.

In addition we have to consider that the direct measurements can be characterized by different precisions therefore, to correctly manage the adjustment we have to introduce the weights of each of them by multiplying each equation (which depend on only one direct measurement) with the weight of the direct measurement itself.

If we assume as weight matrix

$$\boldsymbol{W} = \begin{bmatrix} w_1 & \cdots & 0 \\ 0 & \ddots & 0 \\ 0 & \cdots & w_n \end{bmatrix}$$

the final, and correct, expression of the normal system which sole the problem of the mean estimation of a set of *m* indirect measurements which depend on *n* direct measurements is

$$\boldsymbol{N = A^T \cdot W \cdot A} \qquad \boldsymbol{T_N = A^T \cdot W \cdot T}$$

To estimate the variances of the indirect measurements, the following relations are valid.

First we have to estimate the variance of the distribution which has the weight equal to 1

$$\bar{\sigma}_0^2 = \frac{\boldsymbol{V^T \cdot W \cdot V}}{n - m} \qquad where \quad \boldsymbol{V = A \cdot X - T}$$

The variance-covariance matrix is defined by the following equation:

$$C_{xx} = \bar{\sigma}_0^2 \cdot \boldsymbol{N}^{-1} = \begin{bmatrix} \bar{\sigma}_{X_1}^2 & \cdots & \sigma_{X_1 X_m} \\ \vdots & \ddots & \vdots \\ \sigma_{X_m X_1} & \cdots & \bar{\sigma}_{X_n}^2 \end{bmatrix} \tag{6.41}$$

The variance-covariance matrix is a symmetric matrix which dimensions are *m* x *m*. It do not depend on the measurements but only on the adopted measurement scheme.

The main diagonal contains the variances of the indirect measurement; the other terms express the statistical correlations between the indirect measurements expressed by means of the co-variance coefficients σ_{XiXj}.

If the relationships which connect the indirect measurements to the direct measurements are not linear combinations, the problem can be solved by means of Taylor's series linearization (avoiding the terms of second power) and by considering an approximate solution of the problem obtained with geometric solutions.

The first iteration define a new approximate values for the indirect measurements that can be used to linearise the equations for the second iteration and so on.

When the differences between one iteration and the previous one is smaller than the possible precision of the indirect measurements the process could be ended.

QUESTIONS

1. Define a gross error and explain the possible ways to eliminate it.
2. Define a systematic error and explain the possible ways to eliminate it.
3. Define a random error and the probability.
4. Define direct measurements.
5. Define indirect measurements.
6. Explain the meaning of the central theorem of Statistics.
7. Describe the Gauss's probability distribution.
8. Which is the meaning of the mean of a Gaussian distribution?
9. Which is the meaning of the m.s.e. of a Guassian distribution?
10. List the probabilities that a random extraction x in a Gaussian distribution can be inside a range around the mean of 1σ, 2σ, 3σ
11. Define precision and accuracy of a measurement.
12. Which are the fundamental properties of an estimator? Define them and explain their meaning.
13. Describe the meaning of the maximum likelihood principle.
14. Derive the least square principle for the Gaussian distribution.
15. Demonstrate that the arithmetic mean is a good estimator for the mean of a direct measurement with equal precision.
16. What is a normal system?
17. What is the variance-covariance matrix?
18. Does the variance-covariance matrix depend on the measurements?